EQUIVALENCE OF DOMAINS WITH ISOMORPHIC SEMIGROUPS OF ENDMORPHISMS

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Abstract. For two bounded domains \( \Omega_1, \Omega_2 \) in \( \mathbb{C} \) whose semigroups of analytic endomorphisms \( E(\Omega_1), E(\Omega_2) \) are isomorphic with an isomorphism \( \varphi : E(\Omega_1) \to E(\Omega_2) \), Eremenko proved in 1993 that there exists a conformal or anticonformal map \( \psi : \Omega_1 \to \Omega_2 \) such that \( \varphi f = \psi \circ f \circ \psi^{-1} \), for all \( f \in E(\Omega_1) \).

In the present paper we prove an analogue of this result for the case of bounded domains in \( \mathbb{C}^n \).

1. Introduction

A classical theorem of L. Bers says that every \( \mathbb{C} \)-algebra isomorphism \( H(A) \to H(B) \) of algebras of holomorphic functions in domains \( A \) and \( B \) in the complex plane has the form \( f \mapsto f \circ \theta \), where \( \theta : B \to A \) is a conformal isomorphism, or \( f \mapsto \overline{f} \circ \theta \) with anticonformal \( \theta \). In particular, the algebras \( H(A) \) and \( H(B) \) are isomorphic if and only if the domains \( A \) and \( B \) are conformally equivalent. H. Iss’sa [9] obtained a similar theorem for fields of meromorphic functions on Stein spaces.

A good reference for these results is [5].

Likewise, the question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [12].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) \( E(D) \) of holomorphic endomorphisms of a domain \( D \). A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in \( \mathbb{C} \) can have isomorphic semigroups of endomorphisms. The reason is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [4] proved that for two Riemann surfaces \( D_1, D_2 \), which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms \( E(D_1) \) and \( E(D_2) \) are isomorphic with an isomorphism \( \varphi : E(D_1) \to E(D_2) \), there exists a conformal or anticonformal map \( \psi : D_1 \to D_2 \) such that \( \varphi f = \psi \circ f \circ \psi^{-1} \), for all \( f \in E(D_1) \). In the present paper we investigate the analogue of this result for the case of bounded domains in \( \mathbb{C}^n \). The theorems of Bers and Iss’sa, mentioned above, do not extend to arbitrary domains in \( \mathbb{C}^n \).
For a bounded domain $\Omega$ in $\mathbb{C}^n$ we denote by $E(\Omega)$ the semigroup of analytic endomorphisms of $\Omega$ under composition. In what follows, we say that a map is (anti-) biholomorphic, if it is biholomorphic or antibiholomorphic. We prove the following theorem.

**Theorem 1.** Let $\Omega_1, \Omega_2$ be bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, and suppose that there exists $\varphi : E(\Omega_1) \to E(\Omega_2)$, an isomorphism of semigroups. Then $n = m$ and there exists an (anti-) biholomorphic map $\psi : \Omega_1 \to \Omega_2$ such that

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1).$$

The existence of a homeomorphism $\psi$ satisfying (1) follows from simple general considerations (Section 2). The hard part is proving that $\psi$ is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of $f \in E(\Omega)$ near an attracting fixed point. In several dimensions such linearization theory exists ([1], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 3, 4 we show how to localize the problem. In Sections 5, 6 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 7.

In Section 8 we complete the proof that $\psi$ is (anti-) biholomorphic.

Theorem 1 can be slightly generalized, namely one may assume that $\varphi$ is an epimorphism. In Section 9 we prove the following theorem.

**Theorem 2.** If $\varphi : E(\Omega_1) \to E(\Omega_2)$ is an epimorphism between semigroups, where $\Omega_1, \Omega_2$ are bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, then $\varphi$ is an isomorphism.

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2. **Topology**

For a bounded domain $\Omega$ in $\mathbb{C}^n$ we denote by $C(\Omega)$ the subsemigroup of $E(\Omega)$ consisting of constant maps. An endomorphism $c_z$ is constant if it sends $\Omega$ to a point $z \in \Omega$. The subset $C(\Omega) \subset E(\Omega)$ can be described using only the semigroup structure as follows:

$$c \in C(\Omega) \text{ iff } \forall (f \in E(\Omega)), \quad (c \circ f = c).$$

It is clear that we have a bijection between constant endomorphisms of $\Omega$ and points of this domain as a set: to each $z \in \Omega$ corresponds a unique $c_z \in C(\Omega)$ and vice versa, so we can identify the two. Under this identification, a subset of $\Omega$ corresponds to a subsemigroup of $C(\Omega)$.

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map $\psi$ between $\Omega_1$ and $\Omega_2$ as follows:

$$\psi(z) = w \text{ iff } \varphi c_z = c_w.$$

So defined, $\psi$ satisfies (1). Indeed, let $f \in E(\Omega_1), f(z) = z$. This is equivalent to

$$f \circ c_z = c_z.$$

Applying $\varphi$ to both sides of (4), we have

$$\varphi f \circ c_{\psi(z)} = c_{\psi(z)}.$$
Then, assuming that continuous, and thus monomorphism consider a relation on the set of formal expressions:

We denote the class of injective endomorphisms of \( S \) by \( S \). Then there exists a semigroup \( S \) satisfying the following two properties:

Moreover, the semigroup \( S \) satisfies the following universal property: for every semigroup \( S \) with a monomorphism \( f : S \to S \) such that \( f(\zeta) = 1 \) and \( f \) commutes with all elements of \( S \), there exists a unique monomorphism \( f : S \to S \) such that \( f = i \circ f \).

**Remark 1.** Uniqueness of \( i \) implies that the semigroup \( S \) with the universal property is unique up to an isomorphism.

**Proof.** We construct \( S \) as follows. First we consider formal expressions of the form \( h f^k \), where \( h \in H \) and \( k \) is an integer (it may be positive, negative or zero). Then we define a multiplication on this set: \( h f^k \ast h f^l = h f^{k+l} \). Next we consider a relation on the set of formal expressions: \( h f^k \sim h f^l \) if \( k \leq l \) and \( h f^k \in H \), or \( k \leq l \) and \( h f^k \in H \). It is easy to verify that this is an equivalence relation and it is compatible with the operation \( \ast \). That is, \( x \sim y \), \( u \sim v \) implies \( x \ast u \sim y \ast v \).

Lastly, let \( S \) be the set of equivalence classes with the binary operation induced by \( \ast \). For \( S \) to be a semigroup, we need to show that the binary operation \( \ast \) is associative. Let \( h f^k \ast h f^l = h f^{k+l} \) and \( h f^k \ast h f^l = h f^{k+l} \) and \( h f^k \ast h f^l = h f^{k+l} \). We need to show that \( (h f^k \ast h f^l) \ast h f^m = h f^{k+l+m} \). By the definition of the operation \( \ast \), the last equivalence is the same as \( h f^k h f^l \ast h f^m = h f^{k+l+m} \). Thus, assuming that \( k \leq l \) and \( l \leq m \), we have essentially one possibility to
consider (the others are either similar or trivial): \( k_1 \leq k_1', k_2 \leq k_2', k_3' \leq k_3. \) In this case \( h_1 h_2 h_3 f^{k_3-k_3'} = h_1' h_2' h_3' f^{k_1-k_1'+k_2-k_2}. \) Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup \( H \) is embedded into \( S_f \) via \( i: h \mapsto [hf^0]. \) The element \( i(f) = [id_f], \) where \( id \) is the identity in \( H, \) is invertible in \( S_f \) with the inverse \( [id_f^{-1}]. \)

Clearly, \([id_f]\) commutes with all elements of \( S_f.\)

Now, suppose that \( S_1, i_1 : H \to S_1 \) is a semigroup and a monomorphism, such that \( i_1(f) \) is invertible in \( S_1 \) and commutes with all elements of \( S_1. \) Then we define

\[
i_1([hf^k]) = i_1(h)(i_1(f))^k.
\]

This definition does not depend on a representative of \([hf^k].\) Indeed, suppose

\[
h_1 f^{k_1} \sim h_2 f^{k_2}
\]

and assume \( k_1 \leq k_2. \) Then \( h_1 = h_2 f^{k_2-k_1}, \) and thus \( i_1(h_1) = i_1(h_2) i_1(f)^{k_2-k_1}. \) Hence \( i_1(h_1) i_1(f)^{k_1} = i_1(h_2) i_1(f)^{k_2}. \)

So defined, \( i_1 \) is a homomorphism:

\[
i_1([h_1 f^{k_1}] [h_2 f^{k_2}]) = \hat{i}_1([h_1 h_2 f^{k_1+k_2}]) = i_1(h_1 h_2) i_1(f)^{k_1+k_2} = i_1(h_1) i_1(h_2) i_1(f)^{k_1} i_1(f)^{k_2} = i_1(h_1) i_1(f)^{k_1} i_1(h_2) i_1(f)^{k_2} = \hat{i}_1([h_1 f^{k_1}] [h_2 f^{k_2}])
\]

The relation \( \hat{i}_1 \circ i = i_1 \) holds, since \( \hat{i}_1([hf^0]) = i_1(h) \) for all \( h \in H. \)

Uniqueness of \( i_1 \) is clear. Lemma \( \mathbf{[\ref{l:ext}] \text{ is proved.}} \)

4. Extension of \( \varphi \)

Following \( \mathbf{[\ref{l:ext}]} \), we say that for a bounded domain \( \Omega \) an element \( f \in E(\Omega) \) is \emph{good} at \( z \in \Omega, \) denoted by \( f \in G_z(\Omega), \) if

1. \( z \) is a unique fixed point of \( f; \)
2. \( f(\Omega) \) has compact closure in \( \Omega; \)
3. \( f \) is injective in \( \Omega. \)

Property 3 of a good element was already stated in terms of the semigroup structure of \( \Omega. \) Since the topology on \( \Omega \) was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

\[
(f \circ c_z = c_z) \land ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).
\]

Since \( f \) is an endomorphism of a domain, all eigenvalues \( \lambda \) of its linear part at \( z \) satisfy \( |\lambda| \leq 1. \) Moreover, \( |\lambda| < 1 \) because the closure of \( f(\Omega) \) is a compact set in \( \Omega. \) The injectivity of \( f \) implies \( \mathbf{[\ref{l:ext}]} \) that it is biholomorphic onto \( f(\Omega) \) and the Jacobian determinant of \( f \) does not vanish at any point of \( \Omega. \)

It is clear that for every \( z \in \Omega \) a good element \( f \) at \( z \) exists. For example, we can take \( f(\zeta) = z + \lambda (\zeta - z) \) with sufficiently small \( |\lambda| \).

Consider a good element \( f \in G_z(\Omega) \) and its commutant \( H_f(\Omega) \) in \( E(\Omega) \):

\[
H_f(\Omega) = \{ h \in E(\Omega) : hf = fh \}.
\]

Clearly \( H_f(\Omega) \) is a subsemigroup of \( E(\Omega). \) The element \( f, \) being good (hence injective), satisfies the cancellation property (ii) of Lemma \( \mathbf{[\ref{l:ext}]} \) in \( H_f(\Omega). \) Thus, by Lemma \( \mathbf{[\ref{l:ext}]} \) we have the extension \( S_f \) of \( H_f(\Omega) \) in which \( f \) is invertible and commutes with all elements of \( S_f. \) In the case of analytic endomorphisms we can embed \( H_f(\Omega) \) into the subsemigroup of \( A_z, \) the semigroup of germs of analytic mappings at \( z \) under composition, consisting of elements that commute with the germ of \( f \).
and containing the germ of $f^{-1}$. We use the universal property of Lemma 1 to conclude that $S_f$ is isomorphic to a subsemigroup of $A_2$. We identify $S_f$ with this semigroup, i.e. we consider elements of $S_f$ as germs of analytic mappings at $z$.

In proving that $\psi$ is (anti-) biholomorphic we need to show that it is so in a neighborhood of every point of $\Omega_1$. Since an (anti-) biholomorphic type of a domain is preserved by translations in $\mathbb{C}^n$, it is enough to show that $\psi$ is (anti-) biholomorphic in a neighborhood of $0 \in \mathbb{C}^n$, assuming that $\Omega_1$ and $\Omega_2$ contain $0$ and $\psi(0) = 0$.

Let $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$ be an isomorphism of the semigroups, $f$ a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of $f$. Then clearly $H_g(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma 1 we have the extensions $S_f$, $S_g$ of $H_f(\Omega_1)$ and $H_g(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism $\varphi$ extends to an isomorphism

$$\Phi : S_f \rightarrow S_g.$$ 

5. System of projections and linearization

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. We say that a good element $f \in G_0(\Omega)$ is very good at $0$, and write $f \in V G_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section 4 contains a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

(a) $\forall (i = 1, \ldots, n), \ (p_i \neq 0)$;
(b) $\forall (i = 1, \ldots, n), \ (p_i^2 = p_i)$;
(c) $\forall (i, j = 1, \ldots, n, i \neq j), \ (p_ip_j = 0)$.

There does exist a very good element, since we can take $f$ to be a homothetic transformation at $0$ with sufficiently small coefficient, $p_i$, a projection on the $i$'th coordinate of the standard coordinate system. Clearly, $p_i f = f p_i$ and there exists $k$ such that $p_i f^k \in E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good element $f \in V G_0(\Omega)$, associated semigroups $H_f(\Omega)$, $S_f$ and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), \ hp_i = p_i h, \ i = 1, \ldots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since $f$ belongs to it.

Lemma 2. For every $h \in P_f(\Omega)$ there exists a biholomorphic germ $\theta_h$ at $0 \in \mathbb{C}^n$ such that $\theta_h h = \Lambda \theta_h$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is an invertible diagonal matrix which is similar to $d h(0)$ in $GL(n, \mathbb{C})$.

Proof. The relations $p_i \neq 0$, $p_i^2 = p_i$, $p_i p_j = 0$, $i \neq j$, imply that for $P_i = d p_i(0)$, the linear part of $p_i$ at $0$, we have $P_i \neq 0$, $P_i^2 = P_i$, $P_i P_j = 0$, $i \neq j$. Since the matrices $P_i$ commute, there exists $A$ a matrix $A \in GL(n, \mathbb{C})$ such that $P_i^2 = A P_i A^{-1} = \Delta_i = \text{diag}(0, \ldots, 1, \ldots, 0)$, where the only non-zero entry appears in the $i$'th place.

Since $p_i^2 = p_i$, $i = 1, \ldots, n$, we can use the argument given in [10] to linearize $p_i$, i.e. there exists a biholomorphic germ $\xi_i$ at $0$ such that $\xi_i p_i = P_i \xi_i$, $d \xi_i(0) = \text{id}$, $i = 1, \ldots, n$. The map $\xi_i$ is constructed in [10] as follows:

$$\xi_i = \text{id} + (2P_i - \text{id})(p_i - P_i), \ i = 1, \ldots, n.$$
If we take $\xi'_i = A\xi_i$, we have $\xi'_i p_i = P'_i \xi'_i$. For simplicity of notations, we assume that $\xi_i$ itself conjugates $p_i$ to a diagonal matrix, that is, $P_i = P'_i$ (in this case $P_i$ is not necessarily $d p_i(0)$, but rather $A d p_i(0) A^{-1}$, $d \xi_i(0) = A$). For every $i = 1, \ldots, n$ we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h \xi_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the $i$'th row and the $i$'th column the matrix $H_i$ has only one non-zero entry, $\lambda_i$, which is located at their intersection. Thus $\lambda_i$ has to be an eigenvalue of $H_i$, and hence of the linear part of $h$. In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \to \mathbb{C}^n$ be the embedding $z \mapsto (0, \ldots, z, \ldots, 0)$, where the only non-zero entry is $z$, which is in the $i$'th place; and $\pi_i : \mathbb{C}^n \to \mathbb{C}$, a projection $(z_1, \ldots, z_n) \mapsto z_i$, corresponding to the $i$'th axis. For every $i = 1, \ldots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of 0 in $\mathbb{C}$ into $\mathbb{C}$, and its derivative at 0, $\lambda_i$, is an eigenvalue of $h$. Hence ([3], p. 31) $\pi_i h_i I_i$ is linearized by the unique solution $\eta_{h,i}$ of the Schröder equation

$$\eta(\pi_i h_i I_i) = \lambda_i \eta, \quad \eta(0) = 0, \quad \eta'(0) = 1. \tag{6}$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = id_\mathbb{C}$, we can rewrite (6) as

$$\eta_{h,i} \pi_i h_i P_i I_i = \lambda_i \eta_{h,i} \pi_i P_i I_i, \quad \text{or} \quad \eta_{h,i} \pi_i h_i P_i I_i = \lambda_i \eta_{h,i} \pi_i P_i I_i. \tag{7}$$

But $h_i P_i = P_i h_i$, and so

$$\eta_{h,i} \pi_i P_i I_i = \lambda_i \eta_{h,i} \pi_i P_i I_i. \tag{8}$$

The equation (7), in its turn, is equivalent to

$$\eta_{h,i} \pi_i \xi p_i h = \lambda_i \eta_{h,i} \pi_i \xi p_i. \tag{9}$$

We denote

$$\theta_{h,i} = \eta_{h,i} \pi_i \xi p_i,$$

a map from a neighborhood of 0 in $\mathbb{C}^n$ into $\mathbb{C}$. Then (9) becomes $\theta_{h,i} h = \lambda_i \theta_{h,i}$. Now we define

$$\theta_h = (\theta_{h,1}, \ldots, \theta_{h,n}),$$

which is a germ of an analytic map at 0. This germ linearizes $h$:

$$\theta_h h = (\theta_{h,1} h, \ldots, \theta_{h,n} h) = (\lambda_1 \theta_{h,1}, \ldots, \lambda_n \theta_{h,n}) = \Lambda \theta_h,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ $\theta_h$ is biholomorphic. Indeed,

$$\theta_{h,i} = \eta_{h,i} \pi_i \xi p_i = \eta_{h,i} \pi_i P_i \xi_i, \quad i = 1, \ldots, n.$$

Using the chain rule, we see that $d \theta_h(0) = A$, where $A$ is an invertible diagonal matrix that diagonalizes $P_i$. We conclude that $\theta_h$ is biholomorphic. Lemma 2 is proved.

6. Simultaneous linearization

Using Lemma 2, we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists $\theta_h$ (constructed in Section 5) such that $\theta_h h = \Lambda_h \theta_h$, where $\Lambda_h$ is an invertible diagonal matrix. In particular, we can linearize $f$:

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ $\theta_f$ is biholomorphic at 0, and $\Lambda_f$ is an invertible diagonal matrix.
Lemma 3. For every \( h \in P_f(\Omega) \) we have \( \theta_h = \theta_f \).

Proof. Let us consider the germ

\[
\theta = \Lambda_f^{-1} \theta_f,
\]

which is clearly biholomorphic. We have

\[
\theta h = \Lambda_f^{-1} \theta_h f h = \Lambda_f^{-1} \theta_h f = \Lambda_f^{-1} \Lambda_h \theta_h f = \Lambda_h \Lambda_f^{-1} \theta_h f = \Lambda_h \theta.
\]

Using (10), we write the equation

\[
\theta h = \Lambda_h \theta
\]

in the coordinate form:

\[
(1/\lambda_{f,i}) \theta_{h,i} f h = (\lambda_{h,i}/\lambda_{f,i}) \theta_{h,i} f, \quad i = 1, \ldots, n.
\]

By (9) and the definition of \( \xi_i \),

\[
(1/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i h_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i, \quad i = 1, \ldots, n,
\]

where \( f_i = \xi_i f \xi_i^{-1} \). Using the commutativity relations \( f_i P_i = P_i f_i \), \( h_i P_i = P_i h_i \), which hold since \( \{p_i\} \subset S_f, \ h \in P_f(\Omega) \), we get

\[
(1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i P_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i P_i, \quad \text{or}
\]

\[
(1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i I_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i, \quad i = 1, \ldots, n.
\]

This is the same as

\[
((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i) (\pi_i h_i I_i) = \lambda_{h,i} ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i), \quad i = 1, \ldots, n,
\]

since \( h_i \) locally preserves the \( i \)th coordinate axis \( (h_i P_i = P_i h_i) \). It is easily seen that

\[
((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i) (0) = 0,
\]

\[
((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)'(0) = 1.
\]

A normalized solution to a Schröder equation is unique, though; thus we have

\[
\eta_{h,i} (\pi_i f_i I_i) = \lambda_{f,i} \eta_{h,i}, \quad \eta_{h,i} (0) = 0, \quad \eta_{h,i}'(0) = 1.
\]

Using the uniqueness argument again, we obtain \( \eta_{h,i} = \eta_{f,i} \), and hence \( \theta_h = \theta_f \).

The lemma is proved.

According to Lemma 3, the single biholomorphic germ \( \theta_f \) conjugates the subsemigroup \( P_f(\Omega) \) to some subsemigroup \( D_f \) of invertible diagonal matrices in \( D_n \), the set of all \( n \times n \) diagonal matrices with entries in \( \mathbb{C} \). We show that \( D_f \) contains all invertible diagonal matrices with sufficiently small entries. To do this, first we extend \( \theta_f \) to an analytic map on the whole domain \( \Omega \) using the formula

\[
\theta_f = \Lambda_f^{-1} \theta_f f^l,
\]

where \( l \) is chosen so large that \( \text{Cl}\{f^l(\Omega)\} \) is contained in a neighborhood of 0 where \( \theta_f \) is originally defined and biholomorphic; the symbol \( \text{Cl} \) denotes closure. From the procedure of extending \( \theta_f \) to \( \Omega \) we see that it is one-to-one and bounded in the whole domain.

Now, let \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) be a matrix such that \( \text{Cl}\{\Lambda \theta_f(\Omega)\} \subset W \), where \( W \) is a neighborhood of 0 in \( \mathbb{C}^n \) for which \( \text{Cl}\{\theta_f^{-1} W\} \subset \Omega \). Such a matrix \( \Lambda \) exists since \( \theta_f \) is bounded in \( \Omega \). Consider \( h = \theta_f^{-1} \Lambda \theta_f \), which belongs to \( G_0(\Omega) \). The map \( h \) commutes with \( f \) and all \( p_i \)'s. Indeed, using the formula \( \theta_f f \theta_f^{-1} = \Lambda_f \), we conclude that \( h f = f h \) is equivalent to \( \Lambda \Lambda_f = \Lambda_f \Lambda \), which is a true relation since both matrices \( \Lambda \) and \( \Lambda_f \) are diagonal. The relations \( h p_i = p_i h, \ i = 1, \ldots, n, \) are
verified similarly, using the formula $\theta_f p \theta_f^{-1} = P$, which follows from the definition of $\theta_f$.

7. SOLVING A MATRIX EQUATION

We proved that for an element $f \in VG_0(\Omega)$ there exists a biholomorphic germ $\theta_f$ conjugating the semigroup $P_f(\Omega)$ to a subsemigroup $D_f \subset D_n$, which contains all invertible diagonal matrices with sufficiently small entries.

Let $f \in VG_0(\Omega_1)$ and $g = \varphi f$. Then $g \in VG_0(\Omega_2)$, and there is an isomorphism

$$\Phi : \mathcal{S}_f \rightarrow \mathcal{S}_g.$$ 

For the mappings $f$ and $g$ we have

$$\theta_ff = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,$$

where $\Lambda_f, M_g$ are invertible diagonal matrices.

Let us consider the germ $L = \theta_g \psi \theta_f^{-1}$. This germ conjugates the semigroups $D_f$, $D_g$:

$$L \Lambda L^{-1} = \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1} = \theta_g h \psi \psi^{-1} \theta_g^{-1} = \theta_g \theta_g^{-1} = M,$$

where $h \in P_f$, $\theta_f h = \Lambda \theta_f$; $j = \varphi h$, $\theta_g j = M \theta_g$.

Define $R(\Lambda) = L \Lambda L^{-1}$. Then $R : D_f \rightarrow D_g$,

$$R(\Lambda_1 \Lambda_2) = R(\Lambda_1) R(\Lambda_2), \quad \Lambda_1, \Lambda_2 \in D_f.$$ 

In what follows, we will identify $D_n$ with the multiplicative semigroup $\mathbb{C}^n$ ($D_n \cong \mathbb{C}^n$) in the obvious way and consider a topology on $D_n$ induced by the standard topology on $\mathbb{C}^n$.

We are going to extend $R$ to an isomorphism of $D_n$. First, we denote by $\overline{D_f}$, $\overline{D_g}$ the closures of $D_f$, $D_g$ in $D_n$, and for $\Lambda \in \overline{D_f}$ we set

$$R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \rightarrow \Lambda, \quad \Lambda_k \in D_f.$$ 

This limit exists and does not depend on the sequence $\{\Lambda_k\}$, which follows from the fact that $\psi \pm 1, \theta_f \pm 1, \theta_g \pm 1$ are continuous. The map $R$ is an isomorphism of topological semigroups $\overline{D_f}$ and $\overline{D_g}$ (the inverse of $R$ has a similar representation).

Next, we extend the map $R$ to $D_n$ as

$$R(\Gamma) = R(\Gamma \Lambda) R(\Lambda)^{-1}, \quad \Gamma \in D_n,$$

where $\Lambda \in D_f$ is chosen so that $\Gamma \Lambda \in \overline{D_f}$. This definition does not depend on the choice of $\Lambda$. Indeed, since all matrices in question are diagonal (hence commute), the relation $R(\Gamma \Lambda_1) R(\Lambda_1)^{-1} = R(\Gamma \Lambda_2) R(\Lambda_2)^{-1}$ is equivalent to $R(\Gamma \Lambda_1) R(\Lambda_2) = R(\Gamma \Lambda_2) R(\Lambda_1)$, which holds.

The extended map $R$ is clearly an isomorphism of $D_n$ onto itself. Thus we have

(11) $$R(\Lambda' \Lambda'') = R(\Lambda') R(\Lambda''), \quad \Lambda', \Lambda'' \in D_n.$$ 

Injectivity of $R$ and (11) imply that $R(\Delta_i) = \Delta_i$ for all $i$, where $j = j(i)$ depends on $i$; $j(i)$ is a permutation on $\{1, \ldots, n\}$ (we recall that $\Delta = \text{diag}(0, \ldots, 1, \ldots, 0)$).

This is because $\{\Delta_i\}_{i=1}^n$ is the only system in $D_n$ with the following relations: $\Delta_i \neq 0$, $\Delta_i^2 = \Delta_i$, $\Delta_i \Delta_j = 0$, $i \neq j$. 

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Since all matrices $\Lambda$ and their images $R(\Lambda)$ are diagonal, we can consider the matrix equation (11) as $n$ scalar equations:

\begin{equation}
q_j(\lambda_1^{n_j}, \ldots, \lambda_n^{n_j}) = r_j(\lambda_1^{n_j}, \ldots, \lambda_n^{n_j}), 
\end{equation}

where $r_j$ are components of $R$. If we rewrite the equation $R(\Delta, \Lambda) = \Delta_j R(\Lambda)$ in the coordinate form, we see that $r_j(\lambda_1, \ldots, \lambda_n) = r_j(0, \ldots, \lambda_i, \ldots, 0) = q_j(\lambda_i)$; that is, each $r_j$ depends on only one of the $\lambda_i$’s. For each $j$ the corresponding equation in (12) in terms of the $q_j$’s becomes

\[q_j(\lambda_i^{n_j}) = q_j(\lambda_i^{n_i})q_j(\lambda_i^{n_i}).\]

This equation has (4, p. 130) either the constant solution $q_j(\lambda_i) = 1$, or

\[q_j(\lambda_i) = \lambda_i^{\alpha_i} / \lambda_i^{\beta_i}, \quad \alpha_i, \beta_i \in \mathbb{C}, \quad \alpha_i - \beta_i = \pm 1.\]

Going back to the function $L$, we have

\begin{equation}
L(\lambda_1, \ldots, \lambda_n) = \text{diag}(\lambda_1^{\alpha_1}, \ldots, \lambda_n^{\alpha_n}) L, 
\end{equation}

where $i(j)$ is the inverse permutation to $j(i)$.

Let us choose and fix $(\mu_1, \ldots, \mu_n)$ such that $(1/\mu_1, \ldots, 1/\mu_n)$ belongs to a neighborhood $W_0$ of $0 \in \mathbb{C}^n$ where $L$ is defined, and let $W_1$ be a neighborhood of $0 \in \mathbb{C}^n$ such that $(\mu_1 z_1, \ldots, \mu_n z_n) \in W_0$, whenever $(z_1, \ldots, z_n) \in W_1$. Then from (13) we have

\[L(z_1, \ldots, z_n) = L(\mu_1 z_1, \ldots, \mu_n z_n)(1/\mu_1, \ldots, 1/\mu_n) \]

\[= \text{diag}((\mu_{i(1)} z_{i(1)})^{\alpha_i} (\mu_{i(1)}^{\alpha_i} \mu_{i(1)}^{\beta_i}), \ldots, (\mu_{i(n)} z_{i(n)})^{\alpha_n} (\mu_{i(n)}^{\alpha_n} \mu_{i(n)}^{\beta_n})) \times L(1/\mu_1, \ldots, 1/\mu_n) = B(z_1^{\alpha_1} z_1^{\beta_1}, \ldots, z_n^{\alpha_n} z_n^{\beta_n}), \]

where $B$ is a constant matrix. The last formula is the explicit expression for $L$.

8. Proving that $\psi$ is (anti-) biholomorphic

To prove that $\psi$ is (anti-) biholomorphic is the same as to prove that $L$ is (anti-) biholomorphic, because the relation $L = \theta_g \circ \psi \circ \theta_f^{-1}$ holds. We showed that

\begin{equation}
L(z_1, \ldots, z_n) = B(z_1^{\alpha_1} z_1^{\beta_1}, \ldots, z_n^{\alpha_n} z_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \ldots, n, \]

in a neighborhood $W_1$ of 0. From the representation (13) we see that $L$ is $R$-differentiable and non-degenerate in $W_1 \setminus \bigcup_{i=1}^{n} \{(z_1, \ldots, z_n) : z_k = 0\}$. Since this is true for every point in the domain $\Omega_1$, the map $\psi$ is $R$-differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from $\Omega_1$, as well as its image under $\psi$ from $\Omega_2$. We call the domains obtained in this way $\Omega_1', \Omega_2'$. Now the map $\psi : \Omega_1' \to \Omega_2'$ is $R$-differentiable and non-degenerate everywhere. It is clear that if we prove that $\psi$ is (anti-) biholomorphic between $\Omega_1, \Omega_2$ due to a standard continuation argument (11). So we can think that $\psi$ is $R$-differentiable and non-degenerate in $\Omega_1$ itself. The map $L$ thus has to be $R$-differentiable and non-degenerate at 0. However, this is the case if and only if $\alpha_i + \beta_i = 1, i = 1, \ldots, n$. Together with the equation $\alpha_i - \beta_i = \pm 1$ it gives us that either $\alpha_i = 1, \beta_i = 0$, or $\alpha_i = 0, \beta_i = 1$. 

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It remains to show that either $\alpha_i = 1$ and $\beta_i = 0$, or $\alpha_i = 0$ and $\beta_i = 1$, simultaneously for all $i$. Suppose, by way of contradiction, that we have $L(z_1, \ldots, z_n) = B(\ldots, z_i, \ldots, \overline{z}_j, \ldots)$. Then

$$L^{-1}(w_1, \ldots, w_n) = (\ldots, l_i(w_1, \ldots, w_n), \ldots, l_j(\overline{w}_1, \ldots, \overline{w}_n), \ldots),$$

where $l_i$, $l_j$ are linear analytic functions. Let us look at an endomorphism $f_0$ of $\Omega_1$ in the form

$$f_0 = \theta^{-1}_f \lambda(\ldots, \theta_{f,j}, \ldots) \theta_f,$$

where $\theta_{f,j}$ is in the $i$'th place and $\theta_{f,j}$ is in the $j$'th place; $|\lambda|$ is sufficiently small. Using (11) and the definition of $L$, we have

$$\theta_g \varphi f_0 \theta_g^{-1} = \theta_g \psi f_0 \psi^{-1} \theta_g^{-1} = L \theta_f f_0 \theta_f^{-1} L^{-1}.$$

So,

$$\theta_g \varphi f_0 \theta_g^{-1}(w_1, \ldots, w_n) = B'(\ldots, l_i(w_1, \ldots, w_n), l_j(\overline{w}_1, \ldots, \overline{w}_n), \ldots, l_j(w_1, \ldots, w_n), \ldots)$$

for some constant matrix $B'$. This map, and hence $\varphi f_0$, is not analytic, though, in a neighborhood of 0, which is a contradiction. Thus $L$, and hence $\psi$, is either analytic or antianalytic in a neighborhood of 0.

Theorem (11) is proved completely.

9. Proof of Theorem (12)

Since $\varphi$ is an epimorphism, it takes constant endomorphisms of $\Omega_1$ to constant endomorphisms of $\Omega_2$, which follows from (2). Thus we can define a map $\psi : \Omega_1 \to \Omega_2$ as in (13). Following the same steps as in verifying (11), we obtain (15)

$$\varphi f \circ \psi = \psi \circ f, \text{ for all } f \in E(\Omega_1).$$

We will show that (15) implies bijectivity of $\psi$. The map $\psi$ is surjective. Indeed, let $w \in \Omega_2$, and let $c_w$ be the corresponding constant endomorphism. Since $\varphi$ is an epimorphism, there exists $f \in E(\Omega_1)$ such that $\varphi f = c_w$. If we plug this $f$ into (15), we get

$$\psi f(z) = w$$

for all $z \in \Omega_1$. Thus $\psi$ is surjective.

To prove that $\psi$ is injective, we show that for every $w \in \Omega_2$, the full preimage of $w$ under $\psi$, $\psi^{-1}(w)$, consists of one point.

Assume for contradiction that $S_w = \psi^{-1}(w)$ consists of more than one point for some $w \in \Omega_2$. The set $S_w$ cannot be all of $\Omega_1$, since $\psi$ is surjective. For $z_0 \in \partial S_w \cap \Omega_1$ we can find $z_1 \in S_w$ and $\zeta \notin S_w$ which are arbitrarily close to $z_0$. Let $z_2$ be a fixed point of $S_w$ different from $z_1$. Consider a homothetic transformation $h$ such that $h(z_1) = z_2$, $h(z_2) = \zeta$. Since the domain $\Omega_1$ is bounded, we can choose points $z_1$ and $\zeta$ sufficiently close to each other so that $h$ belongs to $E(\Omega_1)$. Applying (15) to $h$, we obtain

$$\varphi h(w) = \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w;$$

$$\varphi h(w) = \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w.$$

The contradiction shows injectivity of $\psi$. Thus we have proved that $\psi$ is bijective.
According to (15) we have
\[ \varphi f = \psi \circ f \circ \psi^{-1}, \]
for all \( f \in E(\Omega_1) \),
which implies that \( \varphi \) is an isomorphism.

Theorem 2 is proved.

REFERENCES


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