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ON THE DUAL OF ORLICZ–LORENTZ SPACE

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ABSTRACT. A description of the Köthe dual of the Orlicz–Lorentz space $\Lambda_{\varphi,w}$ generated by an Orlicz function φ and a regular weight function w is presented. It is also shown that in the case of separable Orlicz–Lorentz spaces the regularity condition on w is necessary and sufficient for the coincidence of the Banach dual space with the described Köthe dual space.

1. Introduction

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a measure space with the complete and σ-finite measure μ, and let $L^0(\mu)$ denote the space of all μ-equivalence classes of Σ-measurable functions on Ω with the topology of convergence in measure on μ-finite sets.

A Banach space $(E, \|\cdot\|_E)$ is said to be a *Banach function* space on (Ω, μ) if it is a subspace of $L^0(\mu)$ such that there exists $h \in L^0(\mu)$ with $h > 0$ a.e. in Ω and the assumptions that $f \in L^0(\mu)$, $g \in E$ and $|f| \le |g|$ a.e. in Ω imply $f \in E$ and $\|f\|_E \le \|g\|_E$. If in addition the unit ball $B_E = \{f : \|f\|_E \le 1\}$ is closed in $L^0(\mu)$, then we say that E has the *Fatou property*. A Banach function space defined on $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with the counting measure μ is called a *Banach sequence space* (on $\mathbb{N}$).

A Banach function space E on (Ω, μ) is said to be *symmetric* if for every $f \in L^0(\mu)$ and $g \in E$ with $\mu_f = \mu_g$, we have $f \in E$ and $\|f\|_E = \|g\|_E$, where for any $h \in L^0(\mu)$, μ_h is the *distribution function* defined by

$$\mu_h(t) = \mu(\{\omega \in \Omega : |h(\omega)| > t\}),\ t \ge 0.$$

If E is a Banach function space on (Ω, μ), then the *Köthe dual* E' of E is a Banach function space, which can be identified with the space of all functionals possessing an integral representation, that is,

$$E' = \{g \in L^0(\mu) \,:\, \|g\|_{E'} = \sup_{\|f\|_E \le 1} \int_\Omega |fg| d\mu < \infty\}.$$

It is well known that if E has order continuous norm (i.e., $\|f_n\|_E \to 0$ whenever $E \ni f_n \downarrow 0$), then the dual space E^* can be naturally identified with $E^{'}$ ([8]).

In this paper we are interested in the description of the Köthe duals for symmetric *Orlicz–Lorentz* spaces defined on either nonatomic or purely atomic measure space. Since after minor modifications the proofs presented in the paper work in essentially

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the same way in both cases, for simplicity, we consider only the case of Orlicz–Lorentz spaces defined on (I, m), where either $I = (0,1)$ or $I = (0,\infty)$ and m is the Lebesgue measure.

We recall that if $\varphi : [0,\infty) \to [0,\infty)$ is an *Orlicz function* (i.e., a convex function which assumes value zero only at zero) and $w : I \to (0,\infty)$ is a *weight function* (i.e., nonincreasing and locally integrable with respect to the measure m and such that $\int_0^\infty w\,dm = \infty$ if $I = (0,\infty)$), then the *Orlicz–Lorentz function space* $\Lambda_{\varphi,w}$ on (Ω,μ) is the set of all $f \in L^0(\mu)$ such that

$$\int_\Omega \varphi(\lambda f^*) w\,dm < \infty$$

for some $\lambda > 0$, where for any $f \in L^0(\mu)$, f^* denotes the *nonincreasing rearrangement* of f defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}$$

for any $t > 0$ (by convention $\inf \emptyset = \infty$).

In the case of counting measure on $2^{\mathbb{N}}$ the Orlicz–Lorentz sequence space $\lambda_{\varphi,w}$ on $\mathbb{N}$ is defined by

$$\lambda_{\varphi,w} = \Big\{x = \{x(k)\} : \sum_{k=1}^\infty \varphi(\lambda x^*(k))\, w(k) < \infty \text{ for some } \lambda > 0\Big\}.$$

Here $w = \{w(k)\}$ is a *weight sequence*, a nonincreasing sequence of positive reals such that $\sum_{k=1}^\infty w(k) = \infty$.

It is easy to check that $\Lambda_{\varphi,w}$ (resp., $\lambda_{\varphi,w}$) is a symmetric function space (resp., symmetric sequence space) with the Fatou property, equipped with the norm

$$\|f\| = \inf\Big\{\lambda > 0 : \int_I \varphi(f^*/\lambda) w\,dm \le 1\Big\},$$

respectively

$$\|x\| = \inf\Big\{\lambda > 0 : \sum_{k=1}^\infty \varphi(x^*(k)/\lambda) w(k) \le 1\Big\}.$$

Note that if $w \equiv 1$ (resp., $w(k) = 1$ for all $k \in \mathbb{N}$), then $\Lambda_{\varphi,w}$ (resp., $\lambda_{\varphi,w}$) is the Orlicz function space L_φ (resp., Orlicz sequence space ℓ_φ). If $\varphi(t) = t$, then $\Lambda_{\varphi,w}$ (resp., $\lambda_{\varphi,w}$) is the Lorentz space Λ_w (resp., λ_w). We recall that an Orlicz function φ satisfies the Δ_2-*condition* ($\varphi \in \Delta_2$) if there exists $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t > 0$. We will further say that φ is an N-*function* whenever $\lim_{t\to 0} \varphi(t)/t = 0$ and $\lim_{t\to\infty} \varphi(t)/t = \infty$. We refer to [5] and [7] to study the basic properties of Orlicz–Lorentz spaces as well to the references included therein.

In what follows by a *regular weight* we mean a weight function w such that, if we denote $S(t) = \int_0^t w(s)\,ds$ for $t \in I$, then $S(2t) \ge KS(t)$ for any $t > 0$ $\big(t \in (0,1/2)$ in the case $I = (0,1)\big)$, where $K > 1$ is independent of t. In the sequence case a weight $w = \{w(k)\}$ is *regular* if $S(2n) \ge KS(n)$ for any $n \in \mathbb{N}$, where $S(n) = \sum_{k=1}^n w(k)$ and $K > 1$ is a constant independent of n. It is well known and easy to show that w is regular iff there exists $C > 0$ such that $tw(t) \le S(t) \le Ctw(t)$ for all $t \in I$ in the function case and analogously $nw(n) \le S(n) \le Cnw(n)$ for all $n \in \mathbb{N}$ in the sequence case.

Recall that if $\rho : I \to (0,\infty)$ is a concave function, then the *Marcinkiewicz space* M_ρ is defined by

$$M_\rho = \Big\{ f \in L^0(m) : \, \|f\|_{M_\rho} = \sup_{t\in I} \frac{\int_0^t f^*(s)\,ds}{\rho(t)} < \infty \Big\},$$

and the Marcinkiewicz space M_S with $S(t) = \int_0^t w(s)\,ds$ is the Köthe dual of Λ_w, that is,

$$(\Lambda_w)' = M_S$$

with equality of norms (see [3] or [9]). In what follows we will write $f \asymp g$ for nonnegative functions f and g whenever $C_1 f \le g \le C_2 f$ for some $C_j > 0$, $j = 1, 2$.

The problem of the description of dual spaces for Lorentz sequence spaces has been considered in [1] and [2]. It was proved there that the regularity of w is a necessary and sufficient condition, in the case $p = 1$, and a sufficient condition in the case $p > 1$, in order that the Köthe dual of classical Lorentz sequence space $d(w,p) = \lambda_{\varphi,w}$ with $\varphi(t) = t^p$ for $t \ge 0$ to consist exactly of those sequences $\{x(k)\}$ for which $\{x^*(k)/w(k)^{1/p}\} \in \ell_{p'}$, where $1/p + 1/p' = 1$.

In [1] Allen also raises the question of whether the similar description of the Köthe dual space is also true for Lorentz function spaces. Reisner [12] answered this question positively and gave a proof which works in essentially the same way in the sequence and in the function spaces. He also proved that regularity of w is necessary for $p > 1$.

Following the ideas from [12] we prove in this paper, under the assumption that φ is an N-function satisfying the Δ_2-condition, that the regularity of the weight function w is a necessary and sufficient condition for the dual of the Orlicz–Lorentz space $\Lambda_{\varphi,w}$ on (I,m) to consist exactly of those functions f for which f^*/w belongs to the Orlicz function space L_{φ_*} on $(I, w\,dm)$, where φ_* is the *Young conjugate* of φ, i.e.,

$$\varphi_*(t) = \sup\{st - \varphi(s) : \, s \ge 0\}$$

for $t \ge 0$. We also obtain some partial results for φ being an arbitrary Orlicz function. A similar result is also true for the symmetric Orlicz–Lorentz sequence space $\lambda_{\varphi,w}$ on $\mathbb{N}$.

In the proof of the main result we will use the Lozanovskiĭ theorem on the representation of the Köthe dual space for the Calderón-Lozanovskiĭ space (see [11]). Let us recall that if (E_0, E_1) is any couple of Banach function spaces on (Ω,μ) and $\mathcal{U}$ denotes the set of all concave and positively homogeneous functions $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\psi(s,t) = 0$ if and only if $s = t = 0$, then the Calderón–Lozanovskiĭ space $\psi(E_0,E_1)$ generated by the couple (E_0,E_1) and a function $\psi \in \mathcal{U}$ consists of all $f \in L^0(\mu)$ such that $|f| \le \lambda\psi(|f_0|,|f_1|)$ a.e. for some $\lambda > 0$ and $f_j \in E_j$, $j = 0, 1$. The space $\psi(E_0,E_1)$ is a Banach function space on (Ω,μ) (cf. [10], [11]) equipped with the following equivalent norms:

$$\|f\|_\psi = \inf\{\lambda > 0 : \, |f| \le \lambda\psi(|f_0|,|f_1|) \text{ a.e.}, \; \|f_j\|_{E_j} \le 1, \, j = 0, 1\}$$

and

$$\|f\|_\psi^1 = \inf\{\|f_0\|_{E_0} + \|f_1\|_{E_1} : \, |f| = \psi(|f_0|,|f_1|)\},$$

satisfying the inequalities

$$\|f\|_\psi \le \|f\|_\psi^1 \le 2\|f\|_\psi$$

for all $f \in \psi(E_0, E_1)$. In the case of the power function $\psi_\theta(s,t) = s^{1-\theta}t^\theta$ with $0 < \theta < 1$, $\psi_\theta(E_0, E_1)$ is the well-known Calderón space $E_0^{1-\theta}E_1^\theta$ (see [4]).

For any $\psi \in \mathcal{U}$ the *involution* $\widehat{\psi}$ of ψ is defined on $\mathbb{R}_+^2$ by

$$\widehat{\psi}(s,t) = \inf\Big\{\frac{\alpha s + \beta t}{\psi(\alpha,\beta)} \;:\; \alpha,\, \beta > 0\Big\}.$$

Lozanovskiĭ [11] proved the following theorem (see also [10], [13]).

Theorem 1. a) *If E_0, E_1 are two Banach function spaces on the same measure space, then for all $\psi \in \mathcal{U}$ we have*

$$\big(\psi(E_0, E_1), \|\cdot\|_\psi\big)' = \big(\widehat{\psi}(E_0', E_1'), \|\cdot\|_{\frac{1}{\widehat{\psi}}}\big)$$

with equality of the norms.

b) *For every $0 \le f \in L^1(\mu)$ and $\varepsilon > 0$, there exist $0 \le g \in E$, $0 \le h \in E'$ such that $f = gh$ and*

$$\|g\|_E \|h\|_{E'} \le (1+\varepsilon)\|f\|_{L^1}.$$

If E has the Fatou property, we may take $\varepsilon = 0$ in the above inequality.

2. Results

The main aim of this section is to prove a representation theorem for the Köthe dual of Orlicz–Lorentz spaces. The proof for sequence spaces is analogous and more simple than in the function case. Thus we limit ourselves to a proof for function spaces only. We begin with the lemma which will be useful in the sequel.

Lemma 1. *Let $\psi \in \mathcal{U}$ and let $f, g \in L^0(\mu)$ be such that $\mu_f(t) < \infty$ and $\mu_g(t) < \infty$ for every $t > 0$. If $h \in L^0(\mu)$ is such that $|h| \le \psi(|f|, |g|)$ a.e., then for all $t > 0$*

$$h^*(t) \le 2\psi\big(f^*(t/2), g^*(t/2)\big).$$

Proof. Since each $\psi \in \mathcal{U}$ is nondecreasing in each variable, we have $\psi(s,t) \le \max\big\{s/\alpha, t/\beta\big\}\psi(\alpha,\beta)$, and in consequence

$$\psi(s,t) \le \Big(\frac{s}{\alpha} + \frac{t}{\beta}\Big)\psi(\alpha,\beta)$$

for all $s, t \ge 0$ and $\alpha, \beta > 0$. This implies that

$$\psi(s,t) \le \tilde{\psi}(s,t) \le 2\psi(s,t)$$

holds for all $s, t \ge 0$, where

$$\tilde{\psi}(s,t) := \inf_{\alpha,\beta>0}\big(s/\alpha + t/\beta\big)\psi(\alpha,\beta)$$

for all $s, t \ge 0$. Now fix $\alpha, \beta > 0$. Then applying the above inequalities, we conclude that for all α, $\beta > 0$

$$|h| \le \Big(\frac{|f|}{\alpha} + \frac{|g|}{\beta}\Big)\psi(\alpha,\beta) \text{ a.e.}$$

and in view of the inequality $(f+g)^*(t) \le f^*(t/2) + g^*(t/2)$,

$$h^*(t) \le \Big(\frac{f^*(t/2)}{\alpha} + \frac{g^*(t/2)}{\beta}\Big)\psi(\alpha,\beta)$$

for all $t > 0$. Hence

$$h^*(t) \le \tilde{\psi}(f^*(t/2), g^*(t/2)) \le 2\psi(f^*(t/2), g^*(t/2))$$

for all $t > 0$, which completes the proof. $\square$

In what follows, given an Orlicz function φ, we define $I(f) = \int_I \varphi_*(f^*/w)w\,dm$ for $f \in L^0(m)$ and

$$M_{\varphi_*,w} = \{f \in L^0(m) : I(f/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

In the space $M_{\varphi_*,w}$ we define an order monotone and homogenous functional

$$\|f\|_{M_{\varphi_*,w}} = \inf\{\lambda > 0 : I(f/\lambda) \le 1\}.$$

One can show that if w is regular, then the functional $\|\cdot\|_{M_{\varphi_*,w}}$ is a quasinorm. We also observe that if $\varphi(t) = t$, then

$$M_{\varphi_*,w} = \left\{f \in L^0(m) : \|f\|_{M_{\varphi_*,w}} = \sup_{t\in I} \frac{f^*(t)}{w(t)} < \infty\right\}$$

and $M_{\varphi_*,w} = M_S$ with $\|\cdot\|_{M_{\varphi_*,w}} \asymp \|\cdot\|_{M_S}$, by $tw(t) \asymp S(t)$, whenever w is regular. In a similar way we define the sequence space $m_{\varphi_*,w}$ for a sequence weight w.

Before we prove the main theorem we will need the following lemma which, in the case of $\varphi(t) = t^p$ with $1 \le p < \infty$, has been proved in [12] as Lemma 2 with means of much more involved arguments.

Lemma 2. *Let an Orlicz function φ and a weight funtion w be such that $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$. Then there is $K > 0$ so that*

$$\|g\|_{M_{\varphi_*,w}} \le K\|g\|_{(\Lambda_{\varphi,w})'} \quad \textit{for all } g \in (\Lambda_{\varphi,w})'.$$

Proof. For a contradiction assume that there exists a sequence $\{g_k\}$ with

$$\|g_k\|_{(\Lambda_{\varphi,w})'} = 1 \text{ and } \|g_k\|_{M_{\varphi_*,w}} > k2^k \text{ for all } k \in \mathbb{N}.$$

Obviously for the function g defined by

$$g = \sum_{k=1}^{\infty} \frac{1}{2^k}\,|g_k|,$$

we have $g \in (\Lambda_{\varphi,w})' = M_{\varphi_*,w}$. Since $g \ge |g_k|/2^k$, we have $g^*(t) \ge g_k^*(t)/2^k$ for all $k \in \mathbb{N}$ and $t \in I$. Thus from the order monotonicity and homogeneity of the functional $\|\cdot\|_{M_{\varphi_*,w}}$, we get

$$\|g\|_{M_{\varphi_*,w}} \ge k$$

for all $k \in \mathbb{N}$. This contradiction finishes the proof of the lemma. $\square$

Now we are ready to prove the main result of the paper.

Theorem 2. *Let w be a weight function and let either $\varphi(t) = t$ or φ be an N-function. Then the following holds true:*

(i) *If w is a regular weight, then $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$ and $\|\cdot\|_{(\Lambda_{\varphi,w})'} \asymp \|\cdot\|_{M_{\varphi_*,w}}$.*

(ii) *If φ satisfies the Δ_2-condition and $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$, then w is regular.*

Proof. At first we notice that if E is any Banach function space on a measure space (Ω, μ), $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an Orlicz function and $\psi \in \mathcal{U}$ is defined by $\psi(s,t) = 0$ if $t = 0$ and $\psi(s,t) = t\varphi^{-1}(s/t)$ if $t > 0$, then

$$\psi(E, L^\infty) = \{f \in L^0 : \varphi \circ (\lambda f) \in E \text{ for some } \lambda > 0\}$$

and

$$\|f\|_\psi = \inf\{\lambda > 0 : \|\varphi \circ (f/\lambda)\|_E \leq 1\}.$$

In particular, we obtain $\Lambda_{\varphi,w} = \psi(\Lambda_w, L^\infty)$. Hence by Theorem 1, we get

$$(\Lambda_{\varphi,w})' = \widehat{\psi}\big((\Lambda_w)', (L^\infty)'\big)' = \widehat{\psi}(M_S, L^1),$$

with equality of the norms, where $\widehat{\psi}(M_S, L^1)$ is considered with the norm $\|\cdot\|^1_{\widehat{\psi}}$. It is also easy to check that $\widehat{\psi}(s,t) = 0$ for $s = 0$ and $\widehat{\psi}(s,t) = s\varphi_*^{-1}(t/s)$ for $s > 0$. Notice by the assumption that φ is an N-function, φ_* is an Orlicz function and thus $\widehat{\psi} \in \mathcal{U}$.

Now we are ready to prove (i). If $\varphi(t) = t$, then (i) is obvious by the observation stated before Lemma 2. Assume now that φ is an N-function and let $f \in \widehat{\psi}(M_S, L^1)$ with $\|f\|^1_{\widehat{\psi}} \leq 1$. Since $\|f\|_{\widehat{\psi}} \leq \|f\|^1_{\widehat{\psi}}$, we get $\|f\|_{\widehat{\psi}} \leq 1$. This yields

$$|f| \leq \widehat{\psi}(|f_0|, |f_1|) \text{ a.e.}$$

for some $f_0 \in M_S$ and $f_1 \in L^1$ with $\|f_0\|_{M_S} \leq 1$ and $\|f_1\|_{L^1} \leq 1$. In consequence, it follows by Lemma 1 that

$$f^*(t) \leq 2\widehat{\psi}(f_0^*(t/2), f_1^*(t/2))$$

for all $t \in I$. By regularity of w for any $t \in I$,

$$f_0^*(t/2)/Cw(t) \leq tf_0^*(t/2)/S(t) \leq \|f_0^*(\cdot/2)\|_{M_S} \leq 2\|f_0\|_{M_S} \leq 2$$

and hence for $t \in I$,

$$f_0^*(t/2) \leq 2Cw(t).$$

Combining the above inequalities, we obtain

$$f^*(t) \leq 2\widehat{\psi}(2Cw(t), f_1^*(t/2))$$

for all $t \in I$. Therefore

$$f^*(t) \leq 4Cw(t)\varphi_*^{-1}\big(f_1^*(t/2)/2Cw(t)\big)$$

or equivalently

$$\varphi_*\Big(\frac{f^*(t)}{4Cw(t)}\Big)w(t) \leq \frac{1}{2C}f_1^*(t/2)$$

for any $t \in I$. Since $\|f_1\|_{L^1} \leq 1$, $\|f_1^*(\cdot/2)\|_{L^1} \leq 2$, we obtain that $f^* \in M_{\varphi_*,w}$ and $\|f\|_{M_{\varphi_*,w}} \leq 4C$. In consequence we proved that for any $f \in (\Lambda_{\varphi,w})'$, we have

$$\|f\|_{M_{\varphi_*,w}} \leq 4C\|f\|_{(\Lambda_{\varphi,w})'}.$$

Now suppose that $f \in M_{\varphi_*,w}$ and $\|f\|_{M_{\varphi_*,w}} \leq 1$. This implies that

$$\int_I \varphi_*\Big(\frac{f^*(t)}{w(t)}\Big)w(t)\,dt \leq 1.$$

Taking $f_0 = w$ and $f_1 = \varphi_*(f^*/w)w$, we have $f_0 \in M_S$, $f_1 \in L^1$, $\|f_0\|_{M_S} = 1$ and $\|f_1\|_{L^1} \le 1$. Since

$$f^*(t) = \widehat{\psi}(f_0(t), f_1(t))$$

for all $t \in I$, we get $f^* \in \widehat{\psi}(M_S, L^1)$ and $\|f^*\|_{(\Lambda_{\varphi,w})'} = \|f^*\|^1_{\widehat{\psi}} \le 2$. In consequence $f^* \in (\Lambda_{\varphi,w})'$, and since the Köthe dual of a symmetric space is also a symmetric space (see [9]), we get $f \in (\Lambda_{\varphi,w})'$. Combining the above inequalities, we obtain

$$2^{-1}\|f\|_{(\Lambda_{\varphi,w})'} \le \|f\|_{M_{\varphi_*,w}} \le 4C\|f\|_{(\Lambda_{\varphi,w})'}$$

for every $f \in (\Lambda_{\varphi,w})'$, where C is the regularity constant of w.

(ii). Fix $x \in I$ and let $f = \chi_{(0,x)}/x$. Since $\Lambda_{\varphi,w}$ has the Fatou property, it follows by Theorem 1 b) that there exist functions $g \in E := \Lambda_{\varphi,w}$, $h \in E'$ such that $f = gh$ and $\|g\|_E = \|h\|_{E'} = 1$. Thus, we have $\|f/h\|_E = \|h\|_{E'} = 1$. Without loss of generality we can assume that $h = h^*$ and $\operatorname{supp} h = (0, x)$ (cf. [6, pp. 38-41]). In consequence

$$\begin{aligned}\int_0^x \varphi\Big(\frac{1}{xh(t)}\Big) w(x-t)\,dt &= \int_0^x \varphi\Big(\frac{1}{xh(x-t)}\Big) w(t)\,dt \\ &= \int_I \varphi\Big(\Big(\frac{\chi_{(0,x)}}{xh}\Big)^*(t)\Big) w(t)\,dt \le 1.\end{aligned}$$

Moreover, since $\|h\|_{E'} = 1$, we have $\|h\|_{M_{\varphi_*,w}} \le K$ by Lemma 2, and so

$$\int_I \varphi_*\Big(\frac{h(t)}{Kw(t)}\Big) w(t)\,dt \le 1.$$

From $\varphi \in \Delta_2$, there exists $C \ge 2$ such that $\varphi(2t) \le C\varphi(t)$ for all $t > 0$. This implies that $\varphi(\lambda t) \le 2^p \lambda^p \varphi(t)$ for $p = \ln C/\ln 2 \ge 1$ and all $\lambda \ge 1$, $t \ge 0$. Since w is nonincreasing, we also have

$$\frac{w(x-t)}{w(t)} \ge 1 \quad \text{for any} \quad t \in [x/2, x].$$

Combining the above inequalities with the definition of the conjugate function φ_*, we obtain

$$\begin{aligned}\frac{1}{xK}\int_{x/2}^x \Big(\frac{w(x-t)}{w(t)}\Big)^{1/p} dt &\le \int_{x/2}^x \varphi_*\Big(\frac{h(t)}{Kw(t)}\Big) w(t)\,dt \\ &+ \int_{x/2}^x \varphi\Big(\frac{1}{xh(t)}\Big(\frac{w(x-t)}{w(t)}\Big)^{1/p}\Big) w(t)\,dt \\ &\le 1 + 2^p \int_{x/2}^x \varphi\Big(\frac{1}{xh(t)}\Big) w(x-t)\,dt \le 1 + 2^p.\end{aligned}$$

Clearly

$$\frac{1}{x}\int_0^{x/2} \Big(\frac{w(x-t)}{w(t)}\Big)^{1/p} dt \le \frac{1}{2}.$$

Hence for $K_1 = (1 + 2^p)K + 1$,

$$\sup_{x \in I} \frac{1}{x}\int_0^x \Big(\frac{w(x-t)}{w(t)}\Big)^{1/p} dt \le K_1.$$

This is equivalent to regularity of $w^{1/p}$.

In the case when $\varphi(t) = t$, $p = 1$ and obviously w is regular.

Now we assume that $\varphi \in \Delta_2$ is an N-function and we shall show that w is regular. It is easy to check that the regularity of w^a for some $0 < a < \infty$ implies (cf. [12], Lemma 3)

$$w(t) \asymp w(2t) \quad \text{for all } t \in I \text{ with } 2t \in I.$$

Now let $h \in M_S$ with $\|h\|_{M_S} \le 1$. For any $f \in L^1$ with $\|f\|_{L^1} \le 1$, we have

$$\widehat{\psi}(|h|, |f|) \in (\Lambda_{\varphi,w})' = \widehat{\psi}(M_S, L^1)$$

with $\|\widehat{\psi}(|h|, |f|)\|_{(\Lambda_{\varphi,w})'} \le 2$. Now if $(\Lambda_{\varphi,w})' = M_{\varphi_*,w}$, it follows by Lemma 2 that there is $K > 0$ so that

$$\sup_{\|f\|_{L^1} \le 1} \|\widehat{\psi}(|h|, |f|)\|_{M_{\varphi_*,w}} \le 2K.$$

Taking $f = (1/x)\chi_{(0,x)}$ with $x \in I$, we get by the homogeneity of $\widehat{\psi}$

$$\int_0^x \varphi_*\Big(\frac{\widehat{\psi}(xh^*(t), 1)}{2Kxw(t)}\Big) w(t)\, dt \le \int_I \varphi_*\Big(\frac{\widehat{\psi}(|h|, |f|)^*(t)}{2Kw(t)}\Big) w(t)\, dt \le 1.$$

Combining these with $w(t) \le cw(2t)$ (by $w(t) \asymp w(2t)$) gives that for all $x \in I$,

$$\frac{x}{2}\varphi_*\Big(\frac{\widehat{\psi}(xh^*(x), 1)}{2cKxw(x)}\Big) w(x) \le \int_{x/2}^x \varphi_*\Big(\frac{\widehat{\psi}(xh^*(t), 1)}{2Kxw(t)}\Big) w(t)\, dt \le 1.$$

Hence for all $x \in I$,

$$\begin{aligned} xh^*(x)\varphi_*^{-1}\Big(\frac{1}{xh^*(x)}\Big) &= \widehat{\psi}(xh^*(x), 1) \\ &\le 2cKxw(x)\varphi_*^{-1}\Big(\frac{2}{xw(x)}\Big). \end{aligned}$$

Now since for any N-function it holds $u \le \varphi^{-1}(u)\varphi_*^{-1}(u) \le 2u$ for all $u \ge 0$, we get

$$\varphi^{-1}\Big(\frac{2}{xw(x)}\Big) \le 8cK\varphi^{-1}\Big(\frac{1}{xh^*(x)}\Big) \quad \text{for all } x \in I,$$

which in view of the Δ_2-condition implies that

$$\varphi^{-1}\Big(\frac{2}{xw(x)}\Big) \le \varphi^{-1}\Big(\frac{C}{xh^*(x)}\Big)$$

for all $x \in I$ and a constant $C > 0$, and finally it yields that

$$\sup_{x \in I} \frac{h^*(x)}{w(x)} < \infty.$$

In consequence we proved that

$$(\Lambda_w)' = M_S = \Big\{h \in L^0(m) : \frac{h^*}{w} \in L^\infty\Big\}.$$

Thus, by the above proof for $\varphi(t) = t$, it follows that w is regular. □

If the Orlicz function φ is neither linear nor an N-function, we have the following description of the Köthe dual spaces of Orlicz–Lorentz spaces.

Theorem 3. *Let φ be an Orlicz function and let w be a regular weight. Then the following holds true:*

(i) *If $\lim_{t\to 0} \varphi(t)/t > 0$ and $\lim_{t\to 0} \varphi(t)/t < \infty$, then $\varphi(t) \asymp t$ and*

$$(\Lambda_{\varphi,w})' = M_S.$$

(ii) *If* $\lim_{t\to 0} \varphi(t)/t > 0$ *and* $\lim_{t\to\infty} \varphi(t)/t = \infty$*, then there exists an* N*-function* ϕ *such that* $\phi(t) \asymp t^2$ *for* t *small enough and* $\phi(t) \asymp \varphi(t)$ *for* t *large enough, and*

$$(\Lambda_{\varphi,w})' = M_S + M_{\phi_*,w}.$$

(iii) *If* $\lim_{t\to 0} \varphi(t)/t = 0$ *and* $\lim_{t\to\infty} \varphi(t)/t < \infty$*, then there exists an* N*-function* ϕ *such that* $\phi(t) \asymp \varphi(t)$ *for* t *small enough, and* $\phi(t) \asymp t$ *for* t *large enough, and*

$$(\Lambda_{\varphi,w})' = M_S \cap M_{\phi_*,w}.$$

Proof. We will use two particular cases of the Lozanovskiĭ duality result from which it follows that for any couple (E_0, E_1) of Banach function spaces defined on the same measure space, we have

$$(E_0 \cap E_1)' = E_0' + E_1' \quad \text{and} \quad (E_0 + E_1)' = E_0' \cap E_1',$$

where $E_0 \cap E_1$ and $E_0 + E_1$ are equipped with the natural interpolation norms (cf. [3] or [9]).

(i). It is obvious since in this case $\Lambda_{\varphi,w} = \Lambda_w$.

(ii). We may assume without loss of generality that $\varphi(1) = 1$. Then it is easy to check that

$$\varphi(t) \asymp \max\{t, \phi(t)\},$$

where $\phi(t) = t^2$ for $t \le 1$ and $\phi(t) = 2\varphi(t) - 1$ for $t > 1$. Clearly

$$\Lambda_{\varphi,w} = \Lambda_w \cap \Lambda_{\phi,w}.$$

Thus the required result follows by Theorem 2 and the Lozanovskiĭ result mentioned above.

(iii). We define $\varphi_1(t) := \varphi(t)$ if $t \le 1$ and $\varphi_1(t) := \varphi_+'(1)t^2 + (1 - \varphi_+'(1))$ for $t > 1$ and

$$\phi(t) := \int_0^t \min\{s, \varphi_1(s)\}\, \frac{ds}{s}.$$

Again it is easy to check that ϕ satisfies the required conditions. Since $\varphi(t) \asymp \min\{t, \phi(t)\}$, we immediately obtain

$$\Lambda_{\varphi,w} = \Lambda_w + \Lambda_{\phi,w}.$$

Thus again Theorem 2 and the Lozanovskiĭ result apply. □

By [7] (cf. also [5]), we have that $\Lambda_{\varphi,w}$ is separable if and only if φ satisfies the corresponding Δ_2-condition. Thus if the weight function w is regular, then by Theorems 2 and 3 we obtain a description of the Banach dual space of the Orlicz–Lorentz space. In fact the following corollary of Theorem 2 holds true.

Theorem 4. *Let either* $\varphi(t) = t$ *or* φ *be an* N*-function satisfying the* Δ_2*-condition. Then the regularity of the weight function* w *is a necessary and sufficient condition for the coincidence of the Banach dual space of the Orlicz–Lorentz space* $\Lambda_{\varphi,w}$ *with the space* $M_{\varphi_*,w}$.

Analogous results characterizing the Köthe dual or dual spaces may also be stated for Orlicz–Lorentz sequence spaces.

Finally we notice that the description of the dual of the Lorentz space $\Lambda_{p,w}$ given by Reisner [12] is a particular case of the above theorem for $\varphi(t) = t^p$ for $t \ge 0$, $1 \le p < \infty$.

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