REMOVABLE SETS FOR CONTINUOUS SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

TERO KILPELÄINEN AND XIAO ZHONG

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ABSTRACT. We show that sets of \( n - p + \alpha(p - 1) \) Hausdorff measure zero are removable for \( \alpha \)-Hölder continuous solutions to quasilinear elliptic equations similar to the \( p \)-Laplacian. The result is optimal. We also treat larger sets in terms of a growth condition. In particular, our results apply to quasiregular mappings.

1. INTRODUCTION

Throughout this paper we let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( 1 < p < \infty \) a fixed number. Continuous solutions \( u \in W^{1,p}_{\text{loc}}(\Omega) \) of the equation

\[
- \text{div} \mathcal{A}(x, \nabla u) = 0
\]

are called \( \mathcal{A} \)-harmonic in \( \Omega \). Here \( \mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is assumed to verify for some constants \( 0 < \lambda \leq \Lambda < \infty \):

\[
\begin{align*}
\text{the function } x &\mapsto \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \\
\text{and the function } \xi &\mapsto \mathcal{A}(x, \xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n;
\end{align*}
\]

for all \( \xi \in \mathbb{R}^n \) and a.e. \( x \in \mathbb{R}^n \)

\[
(\mathcal{A}(x, \xi) : \xi \geq \lambda |\xi|^p,
\]

\[
|\mathcal{A}(x, \xi)| \leq \Lambda|\xi|^{p-1},
\]

\[
(\mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta)) \cdot (\xi - \zeta) > 0,
\]

whenever \( \xi \neq \zeta \). A prime example of the operators is the \( p \)-Laplacian

\[
-\Delta_p u = -\text{div}(|\nabla u|^{p-2} \nabla u).
\]

In this case, the continuous solutions of (1.1) are called \( p \)-harmonic functions. The main result in this paper is the following theorem.
1.6. Theorem. Let $E \subset \Omega$ be closed and $s > 0$. Suppose that $u$ is a continuous function in $\Omega$, $A$-harmonic in $\Omega \setminus E$ such that
\begin{equation}
|u(x_0) - u(y)| \leq C|x_0 - y|^{(s + p - n)/(p - 1)}
\end{equation}
for all $y \in \Omega$ and $x_0 \in E$. If $E$ is of $s$-Hausdorff measure zero, then $u$ is $A$-harmonic in $\Omega$.

Since sets of $p$-capacity zero are removable for bounded $A$-harmonic functions, Theorem 1.6 is interesting for $s > n - p$ only. Kilpeläinen, Koskela, and Martio [KKM] had a special version of Theorem 1.6, where $u$ was assumed to be flat on $E$ and Hausdorff measure was replaced by a Minkowski content type condition.

1.8. Corollary. Suppose that $u \in C^{0,\alpha}(\Omega)$, $0 < \alpha \leq 1$, is $A$-harmonic in $\Omega \setminus E$. If $E$ is a closed set of $n - p + \alpha(p - 1)$ Hausdorff measure zero, then $u$ is $A$-harmonic in $\Omega$.

The following theorem shows that Corollary 1.8 is optimal. Before stating the theorem, we recall that there is a constant $K$, $0 < K = K(n, p, \lambda) \leq 1$, such that every $A$-harmonic function $h$ in $\Omega$ verifies the local Hölder continuity estimate
\begin{equation}
\text{osc}(h, B(x, r)) \leq c\left(\frac{r}{R}\right)^\kappa \text{osc}(h, B(x, R))
\end{equation}
for each $0 < r < R$ and $B(x, R) \subset \Omega$ [HKM 6.6]. For smooth $A$, in particular for the $p$-Laplacian, we may choose $\kappa = 1$ (see e.g. [K 2.3]).

1.10. Theorem. Let $\kappa$ be as above and $0 < \alpha < \kappa$. Suppose that $E \subset \Omega$ is a closed set with positive $n - p + \alpha(p - 1)$ Hausdorff measure. Then there is $u \in C^{0,\alpha}(\Omega)$ which is $A$-harmonic in $\Omega \setminus E$, but does not have an $A$-harmonic extension to $\Omega$.

For the $p$-Laplacian we have the following sharp result.

1.11. Corollary. Let $0 < \alpha < 1$. A closed set $E$ is removable for $\alpha$-Hölder continuous $p$-harmonic functions if and only if $E$ is of $n - p + \alpha(p - 1)$ Hausdorff measure\footnote{Assume, of course, that $\alpha \geq (p - n)/(p - 1)$.} zero.

Carleson [C] proved Corollary 1.11 for the Laplacian ($p = 2$). As to the quasilinear case, Heinonen and Kilpeläinen [HK 4.5] proved Corollary 1.8 with $\alpha = 1$, and Trudinger and Wang [TW] proved it under the assumption that $u$ has an $A$-superharmonic extension to $\Omega$, which can be dispensed with for small $\alpha$. However, in the general situation the growth condition of Theorem 1.6 yields a more useful result, since $A$-harmonic functions are not in general in $C^{0,\alpha}$ for $\alpha$ close to 1. Koskela and Martio [KM2] proved a weaker version of Corollary 1.13 and 1.8, where Minkowski content is used in place of Hausdorff measure. Buckley and Koskela [BK] also established very special cases of Corollary 1.8. In [K] there is a weaker version of Theorem 1.10.

A mapping $f : \Omega \to \mathbb{R}^n$ is called quasiregular if $f \in W^{1,n}_{\text{loc}}(\Omega)$ and there is a constant $K$ such that
\begin{equation}
|f'(x)|^n \leq K J_f(x)
\end{equation}
for a.e. $x \in \Omega$; here $J_f(x)$ is the Jacobian determinant of $f$ at $x$. The coordinate functions of a quasiregular map $f$ satisfy an equation of type (1.1) with $p = n$ (cf. [HKM Ch. 14]), whence we have:
1.12. Corollary. Let $E \subset \Omega$ be a closed set of $s$-Hausdorff measure zero, $0 < s \leq n$. Suppose that $f : \Omega \to \mathbb{R}^n$ is a continuous mapping, quasiregular in $\Omega \setminus E$. If
\[
|f(x_0) - f(y)| \leq C|x_0 - y|^{s/(n-1)}
\]
for all $y \in \Omega$ and $x_0 \in E$, then $f$ is quasiregular in $\Omega$.

1.13. Corollary. Suppose that $f \in C^{0,\alpha}(\Omega)$ is quasiregular in $\Omega \setminus E$. If $E$ is a closed set of $\alpha(n-1)$-Hausdorff measure zero, then $f$ is quasiregular in $\Omega$.

Koskela and Martio [KM] showed that sets whose Minkowski dimension is less than $\alpha$ are removable for $\alpha$-Hölder continuous quasiregular mappings provided that $\alpha < 1 - 1/n$, and the same for sets of $\alpha n$-Hausdorff measure zero if $\alpha \leq 1/n$.

Our method of proof combines some ideas from [K], [L], and [TW]. We use solutions of equations
\[- \operatorname{div} A(x, \nabla u) = \mu,\]
where $\mu$ is a nonnegative Radon measure from $W^{-1,p'}_{\text{loc}}(\Omega)$, i.e. $u \in W^{1,p}_{\text{loc}}(\Omega)$ and
\[\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu\]
for all $\varphi \in C_{0}^{\infty}(\Omega)$. In particular, we prove the following theorem that improves the main theorem in [K].

1.14. Theorem. Let $\kappa$ be the number given by (1.9). Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a solution of
\[- \operatorname{div} A(x, \nabla u) = \mu,\]
where $\mu$ is a nonnegative Radon measure such that there are constants $M > 0$ and $0 < \alpha < \kappa$ with
\[
\mu(B(x,r)) \leq Mr^{n-p+\alpha(p-1)}
\]
whenever $B(x,3r) \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$. Moreover, $\kappa(n,p,1,1) = 1$, that is, in the case of the $p$-Laplacian any $\alpha < 1$ will do.

Theorem 1.14 is the best possible (see [KM, 4.18], [K, 2.7]).

Finally, we remark here that Corollary 1.11 is not true when $\alpha = 1$. The problem for which sets are removable for Lipschitz continuous $p$-harmonic functions is more delicate. David and Mattila [DM] treated the case $n = p = 2$: a compact set $E$ of finite 1-Hausdorff measure is removable for Lipschitz continuous harmonic functions if and only if $E$ is purely unrectifiable. The other cases remain open.

2. Proof of Theorem 1.6

We need a potential theoretic version of the obstacle problem. Suppose that $\psi$ is a continuous function on $\Omega$ and let the balayage $\hat{\psi} = \hat{\psi} = \hat{\psi}(\Omega)$ be the pointwise infimum of all supersolutions\footnote{I.e. $u \in W^{1,p}_{\text{loc}}(\Omega)$ and $- \operatorname{div} A(x, \nabla u) \geq 0$ in $\Omega$.} to (1.1) that lie above $\psi$ in $\Omega$. Similarly, let $\hat{\psi} = \hat{\psi}(\Omega)$ be the pointwise supremum of all subsolutions that lie below $\psi$ in $\Omega$. Redistribution or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Then $\hat{R}^{\psi} \geq \psi$ is a continuous supersolution in $\Omega$ and $A$-harmonic in $\{\hat{R}^{\psi} > \psi\}$; similar statements hold for $\hat{R}^{\psi}$. For a more thorough discussion see [HKM, Ch. 9]. Next we show the following estimate for the balayage; see [L] for a related result.

2.1. Lemma. Let $K \subset \Omega$ be compact. Suppose that $\psi$ is a continuous function with

$$|\psi(x) - \psi(y)| \leq M|x - y|^\alpha \text{ for all } x \in K \text{ and } y \in \Omega,$$

where $M > 0$ and $\alpha > 0$. Let $u = \hat{R}^{\psi}$ and

$$\mu = - \text{div } A(x, \nabla u),$$

Then

$$\mu(B(x, r)) \leq cr^{n-p+\alpha(p-1)}$$

for all $r < r_0 = \frac{1}{64} \text{ dist}(K, \partial \Omega)$ and $x \in K$; here $c = c(n, p, \Lambda, M, \alpha) > 0$.

Proof. Write

$$I = \{x \in \Omega : \psi(x) = u(x)\}$$

for the contact set.

First, let $x_0 \in I$. We assume, as we may, that $u(x_0) = 0 = \psi(x_0)$. If $r \leq \frac{1}{8} \text{ dist}(x_0, \partial \Omega)$ and

$$\gamma_0 = \text{osc}(\psi, B(x_0, 8r)),$$

then $(u - \gamma_0)^+$ is a subsolution and $u + \gamma_0$ a nonnegative supersolution in $B(x_0, 8r)$. Hence we deduce from the weak Harnack inequalities [HKM, 3.34 and 3.59] that

$$\sup_{B(x_0, r)} (u - \gamma_0) \leq c \left( \int_{B(x_0, 2r)} |(u - \gamma_0)^+|^{p-1} \, dx \right)^{1/(p-1)}$$

$$\leq c \left( \int_{B(x_0, 2r)} (u + \gamma_0)^{p-1} \, dx \right)^{1/(p-1)}$$

$$\leq c \inf_{B(x_0, 2r)} (u + \gamma_0)$$

$$\leq c\gamma_0.$$

Keeping in mind that $u \geq \psi \geq -\gamma_0$ we conclude

$$\text{osc}(u, B(x_0, r)) \leq c\gamma_0 = c\text{osc}(\psi, B(x_0, 8r)).$$

Let $r \leq \frac{1}{32} \text{ dist}(x_0, \partial \Omega)$ and let $\eta \in C_0^\infty(B(x_0, 2r))$ be a usual nonnegative cut-off function with $\eta = 1$ in $B(x_0, r)$ and $|\nabla \eta| \leq 2/r$. Then we obtain by applying the Caccioppoli estimate [HKM, 3.29] to $u - \sup_{B(x_0, 2r)} u$ and (2.2) that

$$\mu(B(x_0, r)) \leq \int_{B(x_0, 2r)} \eta^p \, d\mu = p \int_{B(x_0, 2r)} \eta^{p-1} A(x, \nabla u) \cdot \nabla \eta \, dx$$

$$\leq c \left( \int_{B(x_0, 2r)} |\nabla u|^p \eta^p \, dx \right)^{(p-1)/p} \left( \int_{B(x_0, 2r)} |\nabla \eta|^p \, dx \right)^{1/p}$$

$$\leq c r^{n-p} \text{osc}(u, B(x_0, 2r))^{p-1}$$

$$\leq c r^{n-p} \text{osc}(\psi, B(x_0, 16r))^{p-1}.$$
Now if $x_0 \in I$ is such that
\[ \text{dist}(x_0, K) \leq r \leq 2r_0, \]
we have the estimate
\[ \mu(B(x_0, r)) \leq c r^{n-p+\alpha(p-1)}, \tag{2.3} \]
where $c = c(n, p, M) > 0$.

Finally, for $x_0 \in K$ and $r < r_0$, there are two alternatives. Either $B(x_0, r) \cap I = \emptyset$ and thus $\mu(B(x_0, r)) = 0$, or there is $x \in B(x_0, r) \cap I$. In this latter case
\[ \mu(B(x_0, r)) \leq \mu(B(x, 2r)) \leq c r^{n-p+\alpha(p-1)} \]
by (2.3). The lemma is proven.

\textbf{Remark.} Using (1.9) and (2.2), one can easily prove that if $2 C_0 (\mathbb{R})$, then $^R_2 C_0 (\mathbb{R})$, where $\alpha = \min(\alpha, \kappa)$ and $\kappa > 0$ is the constant such that (1.9) holds (see e.g. [HKM, 6.47]).

\textbf{Proof of Theorem 1.6.} Fix a regular set $D \subset \subset \Omega$, for instance a ball. Let $v = ^R u = ^R u(D)$ and
\[ \mu = - \text{div} \mathcal{A}(x, \nabla v). \]
Let $K \subset E \cap D$ be compact. Since sets of $n - p$ Hausdorff measure zero $\mu E \cap D$ are known to be removable for bounded $\mathcal{A}$-harmonic functions (see e.g. [HKM]), we need only consider the case where $\alpha = (s+p-n)/(p-1) > 0$. Since $s = n-p+\alpha(p-1)$ we infer from (1.7) and Lemma 2.1 that
\[ \mu(B(x, r)) \leq c r^s \]
for all $r \leq r_0$ and $x \in K$. Because $\mathcal{H}^s(K) = 0$, we may cover $K$ by balls $B(x_j, r_j)$ so that
\[ \mu(K) \leq \sum_j \mu(B(x_j, r_j)) \leq c \sum_j r_j^s < \varepsilon, \]
where $\varepsilon > 0$ is given. Consequently, $\mu(E \cap D) = 0$ and therefore $\mu = 0$, which means that $v$ is $\mathcal{A}$-harmonic in $D$ [Mi 3.19].

Next let $w = ^R u(D)$. We similarly find that $w$ is $\mathcal{A}$-harmonic in $D$. Since $v = u = w$ on $\partial D$ by [HKM, 9.26], we have that $v = w$ in $D$ by the uniqueness of $\mathcal{A}$-harmonic functions. Since $w \leq u \leq v = w$,

$u$ is $\mathcal{A}$-harmonic in $D$ and the theorem follows.

\textbf{3. Proof of Theorems 1.14 and 1.10}

We recall that $\kappa$ is the constant such that (1.9) holds for every $\mathcal{A}$-harmonic function $h$ in $\Omega$. Then
\[ \int_{B(x,r)} |\nabla h|^p \, dx \leq c \left( \frac{T}{R} \right)^{n-p+\kappa} \int_{B(x,R)} |\nabla h|^p \, dx, \tag{3.1} \]
for each $0 < r < R$ with $B(x, R) \subset \Omega$; here $c = c(n, p, \Lambda) > 0$ (see e.g. [K 2.1]).
The following lemma provides the key estimate.

3.2. Lemma. Let $u \in W^{1,p}(B(x_0, R))$ be a solution of
\[- \operatorname{div} A(x, \nabla u) = \mu,\]
where $\mu$ is a nonnegative Radon measure such that
\[\mu(B(x_0, r)) \leq c_0 r^{n-p+\alpha(p-1)}\]
for all $0 < r \leq R$. Then for each $0 < r < R$ and $\varepsilon > 0$ we have
\[\int_{B(x_0, r)} |\nabla u|^p \, dx \leq c_1 \left( \frac{r}{R} n^{p+\rho} + \varepsilon \right) \int_{B(x_0, R)} |\nabla u|^p \, dx + c_2 R^{n-p+\rho},\]
where $c_1 = c_1(n, p, \lambda, \Lambda) > 0$ and $c_2 = c_2(n, p, \lambda, \Lambda, \alpha, c_0, \varepsilon) > 0$.

Proof. There is no loss of generality in assuming that $r < R/2$. Let $h$ be the $A$-harmonic function in $B(x_0, R)$ with $u - h \in W^{1,p}_0(B(x_0, R))$. Then
\[
\begin{align*}
\lambda \int_{B(x_0, r)} |\nabla u|^p \, dx &\leq \int_{B(x_0, r)} A(x, \nabla u) \cdot \nabla u \, dx \\
&= \int_{B(x_0, r)} (A(x, \nabla u) - A(x, \nabla h)) \cdot (\nabla u - \nabla h) \, dx \\
&\quad + \int_{B(x_0, r)} A(x, \nabla h) \cdot (\nabla u - \nabla h) \, dx + \int_{B(x_0, r)} A(x, \nabla u) \cdot \nabla h \, dx \\
&\leq \int_{B(x_0, R)} (A(x, \nabla u) - A(x, \nabla h)) \cdot (\nabla u - \nabla h) \, dx \\
&\quad + \lambda \int_{B(x_0, r)} |\nabla h|^{p-1} |\nabla u| + |\nabla h| |\nabla u|^{p-1} \, dx
\end{align*}
\]
where we used the structural assumptions (1.3)-(1.5). Since $h$ is $A$-harmonic with $h - u \in W^{1,p}_0(B(x_0, R))$ and thus quasiminimizes the $p$-Dirichlet integral, we have by using Adams’ inequality (see [AH Thm 7.2.2] or [Z Thm 4.7.2]) that
\[
\begin{align*}
\int_{B(x_0, R)} (A(x, \nabla u) - A(x, \nabla h)) \cdot (\nabla u - \nabla h) \, dx &= \int_{B(x_0, R)} (u - h) \, d\mu \\
&\leq c R^{(p-1)(n-p+\rho)/p} \left( \int_{B(x_0, R)} |\nabla u - \nabla h|^p \, dx \right)^{1/p} \\
&\leq c R^{n-p+\rho} + \frac{\lambda}{2} \int_{B(x_0, R)} |\nabla u|^p \, dx,
\end{align*}
\]
where we also used Young’s inequality. The remaining integrals on the right of (3.3) do not exceed
\[
\begin{align*}
\frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p \, dx + c \int_{B(x_0, r)} |\nabla h|^p \, dx \\
&\leq \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p \, dx + c \left( \frac{r}{R} n^{p+\rho} \right) \int_{B(x_0, R)} |\nabla h|^p \, dx \\
&\leq \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p \, dx + c \left( \frac{r}{R} n^{p+\rho} \right) \int_{B(x_0, R)} |\nabla u|^p \, dx,
\end{align*}
\]
where we also employed (3.1) and the quasiminimizing property of $A$-harmonic functions. Plugging these estimates in (3.3) we arrive at
\[
\int_{B(x_0, r)} |\nabla u|^p \, dx \leq c R^{n-p+\alpha} + c \int_{B(x_0, R)} |\nabla u|^p \, dx + c \left( \frac{r}{R} \right)^{n-p+\alpha} \int_{B(x_0, R)} |\nabla u|^p \, dx.
\]

The lemma follows.

Proof of Theorem 1.14. If $B(x_0, 4R) \subset \Omega$, then by appealing to [G, Lemma III.2.1, p. 86] Lemma 3.2 yields
\[
\int_{B(x_0, r)} |\nabla u|^p \, dx \leq c \left( \frac{r}{R} \right)^{n-p+\alpha}
\]
for $r < R$. Thus $u \in C^{0,\alpha}(\Omega)$ by the Dirichlet growth theorem [G, Theorem III.1.1, p. 64].

Proof of Theorem 1.10. Let $\kappa$ be the number as in Theorem 1.14. Let $K \subset E$ be compact with $H^{n-p+\alpha(p-1)}(K) > 0$. Frostman’s lemma ([AH, 5.1.12], [C]) gives us a nonnegative Radon measure $\mu$ living on $K$ with $\mu(K) > 0$ and $\mu(B(x, r)) \leq r^{n-p+\alpha(p-1)}$. Any solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to
\[- \text{div} A(x, \nabla u) = \mu\]
is $A$-harmonic in $\Omega \setminus E$ [M, 3.19] and $u \in C^{0,\alpha}(\Omega)$ by Theorem 1.14, but $u$ fails to have an $A$-harmonic extension to $\Omega$, since $\mu(K) > 0$.

References


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JYVÄSKYLÄ, P.O. BOX 35, 40351 JYVÄSKYLÄ, FINLAND

E-mail address: terok@math.jyu.fi

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JYVÄSKYLÄ, P.O. BOX 35, 40351 JYVÄSKYLÄ, FINLAND

E-mail address: zhong@math.jyu.fi