ON ONE PROBLEM OF UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING SMALL FUNCTIONS

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Abstract. In this paper, we show that if two non-constant meromorphic functions \( f \) and \( g \) satisfy \( E(a_j, k, f) = E(a_j, k, g) \) for \( j = 1, 2, \ldots, 5 \), where \( a_j \) are five distinct small functions with respect to \( f \) and \( g \), and \( k \) is a positive integer or \( 1 \) with \( k \geq 14 \), then \( f \equiv g \). As a special case this also answers the long-standing problem on uniqueness of meromorphic functions concerning small functions.

1. Introduction and main result

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane \( \mathbb{C} \). We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1]. For any non-constant meromorphic function \( f(z) \), we denote by \( S(r, f) \) any quantity satisfying

\[
S(r, f) = o(T(r, f))
\]

for \( r \to \infty \) except possibly a set of \( r \) of finite linear measure. A meromorphic function \( a(z) \) is called a small function with respect to \( f(z) \) if \( T(r, a) = S(r, f) \). Let \( S(f) \) be the set of meromorphic functions in the complex plane \( \mathbb{C} \) which are small functions with respect to \( f \). Note that \( \mathbb{C} \in S(f) \) and \( S(f) \) is a field (see [2]).

If \( f(z) \) is a non-constant meromorphic function, \( a(z) \in S(f) \cup \{\infty\} \), and \( k \) is a positive integer or \( \infty \), we denote by \( E(a, k, f) \) the set of distinct zeros of \( f(z) - a(z) \) with multiplicities \( \leq k \), where \( f(z) - \infty \) means \( 1/f(z) \) (see [11] p.195). In particular, we denote by \( E(a, \infty, f) \) the set of distinct zeros of \( f(z) - a(z) \), and we denote it simply by \( E(a, f) \).

Let \( f(z) \) and \( g(z) \) be non-constant meromorphic functions and let \( a(z) \in S(f) \cap S(g) \} \cup \{\infty\} \). We denote by \( N_0(r, a, f, g) \) the counting function of common zeros of \( f(z) - a(z) = 0 \) and \( g(z) - a(z) = 0 \) (ignoring multiplicities), each point counted only once. Let

\[
\mathcal{N}_{12}(r, a, f, g) := N(r, a, f) + N(r, a, g) - 2N(r, a, f, g).
\]

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Then $N_{12}(r,a,f,g)$ denotes the counting function of different solutions to $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$ (see [7] p.107). If

$$N(r,a,f) - N_0(r,a,f,g) = 0 \quad \text{and} \quad N(r,a,g) - N_0(r,a,f,g) = 0,$$

we say $f(z)$ and $g(z)$ share $a(z)$ IM. If

$$N(r,a,f) - N_0(r,a,f,g) = S(r,f) \quad \text{and} \quad N(r,a,g) - N_0(r,a,f,g) = S(r,g),$$

we say $f(z)$ and $g(z)$ share $a(z)$ “IM” (see [10] p.254). It is obvious that if $f(z)$ and $g(z)$ share $a(z)$ IM, then $E(a,f) = E(a,g)$, and $N_{12}(r,a,f,g) = 0$; if $f(z)$ and $g(z)$ share $a(z)$ “IM”, then $N_{12}(r,a,f,g) = S(r,f) + S(r,g)$.

In the 1920s, R. Nevanlinna established the following famous second fundamental theorem:

**Theorem A** ([12] p.70). Let $f(z)$ be a non-constant meromorphic function, and let $a_1, a_2, \ldots, a_q$ be $q \geq 3$ distinct elements in $\mathbb{C} \cup \{\infty\}$. Then

$$(q - 2)T(r,f) \leq \sum_{j=1}^{q} N(r,a_j,f) + S(r,f).$$

In 1929, R. Nevanlinna proved the following well-known theorem on the uniqueness of meromorphic functions by making use of Theorem A:

**Theorem B** ([7] p.109, see also [1] p.48). If $f$ and $g$ are meromorphic functions sharing $a_j$ IM for $j = 1, 2, \ldots, 5$, where $a_1, a_2, \ldots, a_5$ are five distinct elements in $\mathbb{C} \cup \{\infty\}$, then $f \equiv g$.

In [7] p.111, R. Nevanlinna proved that Theorem B is sharp. It is natural to ask the following problem, which is the long-standing one:

**Problem A** (see [7] p.77 and p.109, see also [2, 4, 8, 11]). Does Theorem B hold if $a_1, a_2, \ldots, a_5$ are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$?

In recent years, some partial results were obtained on Problem A (see [2, 6, 8, 11]). In this paper, we give a positive answer to Problem A. In fact, we prove more generally the following theorem:

**Theorem 1.** Let $f$ and $g$ be non-constant meromorphic functions and let $a_j$ ($j = 1, 2, \ldots, 5$) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If

$$(1.2) \quad E(a_j,k,f) = E(a_j,k,g) \quad (j = 1, 2, \ldots, 5),$$

where $k$ is a positive integer or $\infty$ with $k \geq 14$, then $f \equiv g$.

By Theorem 1, we obtain the following corollary:

**Corollary 1.** Let $f$ and $g$ be non-constant meromorphic functions and let $a_j$ ($j = 1, 2, \ldots, 5$) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f$ and $g$ share $a_j$ IM ($j = 1, 2, \ldots, 5$), then $f \equiv g$.

It is obvious that Corollary 1 answers the above Problem A in the affirmative.
2. Some Lemmas

Lemma 1 (\cite{11}, see also \cite{2} Lemma 2 or \cite{10} p.185). Let \( f(z) \) be a non-constant meromorphic function and let \( a_1, a_2, \ldots, a_5 \) be five distinct elements in \( S(f) \cup \{\infty\} \). Then

\[
2T(r, f) \leq \sum_{j=1}^{5} N(r, a_j, f) + S(r, f).
\]  

(2.1)

Let \( f(z) \) be a non-constant meromorphic function, \( a(z) \in S(f) \cup \{\infty\} \), and \( k \) be a positive integer. We denote by \( N_k(r, a, f) \) the counting function of distinct zeros of \( f(z) - a(z) \) with multiplicities \( \leq k \), by \( N_{k+1}(r, a, f) \) the counting function of distinct zeros of \( f(z) - a(z) \) with multiplicities \( \geq k+1 \), each point in these counting functions counted only once (see \cite{10} p.190).

Lemma 2. Let \( f(z) \) be a non-constant meromorphic function and let \( a_1, a_2, \ldots, a_5 \) be five distinct elements in \( S(f) \cup \{\infty\} \), and \( k \) a positive integer. Then

\[
\sum_{j=1}^{5} N_{k+1}(r, a_j, f) \leq \frac{3}{k}T(r, f) + S(r, f).
\]  

(2.2)

Proof. By Lemma 1, we can obtain (2.1). Noting that for \( j = 1, 2, \ldots, 5 \)

\[
k N_{k+1}(r, a_j, f) + N(r, a_j, f) \leq N(r, a_j, f) \leq T(r, f) + S(r, f),
\]

we have from (2.1)

\[
k \sum_{j=1}^{5} N_{k+1}(r, a_j, f) + 2T(r, f) \leq 5T(r, f) + S(r, f).
\]

From this we get (2.2).

Lemma 3. Let \( h \) be a non-constant meromorphic function, and let \( a \in S(h) \) and \( a \neq 0 \). Then

\[
m(r, \frac{a'h - ah'}{h - a}) = S(r, h),
\]  

(2.3)

\[
m(r, \frac{a'h - ah'}{h(h - a)}) = S(r, h).
\]  

(2.4)

Proof. Noting

\[
\frac{a'h - ah'}{h - a} = a' - \frac{a(h' - a')}{h - a},
\]  

(2.5)

\[
\frac{a'h - ah'}{h(h - a)} = \frac{h'}{h} - \frac{h' - a'}{h - a}.
\]  

(2.6)

and the lemma of the logarithmic derivative, we obtain Lemma 3.

The following lemma plays an important role in the proof of Theorem 1.

Lemma 4. Let \( f \) and \( g \) be non-constant meromorphic functions and let \( a_1, a_2, \ldots, a_5 \) be five distinct elements in \( \{S(f) \cap S(g)\} \cup \{\infty\} \). If \( f \neq g \), then

\[
N_0(r, a_5, f, g) \leq \sum_{j=1}^{4} N_{12}(r, a_j, f, g) + S(r, f) + S(r, g).
\]  

(2.7)
3. Proof of Lemma 4

If \( \overline{N}_0(r, a_5, f, g) = S(r, f) + S(r, g) \), (2.7) obviously holds. In the following we suppose
\[
(3.1) \quad \overline{N}_0(r, a_5, f, g) \neq S(r, f) + S(r, g).
\]

Set
\[
(3.2) \quad L(w) := \frac{(w - a_1)(a_3 - a_2)}{(w - a_2)(a_3 - a_1)}.
\]

Let \( F(z) := L(f(z)), G(z) := L(g(z)), b_j := L(a_j) \quad (j = 1, 2, \ldots, 5) \). From (3.2) we have \( b_1 = 0, b_2 = \infty, b_3 = 1, \)
\[
(3.3) \quad T(r, F) = T(r, f) + S(r, f), \quad T(r, G) = T(r, g) + S(r, g).
\]

Since \( a_1, a_2, \ldots, a_5 \) are five distinct elements in \( \{S(f) \cap S(g)\} \cup \{\infty\} \), from (3.2) we know that \( b_1, b_2, \ldots, b_5 \) are five distinct elements in \( \{S(F) \cap S(G)\} \cup \{\infty\} \). Thus, \( b_4, b_5 \neq 0, 1, \infty \) and \( b_1 \neq b_5 \). Noting \( f \neq g \), we have
\[
(3.4) \quad F \neq G.
\]

From (3.1) and (3.3), we get
\[
(3.5) \quad \overline{N}_0(r, b_5, F, G) \neq S(r, F) + S(r, G).
\]

Set
\[
(3.6) \quad H := \frac{F'(a'G - aG')(F - G)}{F(F - 1)G(G - a)} - \frac{G'(a'F - aF')(F - G)}{G(G - 1)F(F - a)},
\]
where \( a = b_4 \neq 0, 1 \). Then we have from (3.6)
\[
(3.7) \quad H = \frac{(F - G)Q}{F(F - 1)(F - a)G(G - 1)(G - a)}.
\]

where
\[
(3.8) \quad Q = F'(a'G - aG')(G - 1)(F - a) - G'(a'F - aF')(F - 1)(G - a).
\]

By a simple computation,
\[
Q = a'FF'G - a'FF'G - a(a - 1)FF'G' - a(a - 1)FF'G' + aa'F'G^2 + aF'G^2
\]
\[
- a'F^2GG' + a'FGG' + a(a - 1)F'GG' + aF^2G' - aF'G'.
\]

Suppose that \( H \equiv 0 \). From (3.4) and (3.6) we obtain
\[
(3.10) \quad \frac{F'(a'G - aG')}{(F - 1)(G - a)} = \frac{G'(a'F - aF')}{(G - 1)(F - a)}.
\]

If \( a \) is a constant, noting \( a \neq 1 \), from (3.10) we get \( F \equiv G \), which contradicts (3.4). Thus, \( a \) is not a constant. From (3.10) we have
\[
\frac{F'(a'G - aG')}{G'(a'F - aF')} - 1 = \frac{(F - 1)(G - a)}{(G - 1)(F - a)} - 1.
\]

Thus,
\[
\frac{a'(F' - G')G - (F - G)G'}{G'(a'F - aF')} = \frac{(1 - a)(F - G)}{(G - 1)(F - a)}.
\]
From this we get
\[(3.11) \quad \frac{F' - G'}{F - G} = \frac{(1 - a)G'(a'F - af')}{a'G'(G - 1)(F - a)} + \frac{G'}{G}.\]

By (3.5), we know that there is a point \(z_0\) such that \(z_0\) is a common zero of \(F - b_5\) and \(G - b_5\) that is not a zero or a pole of \(a, a', b_5, b_5 - 1, b_5 - a\). It is obvious that \(z_0\) is a pole of the left-hand side of (3.11), and not a pole of the right-hand side of (3.11), which is a contradiction. Thus,
\[(3.12) \quad H \neq 0.\]

Suppose that \(z_n\) is a common zero of \(F - b_5\) and \(G - b_5\) that is not a zero or pole of \(a, b_5, b_5 - 1, b_5 - a\). It is obvious that \(z_n\) is a zero of \(F - G\), and \(z_n\) is not a pole of \(\frac{Q}{F(F - 1)(F - a)G(G - 1)(G - a)}\), where \(Q\) is given by (3.8). By (3.7), we know that \(z_n\) is a zero of \(H\). Again by (3.12) we obtain
\[(3.13) \quad N_0(r, b_5, F, G) \leq N(r, 0, H) + S(r, F) + S(r, G) \leq m(r, H) + N(r, H) + S(r, F) + S(r, G).\]

From (3.6) we have
\[(3.14) \quad H = \frac{F'}{F - 1} \cdot \frac{a'G - aG'}{G(G - a)} - \left( \frac{F'}{F - 1} - \frac{F'}{F} \right) \cdot \frac{a'G - aG'}{G - a}
- \left( \frac{G'}{G - 1} - \frac{G'}{G} \right) \cdot \frac{a'F - af'}{F - a} + \frac{G'}{G - 1} \cdot \frac{a'F - af'}{F - a}.
\]

Again by Lemma 3 and the lemma of the logarithmic derivative we obtain
\[(3.15) \quad m(r, H) = S(r, F) + S(r, G).\]

Substituting (3.15) into (3.13) we have
\[(3.16) \quad N_0(r, b_5, F, G) \leq N(r, H) + S(r, F) + S(r, G).\]

Next we estimate on \(N(r, H)\).

By (3.6), we know that the poles of \(H\) only possibly occur from the zeros of \(F, G, F - 1, G - 1, F - a\) and \(G - a\), the poles of \(F, G\) and \(a\). Let \(S_0\) be the set of all zeros, 1-points and poles of \(a\), and let for \(j = 1, 2, 3, 4\)
\[A_j := \{ z | F(z) - b_j(z) = 0 \} \setminus S_0, \quad B_j := \{ z | G(z) - b_j(z) = 0 \} \setminus S_0,\]
where \(b_1 = 0, b_2 = \infty, b_3 = 1\) and \(b_4 = a\). Thus, the poles of \(H\) occur possibly only from the set
\[\bigcup_{1 \leq p \leq 4} A_p \bigcup_{1 \leq q \leq 4} B_q \bigcup S_0.\]
Let

\[ S_1 := \bigcup_{1 \leq p \leq 4} \{ A_p \cap B_p \}, \]
\[ S_2 := \{ \bigcup_{1 \leq p \leq 4} A_p \} \setminus \{ \bigcup_{1 \leq q \leq 4} B_q \}, \]
\[ S_3 := \{ \bigcup_{1 \leq q \leq 4} B_q \} \setminus \{ \bigcup_{1 \leq p \leq 4} A_p \}, \]
\[ S_4 := \bigcup_{1 \leq q \leq 4} \{ A_p \cap B_q \}. \]

It is clear that \( S_1 \) is a set of the common zeros of \( F - b_j \) and \( G - b_j \) \( (j = 1, 2, 3, 4) \); \( S_2 \) is a set of the zeros of \( F - b_j \) \( (j = 1, 2, 3, 4) \) that is not the zeros of \( G - b_k \) \( (k = 1, 2, 3, 4) \); \( S_3 \) is a set of the zeros of \( G - b_j \) \( (j = 1, 2, 3, 4) \) that is not the zeros of \( F - b_k \) \( (k = 1, 2, 3, 4) \); \( S_4 \) is a set of the zeros of \( F - b_j \) that is the zeros of \( G - b_k \), where \( 1 \leq j, k \leq 4 \) and \( j \neq k \). From this we have

\[ \bigcup_{1 \leq j \leq 4} S_j = \bigcup_{1 \leq p \leq 4} A_p \bigcup_{1 \leq q \leq 4} B_q. \]

Thus, the poles of \( H \) occur possibly only from the set

\[ \bigcup_{1 \leq j \leq 4} S_j \bigcup S_0. \]

Since \( b_1, b_2, b_3 \) and \( b_4 \) are four distinct elements in \( \{ S(F) \cap S(G) \} \cup \{ \infty \} \), the contribution of \( S_0 \) to \( N(r, H) \) is at most \( S(r, F) + S(r, G) \). We next estimate the contribution of \( \bigcup_{1 \leq j \leq 4} S_j \) to \( N(r, H) \). We discuss four cases:

Case 1. The contribution of \( S_1 \) to \( N(r, H) \).

We distinguish four subcases:

Subcase 1.1. Suppose that \( z_{11} \in A_1 \cap B_1 \), and assume that \( z_{11} \) is a zero of \( F \) of order \( p_1 \) and \( G \) of order \( q_1 \). Then from (3.9), we know that \( z_{11} \) is a zero of \( Q \) of order at least \( p_1 + q_1 - 1 \). Noting that \( z_{11} \) is a zero of \( F - G \), from (3.7) we deduce that \( z_{11} \) is not a pole of \( H \).

Subcase 1.2. Suppose that \( z_{12} \in A_2 \cap B_2 \), and assume that \( z_{12} \) is a pole of \( F \) of order \( p_2 \) and \( G \) of order \( q_2 \). From (3.9), we know that \( z_{12} \) is a pole of \( Q \) of order at most \( 2p_2 + 2q_2 + 1 \). Noting that \( z_{12} \) is a pole of \( F - G \) of order at most \( \max \{ p_2, q_2 \} \), from (3.7), we have that \( z_{12} \) is not a pole of \( H \).

Subcase 1.3. Suppose that \( z_{13} \in A_3 \cap B_3 \). Noting that \( z_{13} \) is a zero of \( F - G \), a simple pole of \( \frac{F'}{G'} \) and \( \frac{G'}{F'} \), from (3.6) we have that \( z_{13} \) is not a pole of \( H \).

Subcase 1.4. Suppose that \( z_{14} \in A_4 \cap B_4 \). From (2.6), we know that \( z_{14} \) is a simple pole of \( \frac{a'F - a'F'}{b'F - b'F'} \) and \( \frac{a'G - a'G'}{b'G - b'G'} \). Noting that \( z_{14} \) is a zero of \( F - G \), from (3.6) we see that \( z_{14} \) is not a pole of \( H \).

From the above, it follows that the points in \( S_1 \) are not poles of \( H \). Thus, the contribution of \( S_1 \) to \( N(r, H) \) is 0.
Case 2. The contribution of $S_2$ to $N(r, H)$.

We distinguish four subcases:

Subcase 2.1. Suppose that $z_{21} \in A_1$ and $z_{21} \notin \bigcup_{1 \leq q \leq 4} B_q$. Then $z_{21}$ is a zero of $F$, not a zero of $G$, $1/G$, $G - 1$ and $G - a$. From (3.6), we have that $z_{21}$ is a pole of $H$ of order at most 1.

Subcase 2.2. Suppose that $z_{22} \in A_2$ and $z_{22} \notin \bigcup_{1 \leq q \leq 4} B_q$. Then $z_{22}$ is a pole of $F$, not a zero of $G$, $1/G$, $G - 1$ and $G - a$. From (3.6), we have that $z_{22}$ is a pole of $H$ of order at most 1.

Subcase 2.3. Suppose that $z_{23} \in A_3$ and $z_{23} \notin \bigcup_{1 \leq q \leq 4} B_q$. Then $z_{23}$ is a zero of $F$, not a zero of $G$, $1/G$, $G - 1$ and $G - a$. From (3.6), we have that $z_{23}$ is a pole of $H$ of order at most 1.

Subcase 2.4. Suppose that $z_{24} \in A_4$ and $z_{24} \notin \bigcup_{1 \leq q \leq 4} B_q$. Then $z_{24}$ is a zero of $F$, not a zero of $G$, $1/G$, $G - 1$ and $G - a$. From (2.6) and (3.6), we have that $z_{24}$ is a pole of $H$ of order at most 1.

From the above, we know that the points in $S_2$ are poles of $H$ of order at most 1.

Case 3. The contribution of $S_3$ to $N(r, H)$.

As with Case 2, we have that the points in $S_3$ are poles of $H$ of order at most 1.

Case 4. The contribution of $S_4$ to $N(r, H)$.

Suppose that $z_4 \in S_4$. Then $z_4 \in A_j$ and $z_4 \in B_q$, where $1 \leq p, q \leq 4$ and $p \neq q$. Without loss of generality we can assume that $z_4 \in A_1$ and $z_4 \in B_2$. Then, $z_4$ is a zero of $F$, and a pole of $G$. From (2.6) and (3.6), we have that $z_4$ is a pole of $H$ of order at most 2. Thus, the points in $S_4$ are poles of $H$ of order at most 2.

Noting that each point of $S_2$ and $S_3$ is counted once, each point of $S_4$ is counted twice in

$$
\sum_{j=1}^{4} N_{12}(r, b_j, F, G).
$$

From the above and (1.1) we obtain that the contribution of $\bigcup_{1 \leq j \leq 4} S_j$ to $N(r, H)$ is at most

$$
\sum_{j=1}^{4} N_{12}(r, b_j, F, G).
$$

Thus,

$$(3.17) \quad N(r, H) \leq \sum_{j=1}^{4} N_{12}(r, b_j, F, G) + S(r, F) + S(r, G).$$

Substituting (3.17) into (3.16) we have

$$
N_0(r, b_5, F, G) \leq \sum_{j=1}^{4} N_{12}(r, b_j, F, G) + S(r, F) + S(r, G),
$$

i.e.,

$$
N_0(r, a_5, f, g) \leq \sum_{j=1}^{4} N_{12}(r, a_j, f, g) + S(r, f) + S(r, g),
$$

and Lemma 4 is proved.
4. THEOREM 2 AND ITS PROOF

Using Lemma 4, we can prove the following Theorem:

**Theorem 2.** Let \( f \) and \( g \) be non-constant meromorphic functions and let \( a_j \) \((j = 1, 2, \ldots, 5)\) be five distinct elements in \( \{S(f) \cap S(g)\} \cup \{\infty\} \). If \( f \neq g \), then

\[
2T(r, f) + 2T(r, g) \leq 9 \sum_{j=1}^{5} N_{12}(r, a_j, f, g) + S(r, f) + S(r, g).
\]

**Proof.** By Lemma 4 we have

\[
N_0(r, a_5, f, g) \leq \sum_{j=1}^{4} N_{12}(r, a_j, f, g) + S(r, f) + S(r, g).
\]

Noting

\[
N(r, a_5, f) + N(r, a_5, g) = 2N_0(r, a_5, f, g) + N_{12}(r, a_5, f, g),
\]

from this and (4.2) we get

\[
N(r, a_5, f) + N(r, a_5, g) \leq 2 \sum_{j=1}^{5} N_{12}(r, a_j, f, g) - N_{12}(r, a_5, f, g) + S(r, f) + S(r, g).
\]

In the same manner as above, we have for \( i = 1, 2, 3, 4 \)

\[
N(r, a_i, f) + N(r, a_i, g) \leq 2 \sum_{j=1}^{5} N_{12}(r, a_j, f, g) - N_{12}(r, a_i, f, g) + S(r, f) + S(r, g).
\]

By Lemma 1, we have

\[
2T(r, f) + 2T(r, g) \leq \sum_{i=1}^{5} N(r, a_i, f) + \sum_{i=1}^{5} N(r, a_i, g) + S(r, f) + S(r, g).
\]

Substituting (4.3) and (4.4) into (4.5) we obtain (4.1), and Theorem 2 is thus proved.

5. PROOF OF THEOREM 1

Suppose that \( f \neq g \). By Theorem 2, we can obtain (4.1).

If \( k = \infty \), by hypothesis we have \( E(a_j, \infty, f) = E(a_j, \infty, g) \) \((j = 1, 2, \ldots, 5)\). Thus,

\[
N_{12}(r, a_j, f, g) = 0 \quad (j = 1, 2, \ldots, 5).
\]

Substituting (5.1) into (4.1) we obtain

\[
2T(r, f) + 2T(r, g) = S(r, f) + S(r, g).
\]

This is a contradiction. Next, we assume that \( k \) is a positive integer.

By hypothesis we have

\[
E(a_j, k, f) = E(a_j, k, g) \quad (j = 1, 2, \ldots, 5).
\]

From this we obtain for \( j = 1, 2, \ldots, 5 \)

\[
N_{12}(r, a_j, f, g) \leq N_{(k+1)}(r, a_5, f) + N_{(k+1)}(r, a_5, g) + S(r, f) + S(r, g).
\]

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Substituting (5.2) into (4.1) we obtain
(5.3)
\[2T(r, f) + 2T(r, g) \leq 9 \sum_{j=1}^{5} N_{(k+1)(r, a_j, f)} + 9 \sum_{j=1}^{5} N_{(k+1)(r, a_j, g)} + S(r, f) + S(r, g).\]

By Lemma 2, we have
(5.4)
\[\sum_{j=1}^{5} N_{(k+1)(r, a_j, f)} \leq \frac{3}{k} T(r, f) + S(r, f),\]
(5.5)
\[\sum_{j=1}^{5} N_{(k+1)(r, a_j, g)} \leq \frac{3}{k} T(r, g) + S(r, g).\]

Substituting (5.4) and (5.5) into (5.3) we obtain
\[2T(r, f) + 2T(r, g) \leq \frac{27}{k} T(r, f) + \frac{27}{k} T(r, g) + S(r, f) + S(r, g),\]
which contradicts the assumption \(k \geq 14\). Thus, \(f \equiv g\).

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**References**


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