A THEOREM ON THE \( k \)-ADIC REPRESENTATION OF POSITIVE INTEGERS

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Abstract. In this paper, a theorem on the asymptotic property of a summation of digits in a \( k \)-adic representation is presented.

Let \( k > 1 \) be a fixed integer. Then any positive integer \( x \) can be uniquely represented by the following \( k \)-adic form:

\[
x = a_1 k^{n_1} + a_2 k^{n_2} + \cdots + a_t k^{n_t},
\]

where \( n_1 > n_2 > \cdots > n_t \geq 0 \) are integers and \( a_1, a_2, \ldots, a_t \) are nonnegative integers not exceeding \( k - 1 \). Define

\[
\alpha(x) = \sum_{i=1}^{t} a_i, \quad A(x) = \sum_{y \leq x} \alpha(y).
\]

In 1940, Bush (1) showed that

\[
A(x) = \frac{k - 1}{2 \log k} x \log x + o(x \log x),
\]

where \( \log \) denotes the natural logarithm. In 1948, Bellman and Shapiro (2) improved this result and proved that

\[
A(x) = \frac{k - 1}{2 \log k} x \log x + O(x \log \log x)
\]

for \( k = 2 \). In 1949, Mirsky (3) showed that the \( O \)-term can be replaced by \( O(x) \) for any \( k \geq 2 \). In 1955, Cheo and Yien (4) gave another proof for the result and obtained:

\[
A(x) = \frac{k - 1}{2 \log k} x \log x + O(x),
\]

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and proved that $O(x)$ cannot be replaced by $O(x^t)$ for any fixed $t < 1$. Their proof relies on the identity
\[
A(x) = \frac{n_1(k-1)}{2} \sum_{i=1}^{t} a_i k^{n_i} - \frac{k-1}{2} \sum_{i=1}^{t} (n_1 - n_i) a_i k^{n_i} \\
+ \frac{1}{2} \sum_{i=1}^{t} a_i (a_i - 1) k^{n_i} + \sum_{i=1}^{t} a_i + \sum_{i=1}^{t} \left( \sum_{j=1}^{i-1} a_j \right) a_i k^{n_i},
\]
where $a_i, n_i,$ and $t$ are as in (1). The first sum equals $\frac{1}{2}(k-1)[\log x/\log k] x$ and the four other sums are shown to be $O(x)$ after complicated mathematical manipulations.

In this paper, we apply a different identity and obtain an estimate on the constant contained in $O(x)$, consequently providing a much simpler proof to the previously known results. The following result is obtained.

**Theorem.** For any integer $k \geq 2$, we have
\[
A(x) = \frac{k-1}{2} \log k x + \theta(x)x,
\]
where
\[
-\frac{5k-4}{8} \leq \theta(x) \leq \frac{k+1}{2}.
\]

To prove this Theorem, we need the following result due to J. L. Lagrange:

**Lemma (5).**
\[
\frac{n - \alpha(n)}{k-1} = \sum_{r=1}^{\infty} \left\lfloor \frac{n}{k^r} \right\rfloor,
\]
where $[a]$ denotes the integral part of the real number $a$.

**Proof of the Theorem.** Using the Lemma, we have
\[
A(x) = \sum_{n \leq x} \left( n - (k-1) \sum_{r=1}^{\infty} \left\lfloor \frac{n}{k^r} \right\rfloor \right) \\
= \frac{1}{2} x(x+1) - (k-1) \sum_{r=1}^{\infty} \sum_{n \leq \frac{x}{k^r}} \left\lfloor \frac{n}{k^r} \right\rfloor \\
= \frac{1}{2} x(x+1) - (k-1) \sum_{1 \leq r \leq \log_k x} \left( \frac{1}{2} \left\lfloor \frac{x}{k^r} \right\rfloor \left( \left\lfloor \frac{x}{k^r} \right\rfloor - 1 \right) k^r \\
+ \left\lfloor \frac{x}{k^r} \right\rfloor \left( x - \left\lfloor \frac{x}{k^r} \right\rfloor k^r + 1 \right) \right) \\
= \frac{1}{2} x(x+1) + \frac{1}{2} (k-1) \sum_{1 \leq r \leq \log_k x} k^r \left\lfloor \frac{x}{k^r} \right\rfloor - (k-1) \sum_{1 \leq r \leq \log_k x} \left\lfloor \frac{x}{k^r} \right\rfloor \left\lfloor \frac{x}{k^r} \right\rfloor k^r \\
- (k-1) \sum_{1 \leq r \leq \log_k x} \left( x \left\lfloor \frac{x}{k^r} \right\rfloor - \frac{1}{2} \left\lfloor \frac{x}{k^r} \right\rfloor^2 k^r \right).
\]
However, we observe that
\[ \sum_{1 \leq r \leq \log k} k^r \left\lceil \frac{x}{k^r} \right\rceil = x\log k x + \sum_{1 \leq r \leq \log k} k^r \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \frac{x}{k^r} \right) \]
\[ = x\log k x - \theta_1(x)x + \sum_{1 \leq r \leq \log k} k^r \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \frac{x}{k^r} \right), \]
\[ \sum_{1 \leq r \leq \log k} \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \frac{1}{2} \left( \frac{x}{k^r} \right)^2 k^r \right) = \frac{1}{2} \sum_{1 \leq r \leq \log k} \left( \frac{x^2}{k^r} - k^r \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \frac{x}{k^r} \right)^2 \right) \]
\[ = \frac{1}{2} x^2 \sum_{1 \leq r \leq \log k} \frac{1}{k^r} - \frac{1}{2} \sum_{1 \leq r \leq \log k} k^r \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \frac{x}{k^r} \right)^2, \]
where \( 0 \leq \theta_1(x) < 1. \) Taking these into (9), we obtain
\[ A(x) = \frac{1}{2} x(x + 1) + \frac{k - 1}{2} x\log k x - \frac{k - 1}{2} \theta_1(x)x - (k - 1) \sum_{1 \leq r \leq \log k} \left\lceil \frac{x}{k^r} \right\rceil \]
\[ - \frac{1}{2} \sum_{1 \leq r \leq \log k} \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \left\lceil \frac{x}{k^r} \right\rceil \right) k^r - \frac{k - 1}{2} x^2 \sum_{1 \leq r \leq \log k} \frac{1}{k^r}, \]
where \( \{a\} \) denotes the fractional part of the real number \( a. \) Using the following inequalities \( 0 \leq x - x^2 \leq 1/4 \) \((0 \leq x \leq 1)\) and \( [a] \leq a, \) we can find \( 0 \leq \theta_2(x) \leq 1\) and \( 0 \leq \theta_3(x) \leq 1\) such that
\[ \sum_{1 \leq r \leq \log k} \left\lceil \frac{x}{k^r} \right\rceil = \theta_2(x) \frac{x}{k - 1}, \]
\[ \sum_{1 \leq r \leq \log k} \left( \left\lfloor \frac{x}{k^r} \right\rfloor - \left\lceil \frac{x}{k^r} \right\rceil \right) k^r = \theta_3(x) \frac{xk}{4(k - 1)}, \]
\[ x^2 \sum_{1 \leq r \leq \log k} \frac{1}{k^r} = \frac{x^2}{k - 1} - \frac{1}{k - 1} \frac{x^2}{k \log k x}. \]
Substituting these into (9), we finally arrive at
\[ A(x) = \frac{k - 1}{2} x\log k x + \left( -\frac{k - 1}{2} \theta_1(x) - \theta_2(x) + \frac{1}{2} - \frac{k}{8} \theta_3(x) + \frac{x}{2k \log k x} \right) x \]
\[ = \frac{k - 1}{2} x\log k x + \theta(x)x, \]
where
\[ -\frac{5k - 4}{8} \leq \theta(x) \leq \frac{k + 1}{2}. \]
This completes the proof.

References


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