

LINEARITY OF DIMENSION FUNCTIONS FOR SEMILINEAR G -SPHERES

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Dedicated to the memory of Professor Katsuo Kawakubo

ABSTRACT. In this paper, we show that the dimension function of every semilinear G -sphere is equal to that of a linear G -sphere for finite nilpotent groups G of order $p^n q^m$, where p, q are primes. We also show that there exists a semilinear G -sphere whose dimension function is not virtually linear for an arbitrary nonsolvable compact Lie group G .

1. INTRODUCTION

Let G be a compact Lie group and V an orthogonal G -representation. The unit sphere $S(V)$ of V is a closed G -manifold whose H -fixed point set is a sphere (or possibly the empty set) for every closed subgroup H of G . We call this G -manifold a linear G -sphere. A *semilinear G -sphere* is defined as a natural generalization of a linear G -sphere, namely it is a (smooth) closed G -manifold such that the H -fixed point set is a homotopy sphere or the empty set for every closed subgroup H of G .

Many authors studied linear G -spheres, in particular K. Kawakubo [9] studied the G -homotopy classification of linear G -spheres for compact abelian groups, and T. tom Dieck [3] studied these spheres for finite p -groups.

T. tom Dieck and T. Petrie [6] introduced the notion of G -homotopy representations and developed the (stable) homotopy theory of G -homotopy representations. E. Laitinen [10] developed the unstable theory of G -homotopy representations. The G -homotopy representation X is defined as a G -CW complex satisfying the following properties (1)–(3):

- (1) For every closed subgroup H , the H -fixed point set X^H is homotopy equivalent to a sphere $S^{\underline{n}(H)-1}$, where we define the (-1) -dimensional sphere S^{-1} as the empty set.
- (2) $\underline{n}(H) - 1 = \dim X^H$ for every closed subgroup H . (We set $\dim \emptyset = -1$.)
- (3) X has finitely many orbit types.

Note that a semilinear G -sphere is an example of a G -homotopy representation since any (smooth) G -manifold has a G -CW complex structure. We say that a

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G -homotopy representation X is linear if X is G -homotopy equivalent to a linear G -sphere.

There exist many nonlinear G -homotopy representations for groups G . (See for example [16], [17].)

For any G -homotopy representation X , we define the integer-valued function $\text{Dim } X$ on the set of closed subgroups by setting $\text{Dim } X(H) = \dim X^H + 1$ for $H \leq G$. This function is called the *dimension function* of X . Clearly the dimension function is a G -homotopy invariant, and the additivity of dimension functions holds, i.e., $\text{Dim } X * Y = \text{Dim } X + \text{Dim } Y$ for G -homotopy representations, where $*$ means join. (Note that $X * Y$ is also a G -homotopy representation up to G -homotopy.) Dimension functions of G -homotopy representations are studied by [4] and [2] for finite groups and by [1] for compact Lie groups. We say that the dimension function \underline{n} is *linear* if $\underline{n} = \text{Dim } S(V)$ for some G -representation V , and that \underline{n} is *virtually linear* (or stably linear) if $\underline{n} = \text{Dim } S(V) - \text{Dim } S(W)$ for some V and W .

The purpose of this paper is to study the dimension functions of semilinear G -spheres from the viewpoint of linearity.

For certain groups G , it is known that a semilinear G -sphere with nonlinear dimension function is constructed by the equivariant surgery. (See Remark 1.3 below.) On the other hand, the following result is known for p -groups.

Theorem 1.1 ([7], [4]). *For any p -group P , every P -homotopy representation has a linear dimension function.*

Our first result is the following.

Theorem A. *Let G be a finite nilpotent group of order $p^n q^m$, where p, q are primes. Then every semilinear G -sphere has a linear dimension function.*

Remark 1.2. Theorem A does not hold for G -homotopy representations unless G is a p -group.

Remark 1.3. Further generalization of Theorem A cannot be expected, because the following results are known.

- (1) There exists a semilinear C_{pqr} -sphere with nonlinear dimension function for cyclic groups C_{pqr} of order pqr , where p, q, r are distinct odd primes ([14], [11]).
- (2) There exists a semilinear $Z_{p,q}$ -sphere with nonlinear dimension function for metacyclic groups $Z_{p,q}$ of order pq , where p, q are distinct odd primes and $q|(p-1)$ ([13]).
- (3) There exists a semilinear S^1 -sphere with nonlinear dimension function for the circle group S^1 ([15]).

We also show the following result on nonlinearity of dimension functions in Section 4.

Theorem B. *Let G be an arbitrary nonsolvable compact Lie group. Then there exists a semilinear G -sphere whose dimension function is not virtually linear.*

2. THE CASE OF C_{pq}

We first note the following fact.

Lemma 2.1. *Let G be a compact Lie group. If a semilinear G -sphere Σ has a G -fixed point, then Σ is linear, i.e., G -homotopy equivalent to a linear G -sphere, and in particular its dimension function is linear.*

Proof. Let x be a G -fixed point of Σ . Take a (small) G -invariant neighborhood U of x which is G -diffeomorphic to $\text{Int } D(V)$, where V is the tangential representation at x . Then the projection $\Sigma \rightarrow \Sigma/(\Sigma - U) \cong S(V \oplus \mathbb{R})$ is a G -homotopy equivalence by the equivariant Whitehead theorem. \square

In this section, we show the following special case of Theorem A.

Proposition 2.2. *Let G be a cyclic group C_{pq} of order pq , where p, q are primes. Then every semilinear G -sphere has a linear dimension function.*

Proof. In case $p = q$, by Theorem 1.1, every semilinear G -sphere has a linear dimension function. We may assume $p > q$. By Lemma 2.1, we may also assume that the action is G -fixed point free. Let Σ be a semilinear C_{pq} -sphere without a C_{pq} -fixed point, and let \underline{n} be the dimension function of Σ .

We first show the inequality $\underline{n}(C_p) + \underline{n}(C_q) \leq \underline{n}(1)$. Put $M := \Sigma^{C_q}$, which is a homotopy sphere with free $C_p(\cong C_{pq}/C_q)$ -action. Put $N := \Sigma - N(\Sigma^{C_p})$, where $N(\Sigma^{C_p})$ is an equivariant open tubular neighborhood of Σ^{C_p} . Then N is considered as a (compact) C_p -manifold with free C_p -action and its homology group $H_*(N; \mathbb{Z})$ is isomorphic to $H_*(S^{\underline{n}(1) - \underline{n}(C_q) - 1}; \mathbb{Z})$ by the Alexander duality. Clearly the natural inclusion $i : M \rightarrow N$ is a C_p -map. If necessary, by taking the join of $S(W)$ and N , $N * S(W)$ is simply connected and $\dim N * S(W) \geq 3$, where W is a free C_p -representation of dimension $m > 2$. Then $N * S(W)$ is a C_p -CW complex which is (nonequivariantly) homotopy equivalent to $S^{\underline{n}(C_q) + m - 1}$. By Proposition 5.9 of [6], one can see that $N * S(W)$ is C_p -homotopy equivalent to a free C_p -homotopy representation X of dimension $\underline{n}(1) - \underline{n}(C_q) + m - 1$, and $M * S(W)$ is also a free C_p -homotopy representation of dimension $\underline{n}(C_p) + m - 1$. Let $f : N * S(W) \rightarrow X$ be a C_p -homotopy equivalence. Then the composite $f \circ (i * id) : M * S(W) \rightarrow N * S(W) \rightarrow X$ is a C_p -map between free C_p -homotopy representations. By the Borsuk-Ulam theorem for free C_p -actions (cf. [10, Proposition 1.14]), we obtain $\underline{n}(C_p) + m - 1 \leq \underline{n}(1) - \underline{n}(C_q) + m - 1$, and hence $\underline{n}(C_p) + \underline{n}(C_q) \leq \underline{n}(1)$.

Next, since $C_{pq/r} = C_{pq}/C_r$ ($r = p$ or q) acts freely on the homotopy sphere Σ^{C_r} , $\underline{n}(C_r) = \dim \Sigma^{C_r} + 1$ is even unless $pq/r = 2$. Then there exists a free $C_{pq/r}$ -representation V_r such that $\dim V_r = \underline{n}(C_r)$. Via the projection $C_{pq} \rightarrow C_{pq/r}$, we regard V_r as a C_{pq} -representation with kernel C_r . Put $k := \underline{n}(1) - \underline{n}(C_p) - \underline{n}(C_q) \geq 0$. Then k is even. In fact, since $C_p(\leq C_{pq})$ (p : odd prime by the assumption $p > q$) acts semi-freely on Σ , it follows that $\underline{n}(1) - \underline{n}(C_p)$ is even, and $\underline{n}(C_q)$ is even as mentioned above. Therefore k is even and so we can take a free C_{pq} -representation U of dimension k . Put $V = V_p \oplus V_q \oplus U$. One can easily see that $\text{Dim } \Sigma = \text{Dim } S(V)$. \square

3. THE GENERAL CASE

Throughout this section, G is a finite group. We first recall well-known results on linear dimension functions. Let $F(G)$ be the free module consisting of integer-valued functions on the set of subgroups of G . Let us denote by $D_+^\ell(G)$ the set of linear dimension functions for G , and by $D^\ell(G)$ the submodule of $F(G)$ generated by $D_+^\ell(G)$. By the additivity of dimension functions, any function in $D^\ell(G)$ has a form of $\underline{n} - \underline{n}'$, where $\underline{n}, \underline{n}'$ are linear dimension functions, and hence $D^\ell(G)$ is the free module consisting of virtually linear dimension functions for G .

We say that G -representations V and W are G -Galois conjugate if there exists an integer k prime to $|G|$ such that $\chi_V(g) = \chi_W(g^k)$ for all $g \in G$, where χ_V denotes the character of V . Note that if V and W are G -Galois conjugate, then

$\text{Dim } S(V) = \text{Dim } S(W)$ and $\text{Ker } V = \text{Ker } W$. Let V_1, V_2, \dots, V_r be the complete representative system of G -Galois conjugate classes of irreducible G -representations, and $\mathcal{B}(G)$ the set of dimension functions of $S(V_i)$.

Proposition 3.1 ([12], [5]). *The set $\mathcal{B}(G)$ is a \mathbb{Z} -basis of $D^\ell(G)$.*

We call $\mathcal{B}(G)$ the *standard basis* of $D^\ell(G)$. By Proposition 3.1, any virtually linear function \underline{n} is presented as

$$\underline{n} = \sum_{\underline{m} \in \mathcal{B}(G)} a(\underline{m}, \underline{n}) \underline{m}.$$

We call the integer $a(\underline{m}, \underline{n})$ the *multiplicity* of \underline{m} in \underline{n} . We also put

$$\mathcal{B}(G, \underline{n}) = \{ \underline{m} \in \mathcal{B}(G) \mid a(\underline{m}, \underline{n}) \neq 0 \}.$$

We define $\text{Ker } \underline{n}$ as $\text{Ker } V$ if $\underline{n} = \text{Dim } S(V)$. For any $\underline{n} \in D^\ell(G)$, we define a function \underline{n}^K by setting $\underline{n}^K(H/K) = \underline{n}(H)$ for $H/K \leq G/K$, where K is any normal subgroup of G . Then $\underline{n}^K \in D^\ell(G/K)$ because $\underline{n}^K = \text{Dim } S(V^K) - \text{Dim } S(W^K)$ if $\underline{n} = \text{Dim } S(V) - \text{Dim } S(W)$. We shall show the following lemmas.

Lemma 3.2. *Let K be an arbitrary normal subgroup of G . If $\underline{m} \in \mathcal{B}(G)$ and $\text{Ker } \underline{m} \not\geq K$, then $\underline{m}^K = 0$ (zero-function).*

Proof. Let $\underline{m} = \text{Dim } S(V)$, where V is an irreducible G -representation. Then V^K is regarded as a G -subrepresentation of V , and $V^K \neq V$ by $\text{Ker } \underline{m} \not\geq K$. Since V is irreducible, we have $V^K = 0$, and hence $\underline{m}^K = 0$. □

Put $B = \{ \underline{m}^K \mid \underline{m} \in \mathcal{B}(G) \text{ such that } \text{Ker } \underline{m} \geq K \}$.

Lemma 3.3. *The set B is a part of the standard basis $\mathcal{B}(G/K)$.*

Proof. If V is an irreducible G -representation whose kernel contains K , then $V^K (= V)$ is an irreducible G/K -representation. It is easy to see that irreducible G -representations V and W are G -Galois conjugate if and only if V^K and W^K are G/K -Galois conjugate. This shows the desired result. □

We now prove Theorem A. Since G is nilpotent of order $p^n q^m$, we may put $G = G_p \times G_q$, where G_p [resp. G_q] is a p - [resp. q -] group. Let \underline{n} be the dimension function of a semilinear G -sphere. Since all dimension functions of G -homotopy representations are virtually linear if (and only if) G is nilpotent ([6], Proposition 10.23), \underline{n} is presented as a linear combination of $\underline{m} \in \mathcal{B}(G)$; $\underline{n} = \sum_{\underline{m} \in \mathcal{B}(G)} a(\underline{m}, \underline{n}) \underline{m}$. In order to prove Theorem A, it suffices to show that all $a(\underline{m}, \underline{n})$ are nonnegative.

The proof is done by induction on the order of G . For any $\underline{m} \in \mathcal{B}(G, \underline{n})$ with nontrivial kernel K , it is seen that $a(\underline{m}, \underline{n}) = a(\underline{m}^K, \underline{n}^K)$ by Lemmas 3.2 and 3.3, and hence $a(\underline{m}, \underline{n})$ is nonnegative by the inductive assumption.

We have to show that $a(\underline{m}, \underline{n})$ is nonnegative when $\text{Ker } \underline{m} = 1$. By Theorem 1.1, Theorem A holds for p -groups, and so we assume that p and q are distinct primes and that G_p and G_q are nontrivial.

The proof is divided into two cases: (1) G has a normal subgroup A isomorphic to $C_p \times C_p$ or $C_q \times C_q$, and (2) otherwise.

Case (1): If there is no $\underline{m} \in \mathcal{B}(G, \underline{n})$ with trivial kernel, there is nothing to do. We assume that there is an $\underline{m} \in \mathcal{B}(G, \underline{n})$ with trivial kernel, namely G has a faithful irreducible G -representation. Let $\underline{m} = \text{Dim } S(U)$, where U is a faithful irreducible G -representation. By assumption, G has a normal subgroup A isomorphic to $C_p \times$

C_p or $C_q \times C_q$. We may assume that A is isomorphic to $C_p \times C_p$. Let A_0, A_1, \dots, A_p be all subgroups of order p in A . By representation theory, the following is known.

Proposition 3.4. *Under the above situation,*

- (1) *One of the A_i , say A_0 , is normal in G and the others have the same normalizer $N = N_G(A_i)$ of index p in G . Furthermore A_0 is in the center $Z(G)$ of G , and N is normal in G and contains A .*
- (2) *The restriction $\text{Res}_N U$ is the direct sum of irreducible N -representations U_1, \dots, U_p such that $\text{Ker Res}_A V_i = A_i$ ($1 \leq i \leq p$).*
- (3) *The irreducible summands U_i and U_j are not N -Galois conjugate for $i \neq j$.*
- (4) *Let W be another faithful irreducible G -representation and let $\text{Res}_N W = W_1 \oplus \dots \oplus W_p$ as in (2). If U and W are not G -Galois conjugate, then U_i and W_j are not N -Galois conjugate for every i and j .*

Proof. Statements (1) and (2) are shown by Proposition 5.12 of [5] and its proof. Statements (3) and (4) are shown by Proposition 5.6 of [18]. □

For any $\underline{n} \in D^\ell(G)$ and any subgroup H of G , we define the restriction $r_H \underline{n} \in D^\ell(H)$ by setting $r_H \underline{n}(L) = \underline{n}(L)$ for $L \leq H$. Put $\underline{m}_i = \text{Dim } S(U_i) \in \mathcal{B}(N)$ ($1 \leq i \leq p$). By Proposition 3.4, $r_N \underline{m} = \sum_i \underline{m}_i$.

Lemma 3.5. *Under the above situation, $a(\underline{m}, \underline{n}) = a(\underline{m}_i, r_N \underline{n})$ ($1 \leq i \leq p$).*

Proof. Since $\underline{n} = \sum_{\underline{\ell} \in \mathcal{B}(G, \underline{n})} a(\underline{\ell}, \underline{n}) \underline{\ell}$, it follows from the additivity of the restriction that

$$(3.1) \quad a(\underline{m}_i, r_N \underline{n}) = \sum_{\underline{\ell} \in \mathcal{B}(G, \underline{n})} a(\underline{\ell}, \underline{n}) a(\underline{m}_i, r_N \underline{\ell}).$$

We first consider the case where $\underline{\ell} \in \mathcal{B}(G, \underline{n})$ has a nontrivial kernel K . We note that the center $Z(G) = Z(G_p) \times Z(G_q)$ of G is cyclic since G has a faithful irreducible G -representation (see for example [8]). Furthermore any (nontrivial) p -group P has a nontrivial center, and any nontrivial normal subgroup of P intersects the center $Z(P)$ nontrivially. Since $N = N_{G_p}(A_i) \times G_q$, $\text{Ker } r_N \underline{\ell} = N \cap K$ is a nontrivial normal subgroup of G , and hence $\text{Ker } \underline{s} \geq N \cap K$ for any $\underline{s} \in \mathcal{B}(N, r_N \underline{\ell})$. On the other hand

$$r_{N/N \cap K}(\underline{m}^{N \cap K}) = (r_N \underline{m})^{N \cap K} = \sum_i \underline{m}_i^{N \cap K}.$$

Since \underline{m} has the trivial kernel, it follows from Lemma 3.2 that $\underline{m}^{N \cap K} = 0$, and hence $\underline{m}_i^{N \cap K} = 0$ for $1 \leq i \leq p$. Thus we have $\text{Ker } \underline{m}_i \not\geq N \cap K$ and $\text{Ker } \underline{s} \neq \text{Ker } \underline{m}_i$. This shows that $a(\underline{m}_i, r_N \underline{\ell}) = 0$ when $\text{Ker } \underline{\ell} \neq 1$.

Next, if $\underline{\ell} \in \mathcal{B}(G, \underline{n})$ has the trivial kernel, by Proposition 3.4 (3) and (4),

$$a(\underline{m}_i, r_N \underline{\ell}) = \begin{cases} 1 & \text{if } \underline{\ell} = \underline{m}_i, \\ 0 & \text{if } \underline{\ell} \neq \underline{m}_i. \end{cases}$$

Thus we obtain $a(\underline{m}, \underline{n}) = a(\underline{m}_i, r_N \underline{n})$ ($1 \leq i \leq p$) from equation (3.1). □

By Lemma 3.5 and the induction hypothesis, we see that $a(\underline{m}, \underline{n})$ is nonnegative for any \underline{m} with trivial kernel. Thus the inductive argument in Case (1) is completed.

Case (2): Assume $p < q$. In this case, as is well known (see [8]), G_p is cyclic, dihedral, quaternion or semi-dihedral, where the last three cases occur only in

case $p = 2$, and G_q is cyclic. All faithful irreducible G_r -representations ($r = p, q$) are G_r -Galois conjugate by investigating the character tables, and hence all faithful irreducible G -representations are G -Galois conjugate. Therefore $\mathcal{B}(G)$ contains only one dimension function with trivial kernel. We denote by \underline{u} this dimension function. Let C be a cyclic subgroup $C = C_p \times C_q$ of order pq in the center $Z(G)$. Then $\mathcal{B}(C)$ consists of four elements $\underline{\ell}_1, \underline{\ell}_2, \underline{\ell}_3, \underline{\ell}_4$ whose kernels are $1, C_p, C_q, C$ respectively. We have to show that $a(\underline{u}, \underline{n})$ is nonnegative.

Lemma 3.6. *Under the above situation, $a(\underline{\ell}_1, r_C \underline{n}) = a(\underline{u}, \underline{n})a(\underline{\ell}_1, r_C \underline{u})$.*

Proof. The dimension function \underline{n} is presented as $\underline{n} = \sum_{\underline{m} \in \mathcal{B}(G, \underline{n})} a(\underline{m}, \underline{n})\underline{m}$, and furthermore $r_C \underline{n} = \sum_{\underline{m} \in \mathcal{B}(G, \underline{n})} a(\underline{m}, \underline{n})r_C \underline{m}$ and $r_C \underline{m} = \sum_i a(\underline{\ell}_i, r_C \underline{m})\underline{\ell}_i$. If $\underline{m} \in \mathcal{B}(G, \underline{n})$ has a nontrivial kernel K , then $\text{Ker } r_C \underline{m} = C \cap K$ is nontrivial since K is a nontrivial normal subgroup of G and $C = C_p \times C_q$ is contained in the center $Z(G)$. (Note that $Z(G_r)$ ($r = p, q$) is a cyclic r -group.) Hence we have $a(\underline{\ell}_1, r_C \underline{m}) = 0$ by Lemma 3.2, and so

$$\begin{aligned} a(\underline{\ell}_1, r_C \underline{n}) &= \sum_{\underline{m} \in \mathcal{B}(G, \underline{n})} a(\underline{m}, \underline{n})a(\underline{\ell}_1, r_C \underline{m}) \\ &= a(\underline{u}, \underline{n})a(\underline{\ell}_1, r_C \underline{u}). \end{aligned}$$

□

Lemma 3.7. *The number $a(\underline{\ell}_1, r_C \underline{u})$ is positive.*

Proof. Suppose $r_C \underline{u} = \sum_{i=1}^4 b_i \underline{\ell}_i$, where $b_i = a(\underline{\ell}_i, r_C \underline{u})$. Since $\text{Ker } \underline{u} = 1$, $(r_C \underline{u})^C = r_{C/C} \underline{u}^C = 0$ and $(r_C \underline{u})^{C_r} = r_{C/C_r} \underline{u}^{C_r} = 0$ for $r = p$ and q . This shows $b_2 = b_3 = b_4 = 0$, and hence $r_C \underline{u} = b_1 \underline{\ell}_1$. Since $r_C \underline{u}(1)$ and $\underline{\ell}_1(1)$ are positive, it follows that $b_1 = a(\underline{\ell}_1, r_C \underline{u})$ is positive. □

By Lemmas 3.6 and 3.7 and the inductive assumption, we obtain that $a(\underline{u}, \underline{n})$ is nonnegative. Thus the inductive argument in Case (2) is completed and Theorem A has been proved. □

4. SEMILINEAR G -SPHERES WITH NONLINEAR DIMENSION FUNCTIONS

In this section we prove Theorem B. Let G be a compact Lie group. For the sake of the proof, we shall show the following.

Lemma 4.1. *Let \underline{n} be a virtually linear dimension function. If $\underline{n}(C) = 0$ for every (finite) cyclic subgroup C of G , then \underline{n} is a zero-function.*

Proof. Let $\underline{n} = \text{Dim } S(V_1) - \text{Dim } S(V_2)$. In the case where G is finite, by representation theory, $\underline{n}(H) = \dim V_1^H - \dim V_2^H = \sum_{g \in H} (\chi_{V_1}(g) - \chi_{V_2}(g))$, where χ_{V_i} is the character of V_i . We also note

$$(4.1) \quad \sum_{g \in H} (\chi_{V_1}(g) - \chi_{V_2}(g)) = \sum_{\substack{C \leq H \\ C: \text{cyclic}}} \sum_{g \in C^*} (\chi_{V_1}(g) - \chi_{V_2}(g)),$$

where C^* denotes the set of generators of C . We set, for any cyclic subgroup C ,

$$\begin{aligned} h(C) &= \sum_{g \in C} (\chi_{V_1}(g) - \chi_{V_2}(g)), \\ k(C) &= \sum_{g \in C^*} (\chi_{V_1}(g) - \chi_{V_2}(g)). \end{aligned}$$

From equation (4.1), we obtain

$$(4.2) \quad h(C) = \sum_{D \leq C} k(D).$$

Applying the Möbius inversion to (4.2), we obtain

$$k(C) = \sum_{D \leq C} \mu(|C/D|)h(D),$$

where μ is the Möbius function. Since, by assumption, $h(D) = \underline{n}(D) = 0$ for any cyclic subgroup D , we obtain $k(C) = 0$ for any cyclic subgroup C . It follows from (4.1) that $\underline{n}(H) = 0$.

We next consider the case where G is a compact Lie group. Since G_0 is normal, $V_i^{G_0}$ is considered as a G -subrepresentation of V_i . We denote by V_{iG_0} the orthogonal complement of $V_i^{G_0}$ in V_i . Since V_{1G_0} and V_{2G_0} are isomorphic as G -representations by Theorem 1.1 of [20], we have $\underline{n} = \text{Dim } S(V_{1G_0}^{G_0}) - \text{Dim } S(V_{2G_0}^{G_0})$, and also $\underline{n}^{G_0} = \text{Dim } S(V_{1G_0}^{G_0}) - \text{Dim } S(V_{2G_0}^{G_0})$ by considering $V_i^{G_0}$ as a G/G_0 -representation. Let K/G_0 be any cyclic subgroup of G/G_0 , $g \in K$ an element representing a generator of K/G_0 , and T the closure of the subgroup generated by g . Then $K/G_0 = TG_0/G_0$, and $\underline{n}^{G_0}(K/G_0) = \underline{n}(T)$. Since $\bigcup_{C \leq T, C: \text{finite cyclic}} C$ is dense in T and a G -representation has only finitely many isotropy types, there exists a finite cyclic subgroup C such that $\underline{n}(T) = \underline{n}(C)$. By assumption, $\underline{n}(C) = 0$ and hence $\underline{n}^{G_0}(K/G_0) = 0$. Since this lemma has been proved for finite groups, by applying to \underline{n}^{G_0} , we obtain $\underline{n}(H) = \underline{n}^{G_0}(HG_0/G_0) = 0$ for every closed subgroup H . \square

By [19], there exists a smooth G -action on a disk D such that D^H is a disk if H is solvable, and the empty set if H is nonsolvable. In particular D has no G -fixed point if G is nonsolvable. We call D a *quasilinear G -disk*. Clearly $D \times D^n$ is also a quasilinear G -disk, where D^n is an n -dimensional disk with trivial G -action, and the boundary $\partial(D \times D^n)$ is a semilinear G -sphere. We put $\Sigma_n = \partial(D \times D^n)$.

The following leads to the proof of Theorem B.

Theorem 4.2. *Let G be a nonsolvable compact Lie group. Then the dimension function of Σ_n is not virtually linear except at most one n .*

Proof. Suppose that $\text{Dim } \Sigma_r$ and $\text{Dim } \Sigma_s$ are virtually linear for two distinct numbers r, s ($r < s$). Put $\underline{n} = \text{Dim } \Sigma_r$, $\underline{m} = \text{Dim } \Sigma_s$ and $\underline{\ell} = \text{Dim } S(\mathbb{R}^{s-r})$. Then $\underline{u} := \underline{m} - \underline{n} - \underline{\ell}$ is also a virtually linear dimension function. For any finite cyclic subgroup C , $\underline{n}(C) = \text{Dim } \partial D(C) + r$, $\underline{m}(C) = \text{Dim } \partial D(C) + s$ and $\underline{\ell}(C) = s - r$. Therefore $\underline{u}(C) = 0$. By Lemma 4.1, we have $\underline{u} = 0$. On the other hand, since Σ_ℓ^G ($\ell = r, s$) is empty, we have $\underline{u}(G) = s - r \neq 0$. This is a contradiction. \square

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