ON THE DISTRIBUTION SINGULAR VALUES
OF TOEPLITZ MATRICES

MILUTIN R. DOSTANIĆ

Communicated by David R. Larson

Abstract. We prove a second order formula concerning distribution of singular
dvalues of Toeplitz matrices in some cases when conditions of the H. Widom
Theorem are not satisfied.

1. Introduction and notation

In 1920 G. Szegö proved a basic result concerning the distribution of the
eigenvalues \( \{ \lambda_k^{(n)} \}_{k=1}^n \) of the Toeplitz matrix
\[
T_n(f) = (\hat{f}_{i-j})_{i,j=0}^{n-1} \left( \hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta \right)
\]
associated with a bounded real valued function \( f \) on the interval \([-\pi, \pi]\): For any
continuous function \( F \) one has
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n F(\lambda_k^{(n)})}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(\theta)) \, d\theta.
\] (1)

An analogous result holds for the singular values \( s_1^{(n)} \geq s_2^{(n)} \geq \ldots \geq s_n^{(n)} \) of not
necessarily selfadjoint Toeplitz matrices \( T_n(f) \). The analogue of (1) is
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n F(s_k^{(n)})}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\|f(\theta)\|) \, d\theta.
\] (2)

Let
\[
K = \left\{ f \in L^\infty(-\pi, \pi) : \sum_{k=\infty}^{\infty} |k| \|f_k\|^2 < \infty \right\},
\]
\[
M = \|f\|_\infty, \quad m = \text{dist}(0, \text{conv}R(f))
\]
where \( R(f) \) denotes the essential range of \( f \) and “conv” denotes the convex hull. For \( f \in K \) we let
\[
\|f\|_K = \left( \sum_{k \in \mathbb{Z}} |k| \|f_k\|^2 \right)^{\frac{1}{2}}.
\]
Then it is easy to see that each singular value of \( T_n(f) \) belongs to the interval \([m, M]\) (see Lemma 1.2 in [4]).

Let \( \gamma_k^{(n)} = \left( \gamma_k^{(n)} \right)^2 \). H. Widom [7] proved a more exact formula than (1) and (2). Namely he proved

**Theorem 1.** If \( f \in K \) and \( G \in C^3[m^2, M^2] \), then

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} G \left( t_k^{(n)} \right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} G \left( \left| f(\theta) \right|^2 \right) d\theta \right)
= \text{tr} \left( \left( G \left( T(f) \right) T(f) \right) + G \left( T(f) T(\bar{f}) \right) - 2T \left( G \left( |f|^2 \right) \right) \right).
\]

We denote by \( T(f) \) the infinite Toeplitz matrix \( \left( T_{ij} \right)_{i,j=0}^{\infty} \) and by \( H(f) = \left( H_{ij} \right)_{i,j=0}^{\infty} \) the infinite Hankel matrix associated with the symbol \( f \).

If \( f \in K \), it is obvious that \( H(f) \) is a Hilbert-Schmidt operator. Moreover, \( |H(f)|_2 \leq ||f||_K \), where \( | \cdot |_2 \) denotes the Hilbert-Schmidt norm of the operator \( H(f) \) acting on \( L^2 \) of the nonnegative integers. For the little bit of the theory of trace class (i.e., nuclear) and the Hilbert-Schmidt operators that will be needed we refer the reader to [3].

It is easy to see that \( T(\bar{f}) = (T(f))^* \) and hence the operators \( T(\bar{f})T(f) \) and \( T(f) T(\bar{f}) \) are selfadjoint. Therefore, operators \( G(T(\bar{f})T(f)) \) and \( G(T(f) T(\bar{f})) \) are defined by the spectral theorem. If \( f \geq 0 \), the operator \( T(f) \) is obviously nonnegative.

In the case of eigenvalues, formulae similar to (3) are established in [1] and [4] but under much more restrictive assumptions on \( G \) (which is assumed to be analytic) and on the symbol \( f \).

The function \( G \) in Theorem 1 is given in terms of the function \( F \) in relation (2) by \( G(\lambda) = F \left( \sqrt{\lambda} \right) \). For \( G \) to belong to \( C^3 \) it is not enough that \( F \) belongs to \( C^3 \) but we also must have \( F'(0) = F''(0) = F'''(0) = 0 \) (in the case when \( m = 0 \)).

It is conjectured in [7] that the condition \( F \in C^3[0, M], F'(0) = 0 \) (in the case \( m = 0 \)) is sufficient for the statement of Theorem 1.

Essentially, in the case \( m = 0 \) it is necessary to prove Theorem 1 when \( F(\lambda) = \lambda^\beta \) and \( \beta \) small enough. In this paper, we shall prove formula (3) in the case \( m = 0 \) and when the conditions of Theorem 1 are not satisfied.

2. Result

**Theorem 2.** Let \( f \in K \) and \( m = 0 \). If \( F(\lambda) = \lambda^\alpha \) (\( \alpha \geq 2 \)), or \( F \in C^6[0, M], F'(0) = 0 \), then the operator

\[
S = F \left( \sqrt{\left( T(f) T(\bar{f}) \right)} \right) + F \left( \sqrt{T(f) T(\bar{f})} \right) - 2T \left( F(|f|) \right)
\]

is of the trace class and

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} F \left( s_k^{(n)} \right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} F \left( |f(\theta)| \right) d\theta \right) = \text{tr} \ S.
\]

**Remark 1.** Theorem 2 is stated for the case \( m = 0 \). If \( m > 0 \), (3) holds according to Theorem 1 (Of course, in that case the condition \( F'(0) = 0 \) is superficial.)
In the proof of Theorem 2 we will use the following Lemma of Lizorkin [5]:

**Lemma 1.** Let \( \alpha \geq 1, \sigma > 0 \). Then

\[
(i \lambda)^\alpha = A \exp \left( -\frac{i \pi \alpha \lambda}{2 \cdot \sigma} \right) \int_{-\infty}^{\infty} e^{-i(\lambda - \sigma)\xi} d\varrho(\xi) \quad (|\lambda| \leq \sigma)
\]

where \( \varrho \) is a nondecreasing function of bounded variation and \( A = \exp(-i \pi \alpha) \), 
\( \varrho(\xi) = \sum_{k<\frac{\alpha}{2\pi}} a_k \), where \( a_k \) are positive numbers related to Fourier coefficients \( c_n \) of the function \((i \lambda)^\alpha e^{i \frac{\varrho}{2\pi}}\) by

\[
c_k = \frac{A a_k e^{-i k \pi}}{\lambda}.
\]

Here the function \((i \lambda)^\alpha e^{i \frac{\varrho}{2\pi}}\) is assumed to be periodically extended from \((-\sigma, \sigma)\) to the entire real line. Our convention is \(z^\gamma = e^{\gamma \ln z}, \ln z = \ln|z| + i \arg z, 0 \leq \arg z < 2\pi\).

**Lemma 2.** If \( 1 < \alpha < 2 \) and \( M > 0 \), then there exists a nondecreasing function \( \varrho_0 \) of bounded variation, such that for each \( \lambda \in [0, M^2] \)

\[
\lambda^\alpha = \int_{-\infty}^{\infty} e^{-i2\pi \alpha - i M^2 t} e^{i \lambda t} d\varrho_0(t)
\]

holds and \( \int_{-\infty}^{\infty} t^2 d\varrho_0(t) < \infty \).

**Proof.** We apply Lemma 1 with \( \sigma = M^2 \). Since \( \lambda \geq 0 \), we have \((i \lambda)^\alpha = e^{i \frac{\varrho}{2\pi}} \cdot \lambda^\alpha\) and thus

\[
\lambda^\alpha e^{-i \frac{\varrho}{2\pi}} = e^{-\pi \alpha} e^{i \lambda \frac{\varrho}{2M^2}} \int_{-\infty}^{\infty} e^{i(\lambda - M^2)\xi} d\varrho(\xi),
\]

i.e.,

\[
\lambda^\alpha = e^{-\frac{2\pi \alpha}{2M^2}} \int_{-\infty}^{\infty} e^{-i M^2 \xi} e^{i \lambda \left(\xi - \frac{\varrho}{2M^2}\right)} d\varrho(\xi).
\]

Substituting \( \xi - \frac{\varrho}{2M^2} = t \), \( \varrho_0(t) \equiv \varrho \left( t + \frac{\varrho}{2M^2} \right) \) in the last formula we get

\[
\lambda^\alpha = \int_{-\infty}^{\infty} e^{-2\pi \alpha} e^{-i M^2 t} e^{i \lambda t} d\varrho_0(t).
\]

The last formula holds for \( \lambda \in [0, M^2] \) and \( \alpha \geq 1 \). Since \( \varrho \) is a function of bounded variation, so is \( \varrho_0 \).

We will show now that for \( 1 < \alpha < 2 \),

(5) \[
\int_{-\infty}^{\infty} t^2 d\varrho_0(t) < \infty.
\]

Since \( \varrho_0 \) is a function of bounded variation, applying the Cauchy inequality to (5) we get

\[
\int_{-\infty}^{\infty} |t| d\varrho_0(t) < \infty.
\]

Since \( \varrho_0(t) = \varrho \left( t + \frac{\varrho}{2M^2} \right) \), in order to prove (5) it is enough to show that

\[
\int_{-\infty}^{\infty} t^2 d\varrho(t) < \infty.
\]
From the way the function $g$ is defined it follows that it suffices to prove
\[ \sum_{n \in \mathbb{Z}} n^2 a_n < \infty \]
where $a_n$ is the sequence of positive numbers from Lemma 1 (with $\sigma = M^2$), i.e., the convergence of the series
\[ (6) \sum_{n \in \mathbb{Z}} (-1)^n n^2 c_n \]
where $c_n$ are the Fourier coefficients of the function $(i \lambda)^{\alpha} e^{i \lambda \frac{x}{2M}}$ on the interval $[-M^2, M^2]$. Since
\[ \int_{-M^2}^{M^2} (i \lambda)^{\alpha} \exp \left( \frac{i \lambda \pi \alpha}{2M^2} \right) \cdot \exp \left( \frac{i \lambda n \pi}{M^2} \right) \, d\lambda = \left( \frac{M^2}{\pi} \right)^{\alpha + 1} \cdot \int_{-\pi}^{\pi} (i x)^{\alpha} e^{\frac{i x \lambda}{2M}} e^{-i n x} \, dx, \]
in order to prove the convergence of the series (6) it is enough to prove that for $1 < \alpha < 2$ the series
\[ \sum_{n \in \mathbb{Z}} (-1)^n n^2 \int_{-\pi}^{\pi} (i x)^{\alpha} e^{\frac{i x \lambda}{2M}} e^{-i n x} \, dx \]
converges.

Since $\alpha > 1$, integrating by parts twice and having in mind the definition of the function $z \mapsto z^\gamma$, we conclude that the convergence of the above series will be established once we prove that the series
\[ \sum_{n \in \mathbb{Z}} (-1)^n A_n \]
converges for $1 < \alpha < 2$. Here $A_n \overset{\text{def}}{=} \int_{-\pi}^{\pi} (i x)^{\alpha} e^{\frac{i x \lambda}{2M}} e^{-i n x} \, dx$. Consider now the behavior of $A_n$ as $n \to \infty$. If $n > 0$, one gets
\[ A_n = \left( n - \frac{\alpha}{2} \right)^{1-\alpha} \left[ -e^{\frac{\pi n \alpha}{2}} \cdot \int_{0}^{\pi(n-\frac{\pi}{2})} t^{\alpha-2} e^{-it} \, dt - e^{\frac{\pi n \alpha}{2}} \cdot \int_{0}^{\pi(n-\frac{\pi}{2})} t^{\alpha-2} e^{it} \, dt \right]. \]
Since for $1 < \alpha < 2$ $\int_{0}^{\infty} t^{\alpha-2} e^{\pm it} \, dt = \Gamma(\alpha - 1) e^{\pm i \frac{\pi}{2}(\alpha - 1)}$ we obtain
\[ A_n = \left( n - \frac{\alpha}{2} \right)^{1-\alpha} \left[ i \Gamma(\alpha - 1)(e^{2\pi i \alpha} - 1) + e^{\frac{\pi n \alpha}{2}} \cdot \int_{\pi(n-\frac{\pi}{2})}^{\infty} t^{\alpha-2} e^{-it} \, dt 
+ e^{\frac{3\pi n \alpha}{2}} \cdot \int_{\pi(n-\frac{\pi}{2})}^{\infty} t^{\alpha-2} e^{it} \, dt \right]. \]
Integrating by parts, we get
\[
\int_{\pi(n-\frac{1}{2})}^{\infty} t^{\alpha-2} e^{-it} dt = (-1)^{n+1} i \left( \pi n - \frac{\alpha}{2} \right)^{\alpha-2} e^{\frac{\pi \alpha}{2}} + O \left( n^{\alpha-3} \right),
\]
and thus,
\[
A_n = \left( n - \frac{\alpha}{2} \right)^{1-\alpha} \left[ i \Gamma(\alpha - 1) (e^{2\pi i \alpha} - 1) + O \left( n^{\alpha-3} \right) \right].
\]
Therefore, the series \( \sum_{n=1}^{\infty} (-1)^n A_n \) converge. In a similar way one shows that the series
\[
\sum_{n=-\infty}^{1} (-1)^n A_n
\]
also converge.

**Remark 2.** From Lemma 2 (by integrating over \( \lambda \)) we obtain the representation
\[
\lambda^2 = \int_{-\infty}^{+\infty} e^{-2\pi i \alpha - iM^2 t} \cdot \frac{e^{i\lambda t} - 1}{it} (\alpha + 1) \, d\rho_0(t),
\]
for \( 1 < \alpha < 2, \lambda \in [0, M^2] \).

Let \( dv = \frac{d\rho_0}{dt} \). If \( 1 < \alpha < 2 \), then the function \( \rho_0 \) does not have a jump at the \( t = 0 \), hence
\[
\int_{-\infty}^{+\infty} |t|^k |d| < \infty \quad \text{for } k = 0, 1, 2, 3
\]
(\( |d| \) is a variation of measure \( \nu \)). From (2), putting \( \beta = \alpha + 1 \), we get
\[
\lambda^2 = \int_{-\infty}^{+\infty} e^{-2\pi i \alpha - iM^2 t} e^{i\lambda t} \, dv(t) - A, \quad 2 < \beta < 3, \lambda \in [0, M^2],
\]
and
\[
A = \int_{-\infty}^{+\infty} e^{-2\pi i \alpha - iM^2 t} \, dv(t).
\]

We write \( P_n \) for the projection operator, defined by
\[
P_n(x_0, x_1, \ldots) = (x_0, x_1, x_2, \ldots, x_{n-1}, 0, 0, \ldots)
\]
from \( l^2 \) to the subspace of \( l^2 \) on which \( T_n(f) \) may be thought of as acting. We identify \( T_n(f) \) with \( P_n T(f) P_n \) in the obvious way. We define an operator \( Q_n \) on \( l^2 \) by
\[
Q_n(x_0, x_1, \ldots) = (x_{n-1}, x_{n-2}, \ldots, x_1, x_0, 0, 0, \ldots).
\]
For \( f \in L^\infty(-\pi, \pi) \) we define \( \hat{f}(\theta) = f(-\theta) \).
Lemma 3. 1) For any \( f, g \in L^\infty(-\pi, \pi) \) we have
\[
T(f \cdot g) - T(f) \cdot T(g) = H(f)H(\bar{g}),
\]
\[
T_n(f \cdot g) - T_n(f) \cdot T_n(g) = P_nH(f)H(\bar{g})P_n + Q_nH(f)H(\bar{g})Q_n.
\]

2) If \( f \in K \), then we have \( \hat{f} \in K \), \( |f|^2 \in K \), \( e^{itf} \in K \) (\( \forall t \in \mathbb{R} \)) and
\[
\left\| |f|^2 \right\|_K \leq \text{const} \|f\|_K,
\]
\[
\left\| e^{itf} \right\|_K \leq \text{const} \cdot |t| \cdot \|f\|_K.
\]

Proof. 1) Routine computation. (Or see [2], Propositions 2.7 and 3.6.)

2) Can be proved in a same way as Proposition 1 in [3]. \( \square \)

Lemma 4. If \( f \in K \), \( 1 < \alpha < 2 \), then the operator \( (T(|f|^2))^\alpha - T(|f|^{2\alpha}) \) is nuclear.

Proof. Integrating the identity
\[
\frac{d}{ds} \left( T\left( e^{is|f|^2} \right) e^{-isT(|f|^2)} \right) = \left[ T\left( e^{is|f|^2} \cdot i|f|^2 \right) - T\left( e^{is|f|^2} \right) T\left( i|f|^2 \right) \right] e^{-isT(|f|^2)}
\]
on the interval \([0, t]\) and multiplying the result by \( e^{isT(|f|^2)} \) on the right, we get
\[
T\left( e^{it|f|^2} \right) - e^{itT(|f|^2)} = \int_0^t H\left( e^{is|f|^2} \right) \cdot H\left( i|f|^2 \right) e^{i(t-s)T(|f|^2)} ds.
\]

Since \( f \in K \), we have \( e^{is|f|^2} \in K \), \( i|f|^2 \in K \). Applying Lemma 3 formula (9) yields
\[
\left| T\left( e^{it|f|^2} \right) - e^{itT(|f|^2)} \right|_1 \leq \int_0^{|t|} \left| H\left( e^{is|f|^2} \right) \cdot H\left( i|f|^2 \right) \right|_1 ds
\]
for all \( t \in \mathbb{R} \), since the operator \( e^{i(t-s)T(|f|^2)} \) is unitary. Here \( \left| \cdot \right|_1 \) denotes the nuclear norm of an operator. Since the operators \( H\left( e^{is|f|^2} \right) \) and \( H\left( i|f|^2 \right) \) are Hilbert-Schmidt, their product is nuclear, and thus, according to Lemma 3 (statement 2) we get
\[
\left| H\left( e^{is|f|^2} \right) \cdot H\left( i|f|^2 \right) \right|_1 \leq c_0 \cdot |s| \cdot \|f\|_{K}^2
\]
\((c_0 \text{ is independent of } s)\), and thus
\[
\left| T\left( e^{it|f|^2} \right) - e^{itT(|f|^2)} \right|_1 \leq c_0 \|f\|_{K}^2 \int_0^{|t|} s ds = \frac{c_0}{2} t^2 \|f\|_{K}^2.
\]

According to Lemma 2
\[
(T(|f|^2))^\alpha = \int_{\mathbb{R}} e^{-2\pi i \alpha - iM^2 t} e^{itT(|f|^2)} d\varrho_0(t),
\]
\[
|f|^{2\alpha} = \int_{\mathbb{R}} e^{-2\pi i \alpha - iM^2 t} e^{it|f|^2} d\varrho_0(t),
\]
and thus,
\[
T(|f|^{2\alpha}) = \int_{\mathbb{R}} e^{-2\pi i \alpha - iM^2 t} T\left( e^{it|f|^2} \right) d\varrho_0(t).
\]
Therefore,
\[
\left( (T(|f|^2))^\alpha - T(|f|^{2\alpha}) \right) = \int_{\mathbb{R}} e^{-2\pi i \alpha - iM^2 t} \left( e^{itT(|f|^2)} - T(e^{it|f|^2}) \right) \, d\varrho_0(t).
\]
Inequality (10) shows that the integral on the right side in the formula above, converges in nuclear norm and thus \((T(|f|^2))^\alpha - T(|f|^{2\alpha})\) is a nuclear operator. Moreover,
\[
\left| \left( (T(|f|^2))^\alpha - T(|f|^{2\alpha}) \right) \right|_1 \leq \frac{C_0}{2} \|f\|_K^2 \int_{\mathbb{R}} t^2 \, d\varrho_0(t) < +\infty \quad \text{(by Lemma 2)}.
\]

**Lemma 5.** If \(1 < \alpha < 2\) and \(f \in K\), then
\[
\lim_{n \to \infty} \text{tr} \left( (T_n(|f|^2))^\alpha - T_n(|f|^{2\alpha}) \right) = 2 \text{tr} \left( (T(|f|^2))^\alpha - T(|f|^{2\alpha}) \right).
\]

**Proof.** In a same way as we proved (9) we get
\[
T_n \left( e^{it|f|^2} \right) - e^{i t T_n(|f|^2)} = \int_0^1 \left[ T_n \left( e^{i(s|f|^2) \, i |f|^2} \right) - T_n \left( e^{i(s|f|^2) \, i |f|^2} \right) \right] e^{i(t-s)T_n(|f|^2)} \, ds
\]
and thus by Lemma 5
\[
\left( T_n \left( e^{i|f|^2} \right) - e^{i t T_n(|f|^2)} \right) = \int_0^1 \left[ P_n H \left( e^{i(s|f|^2) \, i |f|^2} \right) H \left( i |f|^2 \right) P_n \right. \\
+ Q_n H \left( e^{i(s|f|^2) \, i |f|^2} \right) H \left( i |f|^2 \right) Q_n \left] e^{i(t-s)T_n(|f|^2)} \, ds.
\]
From Lemma 2 we obtain
\[
\left( (T_n(|f|^2))^\alpha - T_n(|f|^{2\alpha}) \right) = \int_{\mathbb{R}} e^{-2\pi i \alpha - iM^2 t} \left( e^{itT_n(|f|^2)} - T_n \left( e^{it|f|^2} \right) \right) \, d\varrho_0(t) \quad (1 < \alpha < 2).
\]
It follows from (11) that
\[
\left| T_n \left( e^{i|f|^2} \right) - e^{i t T_n(|f|^2)} \right|_1 \leq \text{const} \, |t|^2, \quad \forall t \in \mathbb{R}.
\]
(const does not depend on \(t\) and \(n\)) and thus, since \( \int_{\mathbb{R}} |t|^2 \, d\varrho_0(t) < \infty \), by the same arguments as in the proof of (14) in [7] and by the Lebesgue theorem on dominant convergence, (11) and (12) give
\[
\lim_{n \to \infty} \text{tr} \left( (T_n(|f|^2))^\alpha - T_n(|f|^{2\alpha}) \right) = 2 \text{tr} \left( (T(|f|^2))^\alpha - T(|f|^{2\alpha}) \right).
\]

**Lemma 6.** If \(f \in K\), \(1 < \alpha < 2\), then the operator \((T(\overline{f})T(f))^{\alpha} + (T(f)T(\overline{f}))^{\alpha} - 2T(|f|^{2\alpha})\) is nuclear and
\[
\lim_{n \to \infty} \text{tr} \left[ (T_n(\overline{f})T_n(f))^{\alpha} - T_n(|f|^{2\alpha}) \right] = \text{tr} \left[ (\overline{T(f)T(\overline{f}))^{\alpha} - 2T(|f|^{2\alpha})} \right].
\]
Proof. By Lemma 2 we have
\[
(T(T(f)) - T_n (|f|^2)^\alpha)
\]
\[
= \int_{-\infty}^{\infty} e^{-2\pi i \alpha - i M^2 t} \left( e^{i t T(f)} - e^{i t T(|f|^2)^\alpha} \right) dt. \tag{13}
\]
Since \( \left| e^{i t T(f)} - e^{i t T(|f|^2)^\alpha} \right| \leq \text{const} \cdot |t| \) \((t \in \mathbb{R})\), and \( \int_{\mathbb{R}} |t| \, dt \leq +\infty \), the integral in \(13\) converges (in nuclear norm) and thus the operator \((T(T(f)) - T_n (|f|^2)^\alpha)\) is nuclear. In a similar way we prove that the operator \((T(f)T(f)) - T_n (|f|^2)^\alpha)\) is nuclear. Thus \((T(T(f)) - T_n (|f|^2)^\alpha) + (T(f)T(f)) - 2T (|f|^2)^\alpha)\) is also nuclear. Therefore, according to Lemma 4, the operator \((T(f)T(f)) - T_n (|f|^2)^\alpha) - 2T (|f|^2)^\alpha)\) is nuclear. Since \(e^{i t T_n (|f|^2)^\alpha)} = \text{const} \cdot |t| \) \((t \in \mathbb{R}, \ d_0 \) is independent of \(n\) and \(t)\) and \(\int_{-\infty}^{\infty} |t| \, dt \leq +\infty\), it follows from
\[
(T(T(f)) - T_n (|f|^2)^\alpha)
\]
\[
= \int_{-\infty}^{\infty} e^{-2\pi i \alpha - i M^2 t} \left( e^{i t T_n (|f|^2)^\alpha)} - e^{i t T_n (|f|^2)^\alpha)} \right) dt.
\]
that \(\lim_{n \to \infty} \text{tr} \left( (T(T(f)) - T_n (|f|^2)^\alpha) \right) \)
\[
= \int_{-\infty}^{\infty} e^{-2\pi i \alpha - i M^2 t} \lim_{n \to \infty} \text{tr} \left( e^{i t T_n (|f|^2)^\alpha)} - e^{i t T_n (|f|^2)^\alpha)} \right) dt.
\]
From relation (14) in \[7\] and from Lemma \[2\] the last equality becomes
\[
\lim_{n \to \infty} \text{tr} \left( (T(T(f)) - T_n (|f|^2)^\alpha) \right) \]
\[
= \text{tr} \left( (T(f)T(f)) - T_n (|f|^2)^\alpha) - 2T (|f|^2)^\alpha) \right). \tag{14}
\]
From (14) and Lemma \[6\] adding, we obtain
\[
\lim_{n \to \infty} \text{tr} \left[ (T(f)T(f)) - T(f)T(f) \right) \]
\[
= \text{tr} \left( (T(f)T(f)) - T(f)T(f) \right) - 2T (|f|^2)^\alpha) \right].
\]

Remark 3. By using representation \[8\] by the same method as the one used for proving Lemmas 4, 5, 6 one can show that:

Lemma 7. If \(f \in K\) and \(2 < \beta < 3\), then the operator \(T \left( |f|^2 \right)^\beta - T \left( |f|^2 \right)^\beta)\) is the trace class and the following holds:
\[
\lim_{n \to \infty} \text{tr} \left[ T_n (|f|^2)^\beta - T_n (|f|^2)^\beta \right] = \text{tr} \left[ T \left( |f|^2 \right)^\beta - T \left( |f|^2 \right)^\beta \right].
\]
Also, The operator \((T(f)T(f)) - T(f)T(f) \right) - 2T (|f|^2)^\beta)\) is the trace class and the following holds:
\[
\lim_{n \to \infty} \text{tr} \left[ \left( T_n (|f|^2)^\beta - T_n (|f|^2)^\beta \right) \right] = \text{tr} \left[ \left( T (|f|^2)^\beta - 2T (|f|^2)^\beta \right) \right].
Note that Theorem 2 holds for the functions $\lambda \mapsto \lambda^2$, $\lambda \mapsto \lambda^4$ and $\lambda \mapsto \lambda^\alpha$ ($\alpha \geq 6$) as a consequence of Theorem 1. In other words,

$$
\lim_{n \to \infty} \left( \sum_{k=1}^{n} \left( \frac{t_k(n)}{k!} \right)^{\alpha} - \frac{n}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^{\alpha} d\theta \right)
= \text{tr} \left[ (T(f)T(f))^\frac{1}{2} + (T(f)T(f))^\frac{1}{2} - 2T(|f|) \right],
$$

for $\alpha = 2, 4$ and $\alpha \geq 6$.

From Lemma 6 and Lemma 7 we obtain that (15) also holds if $2 < \alpha < 4$ and $4 < \alpha < 6$. Therefore, formula (14) holds if $F(\lambda) = \lambda^\alpha$ and $\alpha \geq 2$.

Now let $F \in C^6[0, M^2]$ and $F'(0) = 0$. Then, for the function $F_0(\lambda) = \sum_{k=2}^{6} \frac{F^{(k)}(0)}{k!} \lambda^k$ we have

$$
\lim_{n \to \infty} \left( \sum_{k=1}^{n} F_0 \left( \frac{t_k(n)}{k!} \right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} F_0 \left( |f(\theta)| \right) d\theta \right)
= \text{tr} \left[ F_0 \left( \sqrt{T(f)T(f)} \right) + F_0 \left( \sqrt{T(f)T(f)} \right) - 2T \left( |f| \right) \right].
$$

Let $R(\lambda) = F(\lambda) - F_0(\lambda)$. The function $\lambda \mapsto R(\sqrt{\lambda})$ satisfies the conditions of Theorem 1 and hence

$$
\lim_{n \to \infty} \left( \sum_{k=1}^{n} R \left( \frac{t_k(n)}{k!} \right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} R \left( |f(\theta)| \right) d\theta \right)
= \text{tr} \left[ R \left( \sqrt{T(f)T(f)} \right) + R \left( \sqrt{T(f)T(f)} \right) - 2T \left( |f| \right) \right].
$$

Adding (16) and (17) one gets (14). (The operators on the right-hand side of (16) and (17) are nuclear and so is their sum, i.e., the operator $S$ is nuclear.)

**Remark 4.** The question of whether the condition $F'(0) = 0$ in Theorem 2 is necessary remains open. To answer it affirmatively it is enough to find the example of a function $f \in K$ such that $m = \text{dist}(0, \text{conv}R(f)) = 0$ and

$$
\lim_{n \to \infty} \left( \sum_{k=1}^{n} s_k(n) - \frac{n}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta \right)
- \text{tr} \left[ \sqrt{T(f)T(f)} + \sqrt{T(f)T(f)} - 2T \left( |f| \right) \right] \neq 0.
$$

**Acknowledgment**

The author is grateful to the referee for the suggestion which helped to improve this article.

**References**


MATEMATICKI FAKULTET, UNIVERSITY OF BELGRADE, STUDENSKI TRG 16, 11000 BELGRADE, YUGOSLAVIA

E-mail address: domi@matf.bg.ac.yu