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EXPLICIT CONTINUED FRACTIONS WITH EXPECTED PARTIAL QUOTIENT GROWTH

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ABSTRACT. For $0 < x < 1$ let $[0, a_1(x), a_2(x), \dots]$ be the continued fraction expansion of x. Write

$$L_N(x) = \max_{1 \le n \le N} a_n(x).$$

We construct some numbers x's with

$$\lim_{N \to \infty} \inf N^{-1} L_N(x) \log\log N = 1/\log 2.$$

1. Introduction

For $0 < x < 1$ let $[0, a_1(x), a_2(x), \dots]$ be the continued fraction expansion of x. Write

$$L_N(x) = \max_{1 \le n \le N} a_n(x).$$

W. Philipp [1] proved the following theorem: *For almost all* x

$$\lim_{N \to \infty} \inf N^{-1} L_N(x) \log\log N = 1/\log 2.$$

Previously no explicit numbers were known that satisfied

$$\lim_{N \to \infty} \inf N^{-1} L_N(x) \log\log N = 1/\log 2.$$

In this paper we give a method that may be used to construct examples of such numbers.

2. Main result

Theorem. *Let k, K and M_0 be arbitrary fixed positive numbers with $k > 1$ and $K \ge 2$. Let $x = [0, a_1, a_2, \dots]$ be an irrational number such that*

$$\max\{a_n \mid (\log k)^2 M K^{k^M} < n \le (\log k)^2 (M+1) K^{k^M}\} = [K^{k^{M+1}}]$$

and

$$\max\{a_n \mid (\log k)^2 (M+1) K^{k^M} < n \le (\log k)^2 (M+1) K^{k^{M+1}}\} \le [K^{k^{M+1}}],$$

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where M is a positive integer with $M > M_0$ and $[K^{k^{M+1}}]$ is the integral part of $K^{k^{M+1}}$. Then

$$\lim_{N\to\infty} \inf N^{-1} L_N(x) \log\log N = 1/\log k.$$

Proof. Put $N_0 = [(\log k)^2 M_0 K^{k^{M_0}}]$ and choose $M' > M_0$ such that $L_{N_0}(x) \leq [K^{k^{M'+1}}]$. Let M be any integer greater than M'. We now consider two cases.

Case 1. $(\log k)^2 M K^{k^M} < N \leq (\log k)^2 (M+1) K^{k^M}$.

Suppose that M is a sufficiently large integer. In this case, we find that $L_N(x) \geq [K^{k^M}]$ and

$$\log\log N = M \log k + O(1) \qquad (M \to \infty).$$

Hence

$$N^{-1} L_N(x) \log\log N \geq \frac{[K^{k^M}](M\log k + O(1))}{(\log k)^2 (M+1) K^{k^M}} \qquad (M \to \infty).$$

Case 2. $(\log k)^2 (M+1) K^{k^M} < N \leq (\log k)^2 (M+1) K^{k^{M+1}}$.

In this case, we find that $L_N(x) = [K^{k^{M+1}}]$ and

$$\log\log N = M \log k + O(1) \qquad (M \to \infty).$$

Hence

$$\begin{aligned} &\inf\{N^{-1} L_N(x) \log\log N \mid (\log k)^2 (M+1) K^{k^M} < N \leq (\log k)^2 (M+1) K^{k^{M+1}}\} \\ &\quad = \frac{[K^{k^{M+1}}](M\log k + O(1))}{(\log k)^2 (M+1) K^{k^{M+1}}} \qquad (M \to \infty). \end{aligned}$$

Then by these two cases, we can deduce that

$$\lim_{N\to\infty} \inf N^{-1} L_N(x) \log\log N = 1/\log k.$$

This completes the proof.

3. Example

Let $x = [0, a_1, a_2, \ldots]$ be an irrational number. Let K and M be as in the Theorem, and let $k = 2$ in the Theorem. Define

$$\mu(M) := [(\log 2)^2 M K^{2^M}] + 1.$$

By the Theorem, we define a_n as follows:

$$a_n := \begin{cases} [K^{2^{M+1}}] & \text{if } n = \mu(M) \text{ for } M \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Now putting $K = 2$, we find that $\mu(1) = 2, \mu(2) = 16, \mu(3) = 369, \mu(4) = 125948, \ldots$. Hence, we can construct x as follows:

$$\begin{aligned} x &= [0, a_1, a_2, a_3, \ldots, a_{15}, a_{16}, a_{17}, \ldots, a_{368}, a_{369}, a_{370}, \ldots, a_{125947}, a_{125948}, a_{125949}, \ldots] \\ &= [0, 1, 2^{2^2}, 1, \ldots, 1, 2^{2^3}, 1, \ldots, 1, 2^{2^4}, 1, \ldots, 1, 2^{2^5}, 1, \ldots]. \end{aligned}$$

We can easily see that x is an example of the numbers defined in the Theorem.

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References

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