INDEX OF B-FREDHOLM OPERATORS
AND GENERALIZATION OF A WEYL THEOREM

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Abstract. The aim of this paper is to show that if $S$ and $T$ are commuting
B-Fredholm operators acting on a Banach space $X$, then $ST$ is a B-Fredholm
operator and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$, where $\text{ind}$ means the index. Moreover
if $T$ is a B-Fredholm operator and $F$ is a finite rank operator, then $T + F$
is a B-Fredholm operator and $\text{ind}(T + F) = \text{ind}(T)$. We also show that if $0$
is isolated in the spectrum of $T$, then $T$ is a B-Fredholm operator of index 0 if
and only if $T$ is Drazin invertible. In the case of a normal bounded linear operator
$T$ acting on a Hilbert space $H$, we obtain a generalization of a classical
Weyl theorem.

1. Introduction

This paper is a continuation of our previous works [2], [3], [4], [5]. We consider a
Banach space $X$ and $L(X)$ the Banach algebra of bounded linear operators acting
on $X$. For $T \in L(X)$ we will denote by $N(T)$ the null space of $T$, by $\alpha(T)$
the nullity of $T$, by $R(T)$ the range of $T$ and by $\beta(T)$ its defect. If the range $R(T)$ of $T$ is
closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then $T$ is called an upper semi-Fredholm
(resp. a lower semi-Fredholm) operator. A semi-Fredholm operator is an upper or
a lower semi-Fredholm operator. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called
a Fredholm operator and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$.

Now for a bounded linear operator $T$ and for each integer $n$, define $T_n$ to be the
restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular
$T_0 = T$). If for some integer $n$ the range space $R(T^n)$ is closed and $T_n$ is a Fredholm
(resp. semi-Fredholm) operator, then $T$ is called a B-Fredholm operator (resp. a
semi-B-Fredholm) operator. In this case and from [2] Proposition 2.1 $T_m$ is a
Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_n)$ for each $m \geq n$. This enables us to
define the index of a B-Fredholm operator $T$ as the index of the Fredholm operator
$T_n$ where $n$ is any integer such that $R(T^n)$ is closed and $T_n$ is a Fredholm operator.
Let $\text{BF}(X)$ be the class of all B-Fredholm operators. In [2] the author has studied
this class of operators and has proved [2] Theorem 2.7 that an operator $T \in L(X)$
is a B-Fredholm operator if and only if $T = T_0 \oplus T_1$, where $T_0$ is a Fredholm operator
and $T_1$ is a nilpotent one.

The aim of this paper is to study the properties of the index of B-Fredholm
operators and to derive a generalization of a classical Weyl theorem.
It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let \( A \) be an algebra with a unit. Following \cite{14} we say that an element \( x \) of \( A \) is Drazin invertible of degree \( k \) if there is an element \( b \) of \( A \) such that

\[
x^k bx = x^k, bxb = b, xb = bx.
\]

(1)

Recall that the concept of Drazin invertibility was originally considered by M. P. Drazin in \cite{6} where elements satisfying relation (1) are called pseudo-invertible elements. The Drazin spectrum is defined by \( \sigma_D(a) = \{ \lambda \in \mathbb{C} : a - \lambda I \text{ is not Drazin invertible} \} \) for every \( a \in A \). In the case of a Banach algebra \( A \) and from \cite{5} Theorem 2.3] we know that the Drazin spectrum satisfies the spectral mapping theorem.

In the case of a bounded linear operator \( T \) acting on a Banach space \( X \), it is well known that \( T \) is Drazin invertible if and only if it has a finite ascent and descent (Definition 2.1), which is also equivalent to the fact that \( T = T_0 \oplus T_1 \), where \( T_0 \) is an invertible operator and \( T_1 \) is a nilpotent one. (See \cite{14} Proposition 6 and \cite{12} Corollary 2.2.) In \cite{5} Theorem 3.4] we have shown that a bounded linear operator \( T \) acting on a Banach space \( X \) is a B-Fredholm operator if and only if its projection in the algebra \( L(X)/F_0(X) \) is Drazin invertible, where \( F_0(X) \) is the ideal of finite rank operators in the algebra \( L(X) \). This characterization of B-Fredholm operators shows easily that the class of B-Fredholm operators is stable under finite rank perturbation and the product of two commuting B-Fredholm operators is a B-Fredholm operator \cite{5} Corollary 3.5].

After giving some preliminaries in the second section, we prove in the third section that if \( S, T \) are two commuting B-Fredholm operators, then the product \( ST \) is a B-Fredholm operator and \( \text{ind}(ST) = \text{ind}(S) + \text{ind}(T) \). Moreover if \( T \) is a B-Fredholm operator and \( F \) is a finite rank operator, then \( T + F \) is a B-Fredholm operator and \( \text{ind}(T + F) = \text{ind}(T) \). Those two results give a positive answer to two open questions of \cite{5}. We also show that if 0 is isolated in the spectrum of \( T \), then \( T \) is a B-Fredholm operator of index 0 if and only if \( T \) is Drazin invertible. Then we define B-Weyl operators and the B-Weyl spectrum as follows:

**Definition 1.1.** Let \( T \in L(X) \). Then \( T \) is called a B-Weyl operator if it is a B-Fredholm operator of index 0.

The B-Weyl spectrum \( \sigma_{BW}(T) \) of \( T \) is defined by \( \sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator} \} \).

In Theorem 4.3 we show that for \( T \in L(X) \) we have

\[
\sigma_{BW}(T) = \bigcap_{F \in F_0(X)} \sigma_D(T + F),
\]

and in the case a normal operator \( T \) acting on a Hilbert space \( H \), we show that

\[
\sigma_{BW}(T) = \sigma(T) \setminus E(T),
\]

where \( E(T) \) is the set of isolated eigenvalues of \( T \), which gives a generalization of the classical Weyl theorem. Recall that the classical Weyl theorem \cite{15} asserts that if \( T \) is a normal operator acting on a Hilbert space \( H \), then the Weyl spectrum \( \sigma_W(T) \) is exactly the set of all points in \( \sigma(T) \) except the isolated eigenvalues of finite multiplicity, that is,

\[
\sigma_W(T) = \sigma(T) \setminus \Pi_{00}(T),
\]
where \( \Pi_{\aleph_0}(T) \) is the set of isolated eigenvalues of finite multiplicity and \( \sigma_W(T) \) is the Weyl spectrum of \( T \); that is, \( \sigma_W(T) = \{ \lambda \in \mathbb{C} \text{ such that } T - \lambda I \text{ is not a Fredholm operator of index 0} \} \). It is known from \([8, \text{Theorem 6.5.2}]\) that

\[
\text{ind} T = \bigcap_{F \in F_0(X)} \sigma(T + F).
\]

Henceforth, if \( M \) and \( N \) are two vector spaces, the notation \( M \cong N \) will mean that \( M \) and \( N \) are isomorphic. We also define the infimum of the empty set to be \( \infty \).

2. Preliminaries

**Definition 2.1** \([3]\). Let \( T \in L(X) \) and let \( n \in \mathbb{N} \).

i) The sequence \( (c_n(T)) \) is defined by \( c_n(T) = \dim R(T^n)/R(T^{n+1}) \), and the descent of \( T \) is defined by \( \delta(T) = \inf \{ n : c_n(T) = 0 \} = \inf \{ n : R(T^n) = R(T^{n+1}) \} \).

ii) The sequence \( (c'_n(T)) \) is defined by \( c'_n(T) = \dim N(T^n)/N(T^{n+1}) \), and the ascent \( a(T) \) of \( T \) is defined by

\[
a(T) = \inf \{ n : c'_n(T) = 0 \} = \inf \{ n : N(T^n) = N(T^{n+1}) \}.
\]

iii) The sequence \( (k_n(T)) \) is defined by

\[
k_n(T) = \dim (R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T)).
\]

**Definition 2.2** \([11]\). Let \( T \in L(X) \) and let \( \Delta(T) = \{ n \in \mathbb{N} : \forall m \in \mathbb{N} \text{ } m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T)) \} \). Then the degree of stable iteration \( dis(T) \) of \( T \) is defined as \( dis(T) = \inf \Delta(T) \).

**Definition 2.3** \([7]\). Let \( T \in L(X) \) and let \( d \in \mathbb{N} \). Then \( T \) has a uniform descent for \( n \geq d \) if \( R(T^n) + N(T^n) = R(T) + N(T^d) \) for all \( n \geq d \), in other words, if \( k_n(T) = 0 \) \( (n \geq d) \).

If in addition \( R(T) + N(T^d) \) is closed, then \( T \) is said to have a topological uniform descent for \( n \geq d \).

**Theorem 2.4** \([7, \text{Theorem 4.7}]\). Suppose that \( T \) is a bounded operator with topological uniform descent for \( n \geq d \) on the Banach space \( X \), \( n,d \in \mathbb{N} \), and that \( V \) is a bounded operator that commutes with \( T \). If \( V - T \) is sufficiently small and invertible, then:

(a) \( V \) has closed range and \( k_p(V) = 0 \) for each integer \( p \geq 0 \).

(b) \( c_p(V) = c_d(T) \) for each integer \( p \geq 0 \).

(c) \( c'_p(V) = c'_d(T) \) for each integer \( p \geq 0 \).

**Remark A.** As it has already been observed in \([2]\) it is immediately seen that a B-Fredholm operator is an operator of topological uniform descent. Using this fact and the properties of operators with topological uniform descent, we have the following properties of the index:

i) If \( S,T \in BF(X), ST = TS \) and \( ||T - S|| \) is small, then \( \text{ind}(T) = \text{ind}(S) \).

(See \([7, \text{Theorem 4.6}.] \).)

ii) If \( S,T \in BF(X), ST = TS \) and \( T - S \) is compact, then \( \text{ind}(T) = \text{ind}(S) \).

(See \([7, \text{Theorem 5.8}.] \).)

iii) If \( T \in BF(X), ST = TS, ||T - S|| \) is small and \( T - S \) is invertible, then \( S \) is a Fredholm operator and \( \text{ind}(S) = \text{ind}(T) \).

(See \([7, \text{Theorem 4.7}.] \).) In particular, if \( T \) is a B-Fredholm operator and \( n \) is an integer large enough, then \( T - \frac{1}{n}I \) is a Fredholm operator and \( \text{ind}(T - \frac{1}{n}I) = \text{ind}(T) \).
3. INDEX OF B-FREDHOLM OPERATORS

If \( T \) is a bounded linear operator \( T \) such that both of \( \alpha(T) \) and \( \beta(T) \) are finite, then the range \( R(T) \) of \( T \) is closed and \( T \) is a Fredholm operator. In the following theorem, we prove a similar result for B-Fredholm operators giving a simple characterization of this class of operators.

**Theorem 3.1.** Let \( T \in L(X) \). Then \( T \) is a B-Fredholm operator if and only if there exists an integer \( n \in \mathbb{N} \) such that \( \alpha(T_n) \) and \( \beta(T_n) \) are finite.

**Proof.** Suppose that \( T \) is a B-Fredholm operator and let \( d = \text{dis}(T) \). Then from [2 Proposition 2.6] we know that \( R(T^d) \cap N(T) \) is of finite dimension and \( R(T) + N(T^d) \) is of finite codimension. So \( \alpha(T_d) \) and \( \beta(T_d) \) are both finite.

Conversely suppose that \( T \in L(X) \) and there exist \( n \in \mathbb{N} \) such that \( \alpha(T_n) \) and \( \beta(T_n) \) are both finite. Then \( R(T) + N(T^n) \) is of finite codimension and \( N(T) \cap R(T^n) \) is of finite dimension. Since \( N(T) \cap R(T^n) \) is of finite dimension, the sequence \( (N(T) \cap R(T^n))_p \) is a stationary sequence for \( p \) large enough. This shows that \( d = \text{dis}(T) \in \mathbb{N} \), \( R(T^d) \cap N(T) \) is of finite dimension and \( R(T) + N(T^d) \) is of finite codimension. From [9 Lemma 3.1] we have \( N(T^{d+1}) \approx N(T) \cap R(T^d) \), and from [9 Lemma 3.2] we have \( \frac{\text{ind}(T^d)}{\text{ind}(T)} \approx \frac{\text{ind}(T)}{\text{ind}(T)} \). It then follows that \( \alpha(T) < \infty \) and \( \beta(T) < \infty \). Since the sequences \( (\alpha(T))_p \) and \( (\beta(T))_p \) are stationary sequences then for \( p \geq d \), we have \( \alpha(T) < \infty \) and \( \beta(T) < \infty \). Moreover by [9 Lemma 3.1] we have \( \frac{\text{ind}(T^p)}{\text{ind}(T)} \approx \frac{\text{ind}(T)}{\text{ind}(T)} \). Therefore \( N(T^p) \cap R(T^p) \) is of finite dimension and \( R(T^p) + N(T^p) \) is of finite codimension. In particular the two sets are closed. Using the Neubauer lemma [11 Proposition 2.1.1] it follows that \( R(T^p) \) is closed. Hence \( T_d \) is a Fredholm operator and so \( T \in BF(X) \).

**Theorem 3.2.** Let \( S, T \) be two commuting B-Fredholm operators. Then the product \( ST \) is a B-Fredholm operator and \( \text{ind}(ST) = \text{ind}(S) + \text{ind}(T) \).

**Proof.** From [5 Corollary 3.5] we know that \( ST \) is a B-Fredholm operator. Moreover there exists an integer \( N_0 \) such that for any integer \( n \geq N_0 \), the operators \( T - \frac{1}{n}I \) and \( S - \frac{1}{n}I \) are both Fredholm operators, \( \text{ind}(T - \frac{1}{n}I) = \text{ind}(T) \) and \( \text{ind}(S - \frac{1}{n}I) = \text{ind}(S) \). Moreover for \( n \geq N_0 \) the difference \( ST - (S - \frac{1}{n}I)(T - \frac{1}{n}I) = \frac{1}{n}(S + T - \frac{1}{n}I) \) is of small norm if the integer \( n \) is chosen large enough. Since \( ST \) and \( (S - \frac{1}{n}I)(T - \frac{1}{n}I) \) are B-Fredholm operators, then by the Remark A we have \( \text{ind}(ST) = \text{ind}((S - \frac{1}{n}I)(T - \frac{1}{n}I)) \). Since \( S - \frac{1}{n}I \) and \( T - \frac{1}{n}I \) are both Fredholm operators, then \( \text{ind}((S - \frac{1}{n}I)(T - \frac{1}{n}I)) = \text{ind}(S - \frac{1}{n}I) + \text{ind}(T - \frac{1}{n}I) \). Since \( \text{ind}(S - \frac{1}{n}I) = \text{ind}(S) \) and \( \text{ind}(T - \frac{1}{n}I) = \text{ind}(T) \), then \( \text{ind}(ST) = \text{ind}(S) + \text{ind}(T) \).

**Proposition 3.3.** Let \( T \in L(X) \) be a B-Fredholm operator and let \( F \) be a finite rank operator. Then \( T + F \) is a B-Fredholm operator and \( \text{ind}(T + F) = \text{ind}(T) \).

**Proof.** From [3 Corollary 3.10], it follows that \( T + F \) is a B-Fredholm operator. Moreover there exists an integer \( N_0 \) such that for any integer \( n \geq N_0 \), \( T - \frac{1}{n}I \) and \( T + F - \frac{1}{n}I \) are Fredholm operators, \( \text{ind}(T - \frac{1}{n}I) = \text{ind}(T) \) and \( \text{ind}(T + F - \frac{1}{n}I) = \text{ind}(T + F) \). Since \( F \) is a finite rank operator and \( T - \frac{1}{n}I \) is a Fredholm operator, by the usual properties of the index we have \( \text{ind}(T + F - \frac{1}{n}I) = \text{ind}(T - \frac{1}{n}I) \). So \( \text{ind}(T + F) = \text{ind}(T) \).
Remark B. 1) If $K$ is a compact operator such that $R(K^n)$ is not closed for every positive integer $n$, then $K$ is not a B-Fredholm operator. So if $F$ is a finite rank operator, then $F$ is a B-Fredholm operator, but $K + F$ is not a B-Fredholm operator, otherwise $K = K + F - F$ would be a B-Fredholm operator. Hence the class of B-Fredholm operators is not stable under compact perturbation.

2) Let $T \in L(X)$. It is easily seen that $T$ is a B-Fredholm operator if only if $T^*$ is a B-Fredholm operator. Moreover in this case $\text{ind}(T^*) = -\text{ind}(T)$.

4. B-Fredholm operators of index 0

Lemma 4.1. Let $T \in L(X)$. Then $T$ is a B-Fredholm operator of index 0 if and only if $T = T_0 \oplus T_1$, where $T_0$ is a Fredholm operator of index 0 and $T_1$ is a nilpotent operator.

Proof. If $T$ is a B-Fredholm operator of index 0, then $X = X_0 \oplus X_1$, where $X_0, X_1$ are closed subspaces of $X$, $T_0 = T|_{X_0}$ is a Fredholm operator and $T_1 = T|_{X_1}$ is a nilpotent operator. Moreover we have $\text{ind}(T) = \text{ind}(T_n)$ for $n$ large enough. Since $T_1$ is a nilpotent operator, then for $n$ large enough we have $R(T^n) = R(T_0^n)$ and $T_n = (T_0)_n$. From [2] Proposition 1] we have $\text{ind}(T_0) = \text{ind}((T_0)_n)) = \text{ind}(T_n) = \text{ind}(T) = 0$.

Conversely if $X = X_0 \oplus X_1$, $T_0 = T|_{X_0}$ is a Fredholm operator of index 0 and $T_1 = T|_{X_1}$ is a nilpotent operator, then by the same arguments, $T$ is a B-Fredholm operator of index 0.

Theorem 4.2. Let $T \in L(X)$ be such that 0 is isolated in the spectrum $\sigma(T)$ of $T$. Then $T$ is a B-Fredholm operator of index 0 if and only if $T$ is Drazin invertible.

Proof. If $T$ is a B-Fredholm operator of index 0, then $X = X_0 \oplus X_1$ such that $T_0 = T|_{X_0}$ is a Fredholm operator of index 0 and $T_1 = T|_{X_1}$ is a nilpotent operator. If $T_0$ is invertible, then $T$ is Drazin invertible. If $T_0$ is not invertible, as 0 is isolated in the spectrum of $T$, then it is also isolated in the spectrum of $T_0$. Since $T_0$ is a Fredholm operator of index 0, it follows from [1] Proposition 2] that $T_0 = T_0 \oplus T_{01}$, where $T_{00}$ is invertible and $T_{01}$ is a nilpotent operator. So $T = T_0 \oplus T_{01} \oplus T_1$, with $T_{00}$ invertible and $T_{01} \oplus T_1$ nilpotent. This shows that $T$ is Drazin invertible.

Conversely if $T$ is Drazin invertible, then $T$ is of finite ascent and descent. It follows from [12] Theorem 1.2] that there is an integer $p$ such that $a(T) = d(T) = p$ and $X = R(T^p) \oplus N(T^p)$. Let $X_0 = R(T^p)$ and $X_1 = N(T^p)$. Since $T_0 = T|_{X_0}$ is an invertible operator and $T_1 = T|_{X_1}$ is a nilpotent operator, from the precedent proposition it follows that $T$ is a B-Fredholm operator of index 0.

Theorem 4.3. Let $T \in L(X)$. Then $\sigma_{\text{BW}}(T) = \bigcap_{F \in F_0(X)} \sigma_D(T + F)$.

Proof. Let $\lambda \notin \sigma_{\text{BW}}(T)$. Then $T - \lambda I$ is a B-Fredholm operator of index 0. From Lemma 4.1, we have $T - \lambda I = T_0 \oplus T_1$, where $T_0$ is a Fredholm operator of index 0 and $T_1$ is a nilpotent operator. From [8] Theorem 6.5.2] there exists a finite rank operator $S_0$ such that $T_0 + S_0$ is invertible. Set $S = S_0 \oplus 0$; then $S$ is of finite rank and $(T - \lambda I) + S = T_0 + S_0 \oplus T_1$ is Drazin invertible. Hence $\lambda \notin \bigcap_{F \in F_0(X)} \sigma_D(T + F)$.

Conversely if $\lambda \notin \bigcap_{F \in F_0(X)} \sigma_D(T + F)$, then there is a finite rank operator $F$ such that $(T - \lambda I) + F$ is Drazin invertible. From Proposition 3.3, $(T - \lambda I) = (T - \lambda I) + F - F$ is a B-Fredholm operator and $\text{ind}(T - \lambda I) = \text{ind}((T - \lambda I) + F) = 0$, and $\lambda \notin \sigma_{\text{BW}}(T)$.
From this theorem, we immediately obtain the following characterization of B-Weyl operators.

**Corollary 4.4.** Let \( T \in L(X) \). Then \( T \) is a B-Weyl operator if and only if \( T = S + F \), where \( S \) is Drazin invertible operator and \( F \) is a finite rank operator.

It is known from [13] Theorem 7.7] that if \( \lambda \) is isolated in the spectrum \( \sigma(T) \) of a normal operator \( T \) acting on a Hilbert space \( H \), then \( T - \lambda I \) is Drazin invertible. By the following theorem we give for such an operator, a generalization of a classical Weyl theorem [13].

**Theorem 4.5.** Let \( T \in L(H) \) be a normal operator. Then \( \sigma_{BW}(T) = \sigma(T) \setminus E(T) \), where \( E(T) \) is the set of isolated eigenvalues of \( T \).

**Proof.** If \( \lambda \notin \sigma_{BW}(T) \) and \( \lambda \in \sigma(T) \), then \( T - \lambda I \) is a B-Fredholm operator of index 0. Hence there exists an integer \( n \) such that \( R((T - \lambda I)^n) \) is closed. Since \( (T - \lambda I)^n \) is a normal operator, then from [13] Theorem VI.3.6],

\[
H = R((T - \lambda I)^n) \oplus N((T - \lambda I)^n).
\]

As \( T - \lambda I \) is a normal operator, then from [13] Theorem VI.3.7], \( N((T - \lambda I)^n) = N((T - \lambda I)) \). Hence \( R((T - \lambda I)) = R((T - \lambda I)^n) \) and \( H = R((T - \lambda I)) \oplus N((T - \lambda I)) \). Since \( \lambda \in \sigma(T) \), then \( N(T - \lambda I) \neq 0 \). It follows that \( \lambda \) is an isolated eigenvalue of \( T \).

Conversely if \( \lambda \in E(T) \), then from [10] Theorem 7.1] we have \( X = X_0 \oplus X_1 \), where \( X_0, X_1 \) are closed subspaces of \( X \), \( T_0 = (T - \lambda I)|_{X_0} \) is an invertible operator and \( T_1 = (T - \lambda I)|_{X_1} \) is a quasi-nilpotent operator. Since \( T \) is a normal operator, then \( T_1 \) is also a normal operator. As \( T_1 \) is quasi-nilpotent, it is a nilpotent operator. Therefore \( T - \lambda I \) is Drazin invertible. From Theorem 2.2 it is a B-Fredholm operator of index 0.

\[ \square \]

**References**


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