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A UNIQUENESS RESULT FOR HARMONIC FUNCTIONS

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ABSTRACT. Let $d \geq 2$, $D = \mathbb{R}^d \times (0, \infty)$, and suppose u is harmonic in D and C^2 on the closure of D. If the gradient of u vanishes continuously on a subset of ∂D of positive d-dimensional Lebesgue measure and u satisfies certain regularity conditions, then u must be identically constant.

Suppose $d \geq 2$, $D = \{(x_1, \ldots, x_d, x_{d+1}) : x_{d+1} > 0\}$, and u is a function that is harmonic on $\overline{D}$. Suppose both u and its gradient vanish continuously on a subset of ∂D of positive d-dimensional Lebesgue measure.

What further conditions on u are necessary to guarantee that u is identically zero?

This question dates back to at least the 1950s and is apparently due to L. Bers. An answer to the above question may be viewed as a higher-dimensional analog to Privalov's uniqueness theorem. Under suitable further conditions on u, it may also be considered a problem in unique continuation.

That further conditions on u are necessary may be seen from a result of Bourgain and Wolff [6]. They showed that there exists $\alpha \in (0, 1)$ and a harmonic function u that is $C^{1+\alpha}$ on $\overline{D}$ such that both u and ∇u vanish continuously on a subset of ∂D of positive Lebesgue measure.

On the positive side, previous results that give sufficient conditions for u to be identically zero have fallen into two categories. One includes strong assumptions on the behavior of u in a neighborhood of a single point in ∂D; see [3], [11], [12], [13]. The other category of papers assumes that u is identically 0 in a relatively open set in ∂D (here D may be a less regular domain than a half space) and that the gradient vanishes continuously in a subset of that open set of positive measure; see [1], [2], [7], [9], [10].

In this paper we give a new and quite different sufficient condition on u. As far as we have been able to tell, this is the first sufficient condition given only in terms of the behavior of u and its derivatives on a set of positive Lebesgue measure. Let u_i, u_{ij} denote the first and second partial derivatives of u, respectively. For a nonnegative definite matrix a, let $\lambda_1(a)$ denote the largest eigenvalue of a. We

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define the matrices $a(z), \widetilde{a}(z)$ by

$$a_{ij}(z) = \sum_{k=1}^{d+1} u_{ik}(z) u_{kj}(z), \qquad i,j = 1, \dots, d+1,$$

$$\widetilde{a}(z) = \lambda_1(a(z))^{-1} a(z).$$

The matrix $\widetilde{a}$ is initially defined only when $a(z) \neq 0$.

Theorem 1. *Suppose u is C^2 on $\overline{D}$ and nonconstant. Then there does not exist a subset A of ∂D of positive d-dimensional Lebesgue measure such that*:

(i) $\widetilde{a}(z)$ *has a continuous extension to A (we denote the extension by $\widetilde{a}$ also);*
(ii) $\widetilde{a}(z)$ *is of rank at least three for all $z \in A$;*
(iii) ∇u *vanishes continuously on A.*

We make a few remarks:

(1) We do not assume that u also vanishes on A.

(2) We show in Proposition 3 that if $a(z)$ is not zero, then it must be at least of rank 2. Our theorem does not settle what happens when $a(z)$ is of rank two almost everywhere that ∇u vanishes.

(3) Our conditions are those of continuity and nondegeneracy in terms of $\widetilde{a}(z)$. This is natural from the following point of view. Let W_t be a $(d+1)$-dimensional Brownian motion in D and let $U_t = (u_1(W_t), \dots, u_{d+1}(W_t))$. It is easy to see that the question of whether there can exist a set A satisfying the properties of Theorem 1 when u is nonconstant is equivalent to whether the diffusion U_t can hit zero. The behavior of U_t is completely determined by the coefficients $a_{ij}(W_t)$. The matrix $\widetilde{a}$ describes and governs the behavior of a certain time change of U_t. If this time change of U_t never hits zero, then U_t cannot hit zero either.

(4) It is possible for diffusions in two dimensions with continuous diffusion coefficients to hit 0. It is also possible for diffusions in three and more dimensions with discontinuous coefficients to hit 0. So the conditions on $\widetilde{a}$ might not be too much stronger than what is necessary.

(5) Our technique is probabilistic.

Proposition 2. *Suppose $x_0 > 0$ and*

$$X_t = x_0 + \int_0^t A_s \, dW_s + \int_0^t \frac{B_s}{X_s} \, ds,$$

where W_t is a standard one-dimensional Brownian motion, A_s and B_s are adapted to the σ-fields generated by W, and the second term on the right is the stochastic integral of Itô. If $B_t \geq \frac{1}{2} A_t^2$ for all t almost surely, then with probability one X_t never hits the point 0.

Proof. Let $M_t = \int_0^t A_s \, dW_s$, so that $\langle M \rangle_t = \int_0^t A_s^2 \, ds$. Let us first suppose that $\langle M \rangle_t \to \infty$ a.s. as $t \to \infty$. Let $\tau_t = \inf\{s : \langle M \rangle_s > t\}$, $Y_t = X_{\tau_t}$, $N_t = M_{\tau_t}$, and $C_t = B_{\tau_t}$. It is well known that this time change makes N_t a continuous martingale with $\langle N \rangle_t = t$, and by Lévy's theorem ([4], p. 50), N_t is a Brownian motion. So

$$Y_t = x_0 + N_t + \int_0^t \frac{C_s}{Y_s} \, ds.$$

Moreover

$$\int_0^{\tau_t} \frac{B_s}{X_s}\,ds = \int_0^t \frac{B_{\tau_u}}{A_{\tau_u}^2 X_{\tau_u}}\,du,$$

so

$$C_s = \frac{B_{\tau_s}}{A_{\tau_s}^2} \geq \frac{1}{2}.$$

Let $\varepsilon > 0$. Let Z_t be a Bessel process of index 2 started at x_0 and driven by the Brownian motion N_t, which means that

$$Z_t = x_0 + N_t + \int_0^t \frac{1}{2Z_s}\,ds.$$

Let $f(x)$ be a C^2 function that equals $1/x$ for $x > \varepsilon$ and let $\widetilde{Y}_t$ and $\widetilde{Z}_t$ be processes satisfying

$$\widetilde{Y}_t = x_0 + N_t + \int_0^t C_s f(\widetilde{Y}_s)\,ds,$$
$$\widetilde{Z}_t = x_0 + N_t + \int_0^t \frac{1}{2} f(\widetilde{Z}_s)\,ds.$$

Clearly $Y_t = \widetilde{Y}_t$ and $Z_t = \widetilde{Z}_t$ up until the time each first hits ε. By a stochastic comparison theorem ([8], Theorem VI.1.1), $\widetilde{Y}_t \geq \widetilde{Z}_t$ for t less than the first time $\widetilde{Z}_t$ hits ε. So $Y_t \geq Z_t$ up until the first time Z_t hits ε. Letting $\varepsilon \to 0$ and using the fact that Z_t never hits 0 ([5], Proposition I.7.2), we see that Y_t also never hits 0. Since X_t is a time change of Y_t, then X_t never hits 0 either.

If $\langle M \rangle_t$ does not tend to ∞ as $t \to \infty$, then we have the same formula for Y, except that N_t is now a Brownian motion stopped at a stopping time, and neither N_t nor Y_t changes after that stopping time. Just as above, X_t does not hit 0 in this case either. □

The following proposition is of interest, but is not needed for the proof of Theorem 1.

Proposition 3. *If a is not identically 0, then a has rank at least two.*

Proof. Define the matrix $\sigma_{ij}(x) = u_{ij}(x)$. Since u is harmonic, then $\mathit{trace}\,(\sigma) = 0$. σ is symmetric because u is C^2. Note $a = \sigma^2$. Let $\mu_1, \mu_2, \cdots$ be the eigenvalues of σ, arranged in decreasing order of absolute value. Since $\mathit{trace}\,(\sigma) = 0$, then $\mu_1 + \cdots + \mu_{d+1} = 0$. μ_1 cannot be 0 if $a \neq 0$. Then at least one of $\mu_2, \mu_3, \ldots, \mu_{d+1}$ must be greater than $|\mu_1|/d$ in absolute value. This implies that σ is at least of rank two, and hence a is also. □

For $x \in \mathbb{R}^d$, let $G_h(x) = \{z' = (x', y') : x' \in \mathbb{R}^d, 0 < y' < h, |x - x'| < y'\}$. Let W_t be a Brownian motion in $\mathbb{R}^{d+1}$. For any Borel set F, let $\tau_F = \inf\{t : W_t \notin F\}$, the first exit time of a W_t from F.

Proposition 4. *Let $h > 0$. Suppose B is a subset of ∂D with positive d-dimensional Lebesgue measure and with $\mathit{diam}\,(B) < h$. Let $E = \bigcup_{x \in B} G_h(x)$. If $z \in E$, then $\mathbb{P}^z(\tau_D = \tau_E) > 0$.*

Proof. We need to prove that starting in E there is positive probability that W_t exits E by hitting B. Let $H(z) = \mathbb{P}^z(W_{\tau_D} \in B)$. If $z' = (x', y')$ with $x' \in \mathbb{R}^d$ and $y' > 0$, then by the formula for the Poisson kernel,

$$\mathbb{P}^{z'}(|W_{\tau_D} - x'| < y') = c_1 \int_{B(x',y')} \frac{y'}{((y')^2 + |x' - x|^2)^{(d+1)/2}} dx \geq \delta,$$

where δ is independent of z'. So if $z' \in \partial E - \partial D$ with $y' < h$, there is probability at least δ that, starting at z', the Brownian motion W_t will exit D in $\{x \in \partial D : |x - x'| < y'\} \subset \partial D - B$, and thus $H(z') \leq 1 - \delta$. Since $diam\,(B) < h$, there exists $x_0 \in \partial D$ such that $B \subset B(x_0, h/2)$, where $B(x_0, h/2)$ is the ball in ∂D centered at x_0 with radius $h/2$. Hence $E \subset B(x_0, 3h/2) \times (0, h)$. If $z' = (x', h)$, clearly there is positive probability bounded away from 0 that starting at z' the Brownian motion will exit D in $\partial D - B(x_0, 3h/2)$. Making δ smaller if necessary, we thus have $H(z') \leq 1 - \delta$ whenever $z' \in \partial E - \partial D$.

We are given that H is harmonic in D, so by Doob's optional stopping theorem and the strong Markov property, if $z \in E$,

$$\begin{aligned} H(z) = \mathbb{E}^z H(W_{\tau_D \wedge \tau_E}) &= \mathbb{E}^z[H(W_{\tau_E}); \tau_E < \tau_D] + \mathbb{E}^z[H(W_{\tau_D}); \tau_E = \tau_D] \\ &\leq (1 - \delta)\mathbb{P}^z(\tau_E < \tau_D) + \mathbb{P}^z(\tau_D = \tau_E) \\ &= 1 - \delta\mathbb{P}^z(\tau_E < \tau_D). \end{aligned}$$

By Fatou's theorem, $H(z) \to 1$ as z tends to x within E for almost every point $x \in B$. So there exists $z_0 \in E$ and a neighborhood S of z_0 such that $S \subset E$ and $H \geq 1 - \delta/2$ in S. This implies that $\mathbb{P}^z(\tau_E < \tau_D) \leq \frac{1}{2}$ for $z \in S$. By the support theorem, starting at any point in E there is positive probability of hitting S before exiting E; this and the strong Markov property imply the proposition. $\square$

Proof of Theorem 1. We consider the space of $(d+1) \times (d+1)$ matrices with norm given by $\|b\| = \sup_{\|x\|_2 \leq 1} \|bx\|_2$, where $\|x\|_2$ is the ℓ^2 norm on $\mathbb{R}^{d+1}$. Suppose b is a nonnegative definite matrix whose largest eigenvalue is 1 and has rank at least 3. There is an orthogonal matrix p such that $p^t bp$ is a diagonal matrix with the first diagonal entry equal to 1 and the next two diagonal entries positive. We can then find an invertible matrix q such that $q^t bq$ is a diagonal matrix with the first three diagonal entries equal to one and all the other diagonal entries less than or equal to 1 in absolute value. Note that $trace\,(q^t bq) \geq 3$. By continuity, there is a neighborhood of the matrix b such that if c is a nonnegative definite matrix in this neighborhood of b, then

$$trace\,(q^t cq) \geq 2\lambda_1(c). \tag{1}$$

We can find a countable number of nonnegative definite matrices b_i of rank at least three with neighborhoods V_i and invertible matrices q_i such that the collection $\{V_i\}$ covers the set of nonnegative definite matrices of rank at least 3 and if $c \in V_i$, then (1) holds with q replaced by q_i.

Suppose u satisfies the hypotheses of Theorem 1 and u is nonconstant. Let $A_i = \{x \in A : \widetilde{a}(x) \in V_i\}$. Since A has positive measure, there exists i such that A_i has positive measure. We have that $\widetilde{a}$ extends continuously to ∂D and the eigenvalues of a matrix are continuous functions of the coefficients, so there exists a set $B \subset A_i$ of positive measure and $h > 0$ such that $\widetilde{a}(z) \in V_i$ whenever $x \in B$ and $z \in G_h(x)$. Without loss of generality we may assume that $diam\,(B) < h$. Let $E = \bigcup_{x \in B} G_h(x)$ and pick a point $z_0 \in E$.

By Proposition 4 there is positive probability that started at z_0, a Brownian motion W_t will remain in E up until τ_D. Define a process $U_t = (u_1(W_t), \dots, u_{d+1}(W_t))$. Since there is positive probability that W_t stays in E until hitting ∂D, there is positive probability that W_{τ_D} is in B and thus positive probability that $U_{\tau_D} = 0$. We will show that U_t can never hit 0 before leaving E, which leads to a contradiction, and hence to the conclusion that u must be constant.

Each u_i is harmonic, so $\Delta u_i = 0$, and by Ito's formula,

$$u_i(W_t) = u_i(W_0) + \sum_{j=1}^{d+1} \int_0^t u_{ij}(W_s)\, dW_s^j.$$

Therefore each component of U_t is a continuous martingale and the quadratic variations are given by

$$\langle U^i, U^j \rangle_t = \sum_{k=1}^{d+1} \int_0^t u_{ik}(W_s) u_{jk}(W_s)\, ds = \int_0^t a_{ij}(W_s)\, ds.$$

Note that we have here $a_{ij}(W_s)$ and not $a_{ij}(U_s)$; with the latter a much more delicate analysis would be possible.

For $t < \tau_E$ we have $\widetilde{a} \in V_i$. Let $H_t = q_i^t U_{t \wedge \tau_E}$. Clearly it is possible for U_t to hit 0 while W_t is in B only if H_t ever hits 0. Let $c = q_i^t a q_i$. It is easy to see that $\langle H^i, H^j \rangle_t = \int_0^{t \wedge \tau_E} q_i^t a q_i(W_s)\, ds$. A straightforward calculation using Ito's formula with the function $f(x) = |x|$ (cf. [5], Proposition V.2.1) shows that if $I_t = |H_t|$, then

$$I_t = M_t + \frac{1}{2} \int_0^{t \wedge \tau_E} \frac{trace\,(c(W_s)) - \sum_{i,j=1}^{d+1} H_s^i c_{ij}(W_s) H_s^j / I_s^2}{I_s}\, ds,$$

where

$$\langle M \rangle_t = \int_0^{t \wedge \tau_E} \sum_{i,j=1}^{d+1} \frac{H_s^i c_{ij}(W_s) H_s^j}{I_s^2}\, ds.$$

This means there is a one-dimensional Brownian motion $\widehat{W}_t$ such that $M_t = \int_0^{t \wedge \tau_E} A_s\, d\widehat{W}_s$ with $A_s = (\sum_{i,j=1}^{d+1} (H_s^i c_{ij}(W_s) H_s^j)/I_s^2)^{1/2}$. For $t < \tau_E$ we have $\widetilde{a}(W_t) \in V_i$, and hence

$$trace\,(c(W_t)) \geq 2\lambda_1(c(W_t)) \geq 2 \sum_{i,j=1}^{d+1} \frac{H_t^i c_{ij}(W_t) H_t^j}{I_t^2}.$$

We now apply Proposition 2 and conclude that U_t never hits 0, our contradiction. $\square$

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