THE F-DEPTH OF AN IDEAL ON A MODULE

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Abstract. Let $I$ be an ideal of a Noetherian local ring $R$ and $M$ a finitely generated $R$-module. The f-depth of $I$ on $M$ is the least integer $r$ such that the local cohomology module $H^r_I(M)$ is not Artinian. This paper presents some part of the theory of f-depth including characterizations of f-depth and a relation between f-depth and f-modules.

1. Introduction

Let $(R, \mathfrak{m})$ be a (commutative) Noetherian local ring, $I$ a proper ideal of $R$ and $M$ a finitely generated $R$-module. It is well known that the depth, depth$(I; M)$, of $I$ on $M$, i.e., the length of a maximal $M$-regular sequence in $I$, is the least integer $r$ such that the local cohomology module $H^r_I(M) \neq 0$. Faltings [2] proved that the least integer $r$ such that $H^r_I(M)$ is not finitely generated is

$$\min\{\text{depth}(M_p) + \text{ht}((I + p)/p)|p \notin I\}.$$

Now we consider the problem of what is the least integer $r$ such that $H^r_I(M)$ is not Artinian. In [4], Melkersson showed that, when Supp$(M/IM) \notin \{\mathfrak{m}\}$, the least integer $r$ such that $H^r_I(M)$ is not Artinian is

$$\min\{\text{depth}(IR_p, M_p)|p \in \text{Supp}(M/IM) \setminus \{\mathfrak{m}\}\},$$

and called this integer the f-depth (filter depth) of $I$ on $M$. In this paper, for any proper ideal $I$, we define the f-depth of $I$ on $M$ as the length of a maximal $M$-filter regular sequence in $I$. Then it turns out that our f-depth of $I$ on $M$ is the least integer $r$ such that $H^r_I(M)$ is not Artinian for any proper ideal $I$ and, in the case Supp$(M/IM) \notin \{\mathfrak{m}\}$, our f-depth coincides with the one of Melkersson.

After summarizing some results about filter regular sequences in section 2, we define f-depth in section 3. Then, we characterize f-depth by Ext, Koszul complexes and local cohomology modules. Section 4 contains an equivalent condition using f-depth for an $R$-module to be an f-module which is similar to a Cohen-Macaulay module.
2. Preliminaries on filter regular sequences

Throughout the paper, let \((R, m)\) be a (commutative) Noetherian local ring and \(M\) a finitely generated \(R\)-module. For any submodule \(N\) of \(M\), we use \(N : M \langle m \rangle\) to denote the submodule \(m \in M|m^n M \subseteq N\) for some \(n > 0\).

**Definition 2.1.** Let \(x_1, \ldots, x_n \in m\). If, for \(i = 1, \ldots, n\),
\[ (x_1, \ldots, x_{i-1})M : M x_i \subseteq (x_1, \ldots, x_{i-1})M : M \langle m \rangle, \]
then we say that \(x_1, \ldots, x_n\) is an \(M\)-filter regular sequence.

Notice that \(x \in m\) is \(M\)-filter regular if and only if \(x \not\in \bigcup_{p \in \text{Ass}_R(M) \setminus \{m\}} p\) and \(x_1, x_2, \ldots, x_n\) is an \(M\)-filter regular sequence if and only if \(x_1\) is \(M\)-filter regular and \(x_2, \ldots, x_n\) is an \(M/x_1\)-filter regular sequence. For a finitely generated \(R\)-module \(N\), the length of \(N\) is denoted by \(\ell(N)\). Then \(\ell(N) < \infty\) if and only if \(\dim(N) \leq 0\). Thus \(x_1, \ldots, x_n\) is an \(M\)-filter regular sequence is equivalent to
\[ \ell((x_1, \ldots, x_{i-1})M : M x_i/(x_1, \ldots, x_{i-1}M)) < \infty, \quad i = 1, \ldots, n. \]

On the other hand, we are reminded that \(y_1, \ldots, y_s \in R\) is a poor \(M\)-regular sequence if
\[ (y_1, \ldots, y_i-1)M : M y_i = (y_1, \ldots, y_i-1)M, \quad i = 1, \ldots, s, \]
and, if furthermore \((y_1, \ldots, y_s)M \neq M\), we call \(y_1, \ldots, y_s\) an \(M\)-regular sequence.

Then \(x_1, \ldots, x_n\) is an \(M\)-filter regular sequence if and only if, for any \(p \in \text{Supp}(M) \setminus \{m\}\), \(x_1/1, \ldots, x_n/1\) is a poor \(M_p\)-regular sequence, and, if \(x_1, \ldots, x_n\) is an \(M\)-filter regular sequence, then \(x_1/1, \ldots, x_n/1\) is an \(M_p\)-regular sequence for any \(p \in \text{Supp}(M) \setminus \{m\}\) with \(x_1, \ldots, x_n \in p\).

Since, for any poor \(M\)-regular sequence \(y_1, \ldots, y_s\) and any integers \(i_j > 0, j = 1, \ldots, s, y_1^{i_j}, \ldots, y_s^{i_j}\) is also a poor \(M\)-regular sequence, it follows that if \(x_1, \ldots, x_n\) is an \(M\)-filter regular sequence, then, for any integers \(i_j > 0, j = 1, \ldots, n, x_1^{i_j}, \ldots, x_n^{i_j}\) is also an \(M\)-filter regular sequence. However, a permutation of a filter regular sequence is not necessarily filter regular again. For example, let \(K[x_1, x_2]\) be a polynomial ring and \(K\) a field. Set \(R = K[x_1, x_2]/(x_1, x_2)\) and \(M = R/((x_2)^2)\). Then, as \(0 :_M x_1 = 0\) and \(0 :_{M/x_1} M x_2 \subseteq R/(x_1, x_2^2)\) has finite length, it follows that \(x_1, x_2\) is an \(M\)-filter regular sequence. But, as the prime ideal \((x_2) \in \text{Ass}_R(M) \setminus \{(x_1, x_2)\}\) and \(x_2 \in (x_2)\), we see that \(x_2\) is not \(M\)-filter regular, hence, \(x_2, x_1\) is not an \(M\)-filter regular sequence.

For any \(x_1, \ldots, x_n \in R\), let \(H_i(x_1, \ldots, x_n; M)\) be the \(i\)-th homology module of the Koszul complex \(K_i(x_1, \ldots, x_n; M)\) of \(M\) with respect to \(x_1, \ldots, x_n\). Then, we have the following:

**Proposition 2.2.** If \(x_1, \ldots, x_n\) is an \(M\)-filter regular sequence, then
\[ \ell(H_i(x_1, \ldots, x_n; M)) < \infty, \quad \text{for any } i > 0. \]

**Proof.** By induction on \(n\). If \(n = 1\), then \(H_1(x_1; M) = 0 ;_{M} x_1\) has finite length by definition. Now, assume that \(n > 1\). From the long exact sequence
\[ \cdots \rightarrow H_i(x_1, \ldots, x_{n-1}; M) \rightarrow H_i(x_1, \ldots, x_n; M) \rightarrow H_{i-1}(x_1, \ldots, x_{n-1}; M) \]
\[ \rightarrow (-1)^{i-1} x_n H_{i-1}(x_1, \ldots, x_{n-1}; M) \rightarrow \cdots \rightarrow H_0(x_1, \ldots, x_n; M) \rightarrow 0, \]
we have...
we see that \( \ell(H_i(x_1, \ldots, x_n; M)) < \infty \) for all \( i \geq 1 \) from the induction assumption that \( \ell(H_{i-1}(x_1, \ldots, x_n-1; M)) < \infty \) and \( \ell(H_{i-1}(x_1, \ldots, x_n-1; M)) < \infty \). For the case \( i = 1 \), we have an exact sequence

\[
H_1(x_1, \ldots, x_{n-1}; M) \to H_1(x_1, \ldots, x_n; M) \to H_0(x_1, \ldots, x_{n-1}; M)
\]

\[
= M/(x_1, \ldots, x_{n-1})M \xrightarrow{i} M/(x_1, \ldots, x_{n-1})M.
\]

As \( H_1(x_1, \ldots, x_{n-1}; M) \) and \( 0 : M/(x_1, \ldots, x_{n-1})M \) are finite length, we see that \( \ell(H_1(x_1, \ldots, x_n; M)) < \infty \). Hence, for all \( i > 0 \), \( \ell(H_i(x_1, \ldots, x_n; M)) < \infty \).

Notice that the converse of Proposition 2.2 is not true because the conditions \( \ell(H_i(x_1, \ldots, x_n; M)) < \infty \) for all \( i > 0 \) do not depend on the order of \( x_1, \ldots, x_n \), but the filter regularity does.

### 3. Characterizations of f-depth

In order to show that f-depth is well-defined, we need the following:

**Lemma 3.1.** Let \( I \subseteq m \) be an ideal. If \( \ell(\text{Hom}_R(R/I, M)) < \infty \), then there exists \( x \in I \) which is \( M \)-filter regular.

**Proof.** Assume the contrary. Then \( I \subseteq \bigcup_{p \in \text{Ass}_R(M) \setminus \{m\}} p \), so that \( I \subseteq p \) for some \( p \in \text{Ass}_R(M) \setminus \{m\} \). Hence \( p = \text{ann}_R(m) \) for some \( m \in M \), so \( \dim(Rm) > 0 \). But \( \text{Im} = 0 \), so \( \text{Im} \subseteq 0 : M \text{ I } \cong \text{Hom}_R(R/I, M) \). Thus \( \dim(\text{Hom}_R(R/I, M)) > 0 \), a contradiction.

**Proposition 3.2.** Let \( I \subseteq m \) be an ideal and \( n > 0 \) an integer. Then the following are equivalent:

1. \( \ell(\text{Ext}_R^i(R/I, M)) < \infty \), for all \( i < n \);
2. \( I \) contains an \( M \)-filter regular sequence of length \( n \).

When \( x_1, \ldots, x_n \in I \) is an \( M \)-filter regular sequence,

\[
\text{Ext}_R^i(R/I, M)_p \cong \text{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)_p, \text{ for some } p \in \text{Spec}(R) \setminus \{m\}.
\]

**Proof.** Assume that \( \ell(\text{Ext}_R^i(R/I, M)) < \infty \) for all \( i < n \). We use induction on \( n \) to show that \( I \) contains an \( M \)-filter regular sequence of length \( n \). If \( n = 1 \), then \( I \) contains an \( M \)-filter regular element by Lemma 3.1. Now assume that \( n > 1 \) and the result is true for \( n-1 \). Then, by Lemma 3.1 again, there is \( x_1 \in I \) which is \( M \)-filter regular. From the short exact sequences

\[
0 \to 0 : M x_1 \to M \xrightarrow{x_1} x_1 M \to 0,
\]

\[
0 \to x_1 M \to M/x_1 M \to 0,
\]

we get the long exact sequences

\[
0 \to \text{Hom}_R(R/I, 0 : M x_1) \to \text{Hom}_R(R/I, M) \xrightarrow{x_1} \text{Hom}_R(R/I, x_1 M)
\]

\[
\to \text{Ext}_R^1(R/I, 0 : M x_1) \to \cdots,
\]

\[
0 \to \text{Hom}_R(R/I, x_1 M) \to \text{Hom}_R(R/I, M) \to \text{Hom}_R(R/I, M/x_1 M)
\]

\[
\to \text{Ext}_R^1(R/I, x_1 M) \to \cdots.
\]

As \( \ell(0 : M x_1) < \infty \), we have that \( \ell(\text{Ext}_R^i(R/I, 0 : M x_1)) < \infty \) for any \( i \geq 0 \). Then we see that \( \ell(\text{Ext}_R^i(R/I, x_1 M)) < \infty \) for all \( i < n \) from the first long exact sequence. From the second long exact sequence, we get that \( \ell(\text{Ext}_R^i(R/I, M/x_1 M)) < \infty \) for all \( i < n-1 \). Then, by the induction assumption, there exist \( x_2, \ldots, x_n \in I \) which
is an $M/x_1 M$-filter regular sequence. Hence $x_1, x_2, \ldots, x_n$ is an $M$-filter regular sequence.

Conversely, suppose that $I$ contains an $M$-filter regular sequence of length $n$. Let $x_1, \ldots, x_n \in I$ be an $M$-filter regular sequence. For any $p \in \text{Supp}(M/IM) \setminus \{m\}$, $x_1/1, \ldots, x_n/1$ is an $M_p$-regular sequence. Then, by a well-known property of $M_p$-regular sequences, we have

$$\text{Ext}^i_{R_p}(R_p/IR_p, M_p) = 0, \text{ for all } i < n,$$

and

$$\text{Ext}^n_{R_p}(R_p/IR_p, M_p) \cong \text{Hom}_{R_p}(R_p/IR_p, M_p/(x_1/1, \ldots, x_n/1)M_p).$$

Thus

$$\text{Ext}^i_R(R/I, M)_p = 0, \text{ for all } i < n, \quad \text{Ext}^n_R(R/I, M)_p \cong \text{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)_p.$$

But, for any $p \notin \text{Supp}(M/IM)$ and any $i \geq 0$, it is clear that

$$\text{Ext}^i_R(R/I, M)_p = 0 \text{ and } \text{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)_p = 0.$$

Hence $\ell(\text{Ext}^i_R(R/I, M)) < \infty$ for all $i < n$ and for any $p \in \text{Spec}(R) \setminus \{m\}$,

$$\text{Ext}^n_R(R/I, M)_p \cong \text{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)_p.$$

This completes the proof. \qed

If $x_1, \ldots, x_n$ is a maximal $M$-filter regular sequence in $I$, then, by Lemma 3.1, \text{dim}(\text{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)) > 0$. It follows from Proposition 3.2 that \text{dim}(\text{Ext}^n_R(R/I, M)) > 0$. Hence any two maximal $M$-filter regular sequences in $I$ (if any exist) have the same length.

**Definition 3.3.** Let $I$ be a proper ideal of $R$. The $f$-depth (filter depth) of $I$ on $M$ is defined as the length of any maximal $M$-filter regular sequence in $I$, denoted by $f$-depth$(I, M)$. Here, when the maximal $M$-filter regular sequence in $I$ does not exist, we understand that the length is $\infty$.

Notice that $f$-depth$(I, M) = 0$ if and only if $I \subseteq p$ for some $p \in \text{Ass}_R(M) \setminus \{m\}$ and if $x \in I$ is $M$-filter regular, then

$$f$-depth$(I, M) = f$-depth$(I, M/xM) + 1.$$

Furthermore, by Proposition 3.2, we have that

$$f$-depth$(I, M) = \min \{n | \text{dim}(\text{Ext}^n_R(R/I, M)) > 0\},$$

where, when $\text{dim}(\text{Ext}^n_R(R/I, M)) \leq 0$ for all $n \geq 0$, we understand the right side of the above equality to be $\infty$.

**Proposition 3.4.** Let $I, J$ be proper ideals of $R$. If $\sqrt{I} = \sqrt{J}$, then

$$f$-depth$(I, M) = f$-depth$(J, M).$$

**Proof.** Let $x_1, \ldots, x_n \in I$ be an $M$-filter regular sequence. Then, as $\sqrt{I} = \sqrt{J}$, there exists an integer $\alpha > 0$ such that $x_1^\alpha, \ldots, x_n^\alpha \in J$. But since $x_1^\alpha, \ldots, x_n^\alpha$ is also $M$-filter regular, we see that

$$f$-depth$(I, M) \leq f$-depth$(J, M).$$

Similarly,

$$f$-depth$(J, M) \leq f$-depth$(I, M).$$

Thus $f$-depth$(I, M) = f$-depth$(J, M).$ \qed
Suppose that \( \dim(M/IM) > 0 \). If \( x \in I \) is \( M \)-filter regular, then \( x \notin p \), for any \( p \in \text{Ass}_R(M) \) with \( \dim(R/p) = \dim(M) \). Thus \( \dim(M/xM) = \dim(M) - 1 \). It follows that every \( M \)-filter regular sequence in \( I \) is a subsystem of parameters for \( M \). Furthermore, we have the following:

**Proposition 3.5.** If \( \dim(M/IM) > 0 \), then
\[
\text{depth}(I, M) \leq f\text{-depth}(I, M) \leq \text{ht}_M I,
\]
where \( \text{ht}_M I \) is the infimum of lengths of strictly decreasing chains of prime ideals in \( \text{Supp}(M) \) starting from a prime ideal containing \( I \).

**Proof.** As any \( M \)-regular sequence is an \( M \)-filter regular sequence, we see that \( \text{depth}(I, M) \leq f\text{-depth}(I, M) \). So it remains to show that \( f\text{-depth}(I, M) \leq \text{ht}_M I \).

By assumption, \( \text{Supp}(M/IM) \nsubseteq \{m\} \). Let \( x_1, \ldots, x_n \in I \) be any \( M \)-filter regular sequence. It is enough to show that \( n \leq \text{ht}_M(p) \) for any \( p \in \text{Supp}(M/IM) \setminus \{m\} \). As \( x_1, \ldots, x_n \in p \), \( x_1/1, \ldots, x_n/1 \in R_p \) is an \( M_p \)-regular sequence. Hence \( n \leq \dim(M_p) = \text{ht}_M(p) \), as required. \( \square \)

**Proposition 3.6.** \( f\text{-depth}(I, M) = \infty \) if and only if \( I \) contains a system of parameters for \( M \).

**Proof.** Notice that \( I \) containing a system of parameters for \( M \) is equivalent to \( \dim(M/IM) = 0 \). Thus, by Proposition 3.5, we only need to show that, when \( \dim(M/IM) = 0 \), for any integer \( n > 0 \) we can find an \( M \)-filter regular sequence of length \( n \) in \( I \). But, in this case, we have that \( \ell(\text{Ext}_R^i(R/I, M)) < \infty \) for all \( i \geq 0 \). Then the result follows from Proposition 3.2. \( \square \)

**Proposition 3.7.** Let \( V(I) \) be the set of prime ideals containing \( I \). Then
\[
f\text{-depth}(I, M) = \min\{f\text{-depth}(p, M) | p \in V(I)\}.
\]

**Proof.** If \( \dim(M/IM) = 0 \), then \( f\text{-depth}(I, M) = \infty \). But since the prime ideal containing \( I \) is just \( m \) and \( f\text{-depth}(m, M) = \infty \), the equality holds. Now assume that \( \dim(M/IM) > 0 \). Set \( r = \min\{f\text{-depth}(p, M) | p \in V(I)\} \). As there is some \( p \in V(I) \) such that \( \dim(M/pM) > 0 \), we have that \( f\text{-depth}(p, M) < \infty \) hence, \( r < \infty \). We use induction on \( r \) to show that \( f\text{-depth}(I, M) = r \). If \( r = 0 \), then there exists a prime ideal \( p \supset I \) such that \( f\text{-depth}(p, M) = 0 \). Thus, as \( I \subseteq p \), \( f\text{-depth}(I, M) = 0 \). Suppose that \( r > 0 \). Then, for any \( p \in \text{Ass}_R(M) \setminus \{m\} \), as \( f\text{-depth}(p, M) = 0 \) and \( r > 0 \), we have \( I \nsubseteq p \). Hence \( I \not\subseteq \bigcup_{p \in \text{Ass}_R(M) \setminus \{m\}} p \). Thus there exists \( x_1 \in I \) which is \( M \)-filter regular. Set \( M_1 = M/x_1M \). Then
\[
\min\{f\text{-depth}(p, M_1) | p \in V(I)\} = \min\{f\text{-depth}(p, M) - 1 | p \in V(I)\} = r - 1.
\]
Hence, by the induction assumption, \( f\text{-depth}(I, M_1) = r - 1 \), so \( f\text{-depth}(I, M) = f\text{-depth}(I, M_1) + 1 = r \), as required. \( \square \)

For any finitely generated \( R \)-module \( N \), its \( m \)-adic completion is denoted by \( \widehat{N} \). The following proposition states that \( f\text{-depth} \) does not change after passing to completion.

**Proposition 3.8.** \( f\text{-depth}(I, M) = f\text{-depth}(\widehat{I}, \widehat{M}) \).
Proof. This is because, for any $i \geq 0$,
\[ \dim(\text{Ext}^i_R(R/I, M)) = \dim(\text{Ext}^i_R(R/\widehat{I}, M)) = \dim(\text{Ext}^i_R(\widehat{R}/\widehat{I}, \widehat{M})) \]
\[ \square \]

The following theorems give two characterizations of f-depth.

**Theorem 3.9.** Let $y_1, \ldots, y_n \in I$ such that $I = (y_1, \ldots, y_n)$. Then
\[ \text{f-depth}(I, M) = n - \sup\{i \mid \dim(H_i(y_1, \ldots, y_n; M)) > 0\}, \]
where, if there is no integer $i$ with $\dim H_i(y_1, \ldots, y_n; M) > 0$, we understand that the right side of the above equality is $\infty$.

**Proof.** If $\dim(M/IM) = 0$, then f-depth$(I, M) = \infty$ and $\dim(H_i(y_1, \ldots, y_n; M)) \leq 0$ for any $i$ (since $I \cdot H_i(y_1, \ldots, y_n; M) = 0$), so the theorem is true in this case. Now assume that $\dim(M/IM) > 0$. Let $r = \text{f-depth}(I, M)$. We use induction on $r$. If $r = 0$, then $I \subseteq p$ for some $p \in \text{Ass}_R(M) \setminus \{m\}$. Thus $p = \text{ann}_R(m)$ for some $m \in M$. As $Im = 0$, we see that $m \in 0 :_M I = H_n(y_1, \ldots, y_n; M)$. Then $p \in \text{Ass}_R(H_n(y_1, \ldots, y_n; M))$. But since $p \neq m$, we have that $\dim(H_n(y_1, \ldots, y_n; M)) > 0$, and the equality holds. Suppose that $r > 0$. Let $x \in I$ be an $M$-filter regular element and $M_1 = M/xM$. Then, as $\text{f-depth}(I, M_1) = r - 1$, we have, by the induction assumption, that
\[ \sup\{i \mid \dim(H_i(y_1, \ldots, y_n; M_1)) > 0\} = n - r + 1. \]

Note that, as $\text{Supp}(H_i(y_1, \ldots, y_n; M_1)) \subseteq \text{Supp}(M/IM)$, the above equality is equivalent to $H_i(y_1, \ldots, y_n; M_1)_p = 0$ for all $i > n - r + 1$ and any $p \in \text{Supp}(M/IM) \setminus \{m\}$, and $H_{n-r+1}(y_1, \ldots, y_n; M_1)_p \neq 0$ for some $p \in \text{Supp}(M/IM) \setminus \{m\}$.

For any $p \in \text{Supp}(M/IM) \setminus \{m\}$, as $x \in p$, we see that $x/1$ is $M_p$-regular. From the short exact sequence
\[ 0 \to M_p \xrightarrow{x/1} M_p \to (M_1)_p \to 0, \]
we have a long exact sequence
\[ \cdots \to H_i(y_1/1, \ldots, y_n/1; M_p) \xrightarrow{x/1} H_i(y_1/1, \ldots, y_n/1; M_p) \to H_{i-1}(y_1/1, \ldots, y_n/1; M_p) \xrightarrow{x/1} \cdots. \]
As $H_i(y_1/1, \ldots, y_n/1; M_p)$ is annihilated by $x/1$, the above long exact sequence is split into short exact sequences
\[ 0 \to H_i(y_1/1, \ldots, y_n/1; M_p) \to H_i(y_1, \ldots, y_n/1; (M_1)_p) \to H_{i-1}(y_1, \ldots, y_n/1; M_p) \to 0, \]
i.e.,
\[ 0 \to H_i(y_1, \ldots, y_n; M)_p \to H_i(y_1, \ldots, y_n; M_1)_p \to H_{i-1}(y_1, \ldots, y_n; M)_p \to 0. \]
Then $H_i(y_1, \ldots, y_n; M)_p = 0$ for any $i > n - r$ and any $p \in \text{Supp}(M/IM) \setminus \{m\}$, and $H_{n-r}(y_1, \ldots, y_n; M)_p \neq 0$ for some $p \in \text{Supp}(M/IM) \setminus \{m\}$. Hence sup\{i \mid \dim(H_i(y_1, \ldots, y_n; M)) > 0\} = n - r. The theorem follows. \[ \square \]

**Theorem 3.10** ([4, Theorem 3.1]). For any proper ideal $I$ of $R$,
\[ \text{f-depth}(I, M) = \min\{r \mid H^r_I(M) \text{ is not Artinian}\}. \]
Proof. If \( \dim(M/IM) = 0 \), then \( \sqrt{I + \operatorname{ann}_R(M)} = \mathfrak{m} \), hence, \( H^r_I(M) \cong H^r_{\mathfrak{m}}(M) \) is Artinian for any \( r \geq 0 \). Thus \( \min\{r | H^r_I(M) \text{ is not Artinian} \} = \infty \). In this case, \( \text{f-depth}(I, M) = \infty \) and the result is true.

Now we assume that \( \dim(M/IM) > 0 \). Let \( n = \text{f-depth}(I, M) \). Then \( n = \min\{i | \dim(\operatorname{Ext}^i_R(R/I, M)) > 0 \} \). Note that \( \dim(\operatorname{Ext}^i_R(R/I, M)) \leq 0 \) for all \( i < n \) and \( \dim(\operatorname{Ext}^n_R(R/I, M)) > 0 \) is equivalent to \( \operatorname{Ext}^n_R(R/I, IR_p, M_p) = 0 \) for any \( i < n \) and any \( p \in \operatorname{Supp}(M/IM) \setminus \{m\} \), but, \( \operatorname{Ext}^n_R(R/I, IR_p, M_p) \neq 0 \) for some \( p \in \operatorname{Supp}(M/IM) \setminus \{m\} \); i.e., \( \operatorname{depth}(IR_p, M_p) \geq n \) for any \( p \in \operatorname{Supp}(M/IM) \setminus \{m\} \) and \( \operatorname{depth}(IR_p, M_p) = n \) for some \( p \in \operatorname{Supp}(M/IM) \setminus \{m\} \). It follows that

\[
 n = \min\{\operatorname{depth}(IR_p, M_p) | p \in \operatorname{Supp}(M/IM) \setminus \{m\}\}.
\]

Then, by [4, Theorem 3.1], \( n = \min\{r | H^r_I(M) \text{ is not Artinian} \} \).

\[ \square \]

4. F-DEPTH AND F-MODULES

A finitely generated \( R \)-module \( M \) is called an f-module if every system of parameters for \( M \) is an \( M \)-filter regular sequence. F-modules were introduced in [4] as a generalization of Cohen-Macaulay modules.

The following theorem gives a characterization of f-modules by f-depth.

**Theorem 4.1.** \( M \) is an f-module if and only if, for any \( p \in \operatorname{Supp}(M) \setminus \{m\} \),

\[
 \text{f-depth}(p, M) = \dim(M) - \dim(R/p).
\]

**Proof.** Suppose that \( M \) is an f-module. Let \( p \in \operatorname{Supp}(M) \setminus \{m\} \) and \( x_1, \ldots, x_r \) be a maximal \( M \)-filter regular sequence in \( p \). Then

\[
p \subseteq \bigcup_{q \in \operatorname{Ass}_R(M/(x_1, \ldots, x_r)M) \setminus \{m\}} q;
\]

hence, \( p \subseteq q \) for some \( q \in \operatorname{Ass}_R(M/(x_1, \ldots, x_r)M) \setminus \{m\} \). As \( M \) is an f-module and \( x_1, \ldots, x_r \) is a subsystem of parameters for \( M \), we have, by [4 (2.5)], that \( \dim(R/q) = \dim(M/(x_1, \ldots, x_r)M) \). But since \( p \in \operatorname{Supp}(M/(x_1, \ldots, x_r)M) \), we see that \( p = q \), hence

\[
 \text{f-depth}(p, M) = r = \dim(M) - \dim(R/p).
\]

Conversely, suppose that

\[
 \text{f-depth}(p, M) = \dim(M) - \dim(R/p)
\]

for all \( p \in \operatorname{Supp}(M) \setminus \{m\} \). We use induction on \( d := \dim(M) \) to show that \( M \) is an f-module. The case \( d = 0 \) is trivial. Suppose that \( d > 0 \). Let \( x_1, x_2, \ldots, x_d \) be a system of parameters for \( M \). Then \( x_1 \) is \( M \)-filter regular, otherwise \( x_1 \in p \) for some \( p \in \operatorname{Ass}_R(M) \setminus \{m\} \), hence

\[
 \text{f-depth}(p, M) + \dim(R/p) = \dim(R/p) \leq \dim(M/x_1M) = \dim(M) - 1,
\]

a contradiction. Set \( M_1 = M/x_1M \). Then \( \dim(M_1) = d - 1 \) and \( x_2, \ldots, x_d \) is a system of parameters for \( M_1 \) and, for any \( p \in \operatorname{Supp}(M_1) \setminus \{m\} \), as \( x_1 \in p \), we have that

\[
 \text{f-depth}(p, M_1) = \text{f-depth}(p, M) - 1 = \dim(M) - \dim(R/p) - 1
\]

\[
= \dim(M_1) - \dim(R/p).
\]

Thus, by induction assumption, \( x_2, \ldots, x_d \) is an \( M_1 \)-filter regular sequence. Then \( x_1, x_2, \ldots, x_d \) is an \( M \)-filter regular sequence and \( M \) is an f-module. \( \square \)
Remark 4.2. Suppose that $M$ is an $f$-module. Let $I$ be a proper ideal of $R$ such that $I \supseteq \text{ann}_R(M)$ and $\sqrt{I} \neq m$. Then, by Proposition 3.7 and Theorem 4.1, we have that

$$f\text{-depth}(I, M) = \min \{f\text{-depth}(p, M) \mid p \in \text{Supp}(M/IM) \setminus \{m\}\} = \dim(M) - \max \{\dim(R/p) \mid p \in \text{Supp}(M/IM) \setminus \{m\}\} = \dim(M) - \dim(R/I).$$

Note that $ht_M(I) \leq \dim(M) - \dim(R/I)$ and it follows from Proposition 3.5 that

$$f\text{-depth}(I, M) = ht_M(I) = \dim(M) - \dim(R/I).$$

References


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