DIVISION ALGEBRAS OVER $C_2$- AND $C_3$-FIELDS

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Abstract. Using elementary methods we prove a theorem of Rost, Serre, and Tignol that any division algebra of degree 4 over a $C_3$-field containing $\sqrt{-1}$ is cyclic. Our methods also show any division algebra of degree 8 over a $C_2$-field containing $\sqrt{-1}$ is cyclic.

Introduction

Throughout the discussion $A$ will always denote a central simple algebra over a field $F$ which is fixed. We write $D$ for the underlying division algebra. We want to study the structure of $A$ in terms of the assumption that $F$ is a $C_m$-field for some $m$; we recall that this means every form of degree $d$ in $n > d^m$ variables has a nontrivial zero. $C_m$-fields were discovered by Tsen [9] and rediscovered by Lang [6]. Greenberg [5] contains an excellent treatment. The Tsen-Lang theorem shows the Brauer group over a $C_1$-field is trivial. Examples of $C_m$-fields include all fields of transcendence degree $m$ over an algebraically closed field. More generally, by [5, Theorem 3.6], if $F$ is a $C_m$-field and $K/F$ is a field extension of transcendence degree $t$, then $K$ is a $C_{m+t}$-field.

Since the general structure theory [2] shows $A$ is a tensor product of algebras of prime power index, one may assume index$(A)$ is a power of a prime number $p$. The motivation of this research was the following question which I heard from Zinovy Reichstein:

Question A. Over the purely transcendental field $F = \mathbb{C}(x, y)$ or $F = \mathbb{C}(x, y, z)$, is the tensor product of two cyclic algebras of index $p$ necessarily cyclic? As Reichstein pointed out, this is true for $p = 2$ over arbitrary $C_2$-fields containing $\sqrt{-1}$, seen by solving the equations

$$\text{tr}(a) = \text{tr}(a^2) = \text{tr}(a^3) = 0.$$ 

This is the first question one would ask in trying to find a noncyclic algebra over a field of low transcendence degree over an algebraically closed field. Note that over the transcendental field extension $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of $\mathbb{C}$ one has the noncyclic algebra
algebra which is the tensor product of two symbol algebras \((\lambda_1, \lambda_2) \otimes (\lambda_3, \lambda_4)\), so if we want a cyclicity theorem we must study \(C_m\)-fields for \(m \leq 3\).

Note that Albert [7] has produced a noncyclic division algebra of degree 4 and exponent 2 over \(F = \mathbb{R}(x, y)\); any such algebra must be the tensor product of two quaternion algebras.

We prove cyclicity of all division algebras of degree 4 over \(C_3\)-fields containing \(\sqrt{-1}\), and of division algebras of degree 8 over \(C_2\)-fields containing \(\sqrt{-1}\). The former result follows as an immediate consequence of a lovely new theorem of Rost-Serre-Tignol [7, Theorem 1], but our proof is elementary and direct, yielding an explicit element whose fourth power is central, even without the presence of \(\sqrt{-1}\). For the reader’s convenience, the statement of the Rost-Serre-Tignol Theorem is:

Let \(E\) be a central simple \(F\)-algebra of degree 4, with \(-1 \in F^\times\). Let \(n_Q\) be the norm form of the quaternion algebra \(Q\) Brauer-equivalent to \(E \otimes_F E\). The trace form \(q_E : E \to F\), defined by \(q_E(x) = \text{Tr}_E(x^2)\), satisfies the following equation in the Witt ring \(WF\):

\[
q_E = n_Q + q_4
\]

for some (uniquely determined) 4-fold Pfister form \(q_4\). The form \(q_4\) is hyperbolic if and only if \(E\) is cyclic, i.e. \(E\) is split by some Galois \((\mathbb{Z}/4\mathbb{Z})\)-algebra over \(F\).

§1. Division algebras of degree 4 over \(C_3\)-fields

**Theorem 1.1.** Over a \(C_3\)-field of characteristic \(\neq 2\), if \(\deg(D) = 4\), then there is an element \(c\) such that \(c^4 \in F\) but \(c^2 \notin F\). In particular if \(\sqrt{-1} \in F\), then \(D\) is cyclic.

**Proof.** Take an element \(a\) such that \(a^2 \in F\) but \(a \notin F\). (We know that \(D\) contains such elements by [2, Theorem 11.9].) Now let \(V = \{b : ab = -ba\}\), easily seen to be a subspace of dimension \(\frac{d^2}{4} = 8\). (One way of seeing this is to note that \(V \supseteq \{ad - da : d \in D\}\).) Let \(V' = V + Fa\), which has dimension 9. For any \(c = b + \gamma a \in V'\), we have

\[
c^2 = b^2 + \gamma^2 a^2 + b\gamma a + \gamma ab = b^2 + \gamma^2 a^2.
\]

Recalling \(a^2 \in F\), we see that \(c^2 \in F[b^2]\). But \(F[b^2] \subset F[b]\) for \(b \neq 0\), since \(b \notin F[b^2]\) (because \(b^2\) commutes with \(a\)), so \(c^2\) is quadratic over \(F\). The quadratic form given by \(\text{tr}(c^2) = 0\) has a nontrivial solution over a \(C_3\)-field since \(\dim V' = 9 > 2^3 = 8\). But such \(c\) satisfies

\[
c^4 = (c^2)^2 \in F
\]

although \(c^2 \notin F\).

**Remark.** If \(\sqrt{-1} \in F\), then any such \(c\) of Theorem 1.1 can be recovered via the proof. Indeed, take \(c'\) such that \(c'c = \sqrt{-1} cc'\), and let \(a = (c')^2\). Then \(ac = -ca\), so we could take \(b = c\) and \(\gamma = 0\).

**Remark 1.2.** One can refine this result by means of the Amitsur-Saltman construction [8] of an arbitrary central simple algebra of degree 4 in terms of a maximal subfield \(K\) Galois over the center \(F\). We can write \(K = F(a_1, a_2)\) where \(a_2^2 \in F\), and one has \(z_i \in K\) such that \(z_ia_i^{-1} = -a_i\) and \(z_ia_2^{-i} = a_2^{-i}z_i\), for \(i = 1, 2\).
Suppose \( F \) has a \( C_2 \)-subfield \( F_0 \) such that \( a_1^2 \in F_0 \) and \( z_1^2 \in F_0(a_1, a_2) \). Then letting \( K_0 = F_0(a_1, a_2) \) and

\[
V'' = F_0a_1 + K_0z_1,
\]

which has dimension 5 over \( F_0 \), one sees \( V'' \) has an element \( v \neq 0 \) such that \( \text{tr} v^2 = 0 \). But \( v^2 \in F_0(a_2) \). (Indeed, for any \( k \in K_0 \) one has \((kz_1)^2 \in K_0 \) is fixed under conjugation by \( z_1 \), so is in \( F_0(a_2) \).) Consequently

\[
v^2 \in F_0a_2
\]

and \( v^4 \in F_0 \). The more precise form for this \( v \) will be used in later work.

§2. Division algebras of degree 4 over \( C_2 \)-fields

Along the same lines, we can study division algebras of degree 8 over \( C_2 \)-fields. Artin-Harris and Artin-Tate [4] Theorem 6.2 and Appendix] proved over a \( C_2 \)-field that \( \text{index}(A) = \exp(A) \) for any central simple algebra \( A \) such that \( \text{index}(A) \) is a power of 2 or 3. Since their theorem was proved before the major structure theorems of Merkurjev and Suslin, their statement was somewhat weaker, but examining [4, Theorem 6.2, Step I, and Appendix] one can piece together the components of their arguments for a straightforward proof over an arbitrary \( C_2 \)-field \( F \). Let us sketch this proof for \( \text{index}(A) \) a power of 2:

1. The tensor product of \( m \) quaternion algebras over \( F \) has index 2. Indeed, this is true for \( m = 2 \) by a degree counting argument which shows that any two quaternion algebras over a \( C_2 \)-field have a common (quadratic) subfield; the assertion for arbitrary \( m \) follows by induction.

2. The case \( \exp(A) = 2 \) follows by (1) by Merkurjev’s theorem that every algebra of exponent 2 is similar to a tensor product of quaternion algebras. Note that for \( \text{index}(A) \leq 8 \), one could substitute the much more basic theorems of Albert and Tignol [8].

3. The case of arbitrary exponent \( 2^q \) is achieved by applying induction to \( A \otimes A \), which has exponent \( 2^{q-1} \) and therefore has a splitting field \( L \) of dimension \( 2^{q-1} \). Then \( A \otimes L \) has exponent 2, so one concludes with (2) and Albert’s index reduction formula [2, Theorem 4.20].

**Theorem 2.1.** Suppose \( F \) is a \( C_2 \)-field of characteristic \( \neq 2 \) containing \( \sqrt{-1} \). If \( \deg(D) = 8 \), then \( D \) contains an element \( c \) such that \( c^8 \in F \) but \( c^4 \notin F \). In particular if \( \sqrt{-1} \in F \), then \( D \) is cyclic.

**Proof.** Since \( \exp(D) = \text{index}(D) = 8 \), we have index \( D^{\otimes 2} = 4 \). Hence, by Theorem 1.1 there is an element \( a \in D^{\otimes 2} \) such that \( a^4 \in F \) but \( a^2 \notin F \). Hence \( L = F[a] \) splits \( D^{\otimes 2} \), so \( D \otimes_F L \) has exponent 2, and thus is a quaternion algebra. By Albert’s index reduction formula [2, Theorem 4.20], \( L \) is isomorphic to a subfield of \( D \). Thus we may assume \( a \in D \). Now, writing \( i = \sqrt{-1} \), let \( V = \{ b : ab = iba \} \), easily seen to be a subspace of dimension \( \frac{2^2}{4} = 16 \). Let \( V' = V + Fa \), which has dimension 17. For any \( c = b + \gamma a \in V' \), we have

\[
c^2 = b^2 + \gamma^2 a^2 + b\gamma a + \gamma ab = b^2 + \gamma^2 a^2 + (1 + i)\gamma ba,
\]

so

\[
c^4 = (c^2)^2 = b^4 + \gamma^4 a^4 + (1 + i)^2 \gamma^2 (ba)^2 + 2\gamma^2 a^2 b^2 = b^4 + \gamma^4 a^4.
\]
Since $a^4 \in F$ we see that $c^4 \in F[b^4]$. But $F[b^2] \subset F[b^4] \subset F[b]$ when $b \neq 0$, so $c^4$ is quadratic over $F$. The quartic form given by $\text{tr}(c^4) = 0$ has a nontrivial solution over a $C_2$-field since $\dim V' = 17 > 2^4 = 16$. But such $c$ satisfies

$$c^8 = (c^4)^2 \in F$$

although $c^4 \notin F$.

**Remark 2.2.** If $\sqrt{-1} \in F$, then any such $c$ of Theorem 2.1 can be recovered via the proof. Indeed, take $c'$ such that $c'c = \sqrt{-1}cc'$, and let $a = (c')^2$. Then $ac = ica$, so we could take $b = c$ and $\gamma = 0$.

**References**


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