PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 7, Pages 2115–2123
S 0002-9939(01)06308-0
Article electronically published on December 20, 2001

SOME VARIATIONAL FORMULAS ON ADDITIVE FUNCTIONALS OF SYMMETRIC MARKOV CHAINS

DAEHONG KIM, MASAYOSHI TAKEDA, AND JIANGANG YING

(Communicated by Claudia M. Neuhauser)

Abstract. For symmetric continuous time Markov chains, we obtain some formulas on total occupation times and limit theorems of additive functionals by using large deviation theory.

1. Introduction

In [11], we extended Donsker-Varadhan's type large deviation of symmetric Markov processes with finite lifetime and established the full large deviation principle; let $\mathbf{M} = (P_x, X_t, \zeta)$ be an m-symmetric Markov process on a locally compact separable metric space X. Here ζ is the lifetime of $\mathbf{M}$ and m is a Radon measure with full support. If the process $\mathbf{M}$ explodes rapidly in the sense that the 1-resolvent of the identity function 1, $R_1 1(x)(= 1 - E_x(\exp(-\zeta)))$, belongs to $C_\infty(X)$, the space of continuous functions vanishing at infinity, then the full large deviation principle is derived (Theorem 3.1 and Theorem 4.4 in [11]). As a corollary, the following variational formula on the lifetime ζ is obtained:

$$\lim_{t\to\infty} \frac{1}{t} \log P_x(t < \zeta) = -\inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F},\ \int_X u^2 dm = 1\right\} \tag{1}$$

for any $x \in X$, where $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form on $L^2(X;m)$ generated by $\mathbf{M}$. As an application of formula (1), we considered in [12] tail probabilies of A_{τ_D}, where A_t is a positive continuous additive functional of the Brownian motion and τ_D is the exit time from a regular domain D.

In this paper, we shall treat the same problem for symmetric continuous time Markov chains. Our results are also formulated as an application of formula (1) but contain the accurate calculations of its exponential decay rates represented by Dirichlet forms. Let I be a countable set with the discrete topology. Let $\mathcal{Q} = (q_{ij})$ be an $I \times I$ matrix such that $m_i q_{ij} = m_j q_{ji}$ for some strictly positive function m_i on I. Denote by $\mathbf{M} = (P_i, X_t)$ the minimal $\mathcal{Q}$-process with lifetime ζ, $\zeta = \inf\{t > 0 : X_t = \Delta\}$. Here Δ is the one-point compactification of I. We shall show our main theorem in section 2.

Received by the editors May 20, 2000 and, in revised form, January 29, 2001.
2000 *Mathematics Subject Classification.* Primary 60F10, 60J20; Secondary 31C25.
Key words and phrases. Additive functional, Dirichlet form, large deviation, symmetric Markov chain.
The first author's research was supported in part by Brain Korea 21.

Theorem 1.1. *Let μ be a non-negative function on I such that*

$$\lim_{i\to\Delta} E_i\left(\exp\left(-\int_0^\zeta \frac{\mu}{m}(X_t)\,dt\right)\right) = 1.$$

We have for any $i \in I$,

$$\begin{aligned}\lim_{\beta\to\infty}\frac{1}{\beta}\log P_i\left(\int_0^\zeta \frac{\mu}{m}(X_t)\,dt > \beta\right) \\ = -\inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F},\ \sum_{i\in I} u^2(i)\mu_i = 1\right\},\end{aligned} \tag{2}$$

where $(\mathcal{E},\mathcal{F})$ is the Dirichlet form associated with the Markov chain $\mathbf{M}$.

In section 3, we shall consider concrete examples as an application of Theorem 1.1, and extend D. Freedman's theorem on stochastic interpretation of q_{ij} in our setting.

Let us consider the discontinuous additive functional of the form

$$A_t^{\mu,F} = A_t^\mu + \sum_{0<s\le t} F(X_{s-}, X_s),$$

where A_t^μ is the positive continuous additive functional of the form (5) below for a non-negative function μ on I. Under certain conditions on $\mathbf{M}$ and F, we can see from [7] that

$$\lim_{t\to\infty}\frac{1}{t}\log E_i\left(\exp\left(A_t^{\mu,F}\right)\right) = -\inf\left\{\mathcal{E}^{\mu,F}(u,u) : \sum_{i\in I} u^2(i)m_i = 1\right\}, \tag{3}$$

where $\mathcal{E}^{\mu,F}$ is the symmetric form on $L^2(I;m)$ generated by the semigroup

$$p_t^{\mu,F} f(i) = E_i\left(\exp\left(A_t^{\mu,F}\right) f(X_t)\right).$$

In section 4, we obtain the large deviation principle for $A_t^{\mu,F}/t$ by combining (3) with the Gärtner and Ellis theorem ([3]). In particular, we shall show that

$$\frac{A_t^{\mu,F}}{t} \text{ exponentially converges to } \frac{\mu(I) + \sum_{i\ne j} F(i,j)q_{ij}m_i}{m(I)}.$$

2. Proof of the main theorem

Let I be a countable set with the discrete topology. Let $\mathcal{Q} = (q_{ij})$ be an $I \times I$ matrix such that

$$q_{ij} \ge 0 \quad (i \ne j), \qquad \sum_{k\ne i} q_{ik} \le -q_{ii} < \infty, \quad \forall i \in I$$

and $m_i q_{ij} = m_j q_{ji}$ for some strictly positive function m_i on I. Let $\mathcal{E}$ be the Dirichlet form on $L^2(I;m)$ defined by

(4)

$$\mathcal{E}(u,v) = \frac{1}{2}\sum_{i\ne j}(u(j)-u(i))(v(j)-v(i))q_{ij}m_i + \sum_i u(i)v(i)\left(-q_{ii} - \sum_{j\ne i} q_{ij}\right)m_i.$$

Denote by $\mathcal{F}^r$ the collection of functions u on I such that $\mathcal{E}(u,u)<\infty$. Let $\mathcal{F}$ be the set of $L^2(I;m)$-functions u in $\mathcal{F}^r$ for which there exists u_n $(n=1,2,\dots,)$ with finite support such that

$$u_n \to u \quad \text{and} \quad \sup_n \mathcal{E}(u_n,u_n)<\infty.$$

Then $(\mathcal{E},\mathcal{F})$ becomes a regular Dirichlet form on $L^2(I;m)$. Denote by $\mathbf{M}=(\Omega,X_t,P_i,\zeta)$ the Hunt process generated by $(\mathcal{E},\mathcal{F})$, which is nothing but the minimal $\mathcal{Q}$-process constructed by W. Feller (Theorem 17.2 in [10]). Here Ω is the space of all right continuous maps of $[0,\infty)$ into $I\cup\{\Delta\}$, the one-point compactification of I. When I is finite, Δ is regarded as an isolated point. ζ is the lifetime of $\mathbf{M}$, $\zeta=\inf\{t>0: X_t=\Delta\}$. Let us denote by $\{p_t\}_{t\geq 0}$ and $\{R_\alpha\}_{\alpha>0}$ the semigroup and resolvent of $\mathbf{M}$ respectively, i.e., $p_tf(i)=E_i(f(X_t)),\ R_\alpha f(i)=E_i(\int_0^\zeta e^{-\alpha t}f(X_t)dt)$. We then make following assumption:

(**A**) (Irreducibility of $\mathbf{M}$.) For any $i,j\in I$, $P_i(\sigma_j<\zeta)>0$, where σ_j is the first hitting time of j, $\sigma_j=\inf\{t>0: X_t=j\}$.

Let $\mathcal{B}^+$ be the set of all non-negative functions on I. We introduce the subspace $\mathcal{K}$ of $\mathcal{B}^+$,

$$\mathcal{K}=\left\{\mu\in\mathcal{B}^+ : \lim_{i\to\Delta} E_i\left(\exp\left(-A^\mu_\zeta\right)\right)=1\right\},$$

where A^μ_t is the positive continuous additive functional of the form

$$A^\mu_t=\int_0^t \frac{\mu}{m}(X_s)\,ds. \tag{5}$$

Let us denote by τ_t $(t\geq 0)$ the right continuous inverse of A^μ_t,

$$\tau_t=\inf\left\{s>0: A^\mu_s>t\right\}.$$

The time changed process Y^μ_t of X_t with respect to A^μ_t is defined by

$$Y^\mu_t=X_{\tau_t}.$$

Let $F=\{i\in I:\mu_i>0\}$. The process Y^μ_t is then a μ-symmetric Hunt process on F with lifetime A^μ_ζ (Theorem 6.2.3 in [6] and Theorem 65.9 in [9]). Set

$$H_Fu(i)=E_i(u(X_{\sigma_F}):\sigma_F<\zeta),$$

where $\sigma_F=\inf\{t>0: X_t\in F\}$. Then the time changed process Y^μ_t of X_t generates the following Dirichlet form $(\check{\mathcal{E}},\check{\mathcal{F}})$ on $L^2(F;\mu)$:

$$\begin{cases} \check{\mathcal{F}}=\{\varphi\in L^2(F;\mu):\varphi=u \text{ on } F \text{ for some } u\in\mathcal{F}\},\\ \check{\mathcal{E}}(\varphi,\varphi)=\mathcal{E}(H_Fu,H_Fu),\ \varphi\in\check{\mathcal{F}},\ \varphi=u \text{ on } F\ \ u\in\mathcal{F}_e. \end{cases} \tag{6}$$

Here $\mathcal{F}_e$ stands for the extended Dirichlet space of $(\mathcal{E},\mathcal{F})$ (Theorem 6.2.1 in [6]).

Let $\{G^\mu_\alpha(i,j)\}_{\alpha\geq 0}$ be the resolvent kernel of Y^μ_t,

$$G^\mu_\alpha(i,j)=E_i\left(\int_0^\infty e^{-\alpha t}I_{\{j\}}(Y^\mu_t)dt\right).$$

Then by the definition of the time changed process and the fact that A^μ_ζ is the lifetime of Y^μ_t, we have the following:

$$G^\mu_0(i,j)=E_i\left(\int_0^\zeta I_{\{j\}}(X_t)dt\right)\frac{\mu_j}{m_j} \quad \text{for } i,j\in F \tag{7}$$

and

$$E_i\left(\int_0^\infty e^{-t}I_F(Y_t^\mu)dt\right)=E_i\left(\int_0^{A_\zeta^\mu}e^{-t}dt\right)=1-E_i\left(\exp\left(-A_\zeta^\mu\right)\right). \tag{8}$$

Since $G_0^\mu(i,j)>0$ for any $i,j\in F$ due to (7), the process Y_t^μ is irreducible. Moreover, equation (8) says that for $\mu\in\mathcal{K}$, the 1-resolvent G_1^μ of the identity function 1_F vanishes at infinity Δ. Therefore, we obtain Theorem 1.1 by exactly the same argument as Theorem 3.1 in [12].

Remark 2.1. (i) We can easily see that the statement of Theorem 4.1 in [11] still holds even if P_x is replaced by $\sup_{x\in X}P_x$. As a result, we see that

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log\sup_{i\in I}P_i\left(A_\zeta^\mu>\beta\right)\le-\inf\left\{\mathcal{E}(u,u):u\in\mathcal{F},\ \sum_{i\in I}u^2(i)\mu_i=1\right\}.$$

(ii) For $\mu\in\mathcal{B}^+$, assume that the support of μ is a finite set. Then μ belongs to the class $\mathcal{K}$.

(iii) If $\lim_{i\to\Delta}E_i\left(A_\zeta^\mu\right)=0$, then μ belongs to the class $\mathcal{K}$.

3. Examples

For a Borel subset B of $I\times I$, let J_B be the terminal time defined by

$$J_B=\inf\{t>0:(X_{t-},X_t)\in B\}$$

and X_t^B the killed process defined by

$$X_t^B=\begin{cases}X_t & \text{on } t<J_B,\\ \Delta & \text{on } t\ge J_B\end{cases}\qquad (X_\infty^B=\Delta).$$

The process X_t^B is then a Hunt process on the state space

$$I^B=\{x\in I:P_x(J_B>0)=1\}.$$

Note that X_t^B is not always symmetric; however, we see from Theorem 3.10 in [13] that it is nearly symmetric. More precisely, let us define the bilinear form $(\mathcal{E}^B,\mathcal{F}^B)$ on $L^2(I^B;m)$ by

$$\begin{cases}\mathcal{E}^B(u,v)=\mathcal{E}(u,v)+\sum_{(i,j)\in B}u(i)v(j)q_{ij}m_i,\\ \mathcal{F}^B=\mathcal{F}\cap\left\{u:\sum_{(i,j)\in B}u^2(i)q_{ij}m_i<\infty,\ \sum_{(i,j)\in B}u^2(j)q_{ij}m_i<\infty\right\}.\end{cases}$$

Then, $(\mathcal{E}^B,\mathcal{F}^B)$ is the non-symmetric Dirichlet form generated by the process X_t^B.

Let K be a finite subset of I^B and consider the time changed process Y_t^K of X_t^B with respect to $\int_0^t I_K(X_s)ds$,

$$Y_t^K=X_{\tau_t}^B,\qquad \tau_t=\inf\left\{s>0:\int_0^s I_K(X_u)du>t\right\}.$$

Then, Y_t^K is a Hunt process on K with lifetime $\zeta=\int_0^{J_B}I_K(X_s)ds$. Set $H_Ku(j)=E_j(u(X_{\sigma_K}):\sigma_K<J_B)$, where $\sigma_K=\inf\{t>0:X_t\in K\}$. The Dirichlet form $(\check{\mathcal{E}}^B,\check{\mathcal{F}}^B)$ on $L^2(K;m)$ generated by the time changed process Y_t^K is as follows:

$$\begin{cases}\check{\mathcal{F}}^B=\left\{\varphi\in L^2(K;m):\varphi=u \text{ on } K \text{ for some } u\in\mathcal{F}_e^B\right\},\\ \check{\mathcal{E}}^B(\varphi,\varphi)=\mathcal{E}^B(H_Ku,H_Ku),\ \varphi\in\check{\mathcal{F}}^B,\ \varphi=u \text{ on } K\ \ u\in\mathcal{F}_e^B.\end{cases}$$

Here $\mathcal{F}_e^B$ stands for the extended Dirichlet space of $(\mathcal{E}^B, \mathcal{F}^B)$. Now by applying Theorem 1.1 and Remark 2.2 (ii) to X_t^B, we obtain the asymptotic formula for the total occupation times as follows:

Proposition 3.1. *Assume that X_t^B is irreducible and that Y_t^K is symmetric with respect to m. Then, for any $i \in I^B$*

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log P_i \left(\int_0^{J_B} I_K(X_s)ds > \beta \right)$$

$$= \; -\inf \left\{ \check{\mathcal{E}^B}(u,u) : u \in \check{\mathcal{F}^B},\ \sum_{i\in K} u^2(i)m_i = 1 \right\}. \tag{9}$$

Now, let us consider some concrete examples of (9), for which we mainly concentrate on the calculations of exponential decay rates represented by Dirichlet forms.

Example 3.1. Let $B = I \times D$ $(D \subset I)$ and $K \subset I \setminus D$. Then, J_B is the hitting time σ_D of D, $\sigma_D = \inf\{t > 0 : X_t \in D\}$, and $H_K u(j) = E_j(u(X_{\sigma_K}) : \sigma_K < \sigma_D)$. If X_t^B is irreducible, then the assumptions in Proposition 3.1 are fulfilled. Hence, we have

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log P_j \left(\int_0^{\sigma_D} I_K(X_s)ds > \beta \right)$$

$$= \; -\inf \left\{ \mathcal{E}^B(H_K u, H_K u) : \sum_{i\in K} u^2(i)m_i = 1 \right\} \tag{10}$$

for any $j \in I \setminus D$. In particular, let K be a one point set $\{a\}$. Then, since

$$H_{\{a\}}u(j) = E_j(u(X_{\sigma_{\{a\}}}) : \sigma_{\{a\}} < \sigma_D) = u(a)P_j(\sigma_{\{a\}} < \sigma_D)$$

and $\mathrm{Cap}^{I\setminus D}(\{a\}) = \mathcal{E}^B(P_j(\sigma_{\{a\}} < \sigma_D), P_j(\sigma_{\{a\}} < \sigma_D))$ on account of Theorem 4.3.3 in [6], we obtain

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log P_j \left(\int_0^{\sigma_D} I_{\{a\}}(X_s)ds > \beta \right)$$

$$= \; -\inf \left\{ \mathcal{E}^B(H_{\{a\}}u, H_{\{a\}}u) : u^2(a)m_a = 1 \right\}$$

$$= \; -\frac{\mathrm{Cap}^{I\setminus D}(\{a\})}{m_a}, \tag{11}$$

where $\mathrm{Cap}^{I\setminus D}(\{a\})$ is the 0-order capacity defined by $\mathrm{Cap}^{I\setminus D}(\{a\}) = \inf\{\mathcal{E}^B(u,u) : u \in \mathcal{F},\ u(a) = 1,\ u(i) = 0\ \forall i \in D\}$ ([6]). Indeed, let $G(i,j)$ be the green function of the part process on $I \setminus D$ and $F(a) = P_a(\hat{\sigma}_{\{a\}} < \infty), \sigma_{\{a\}} = \inf\{t > \sigma_{I\setminus\{a\}} : X_t = a\}$. Denote by $R(a)$ the expected number of visits to a with respect to P_a. Then, since

$$R(a) = 1 + F(a) + F^2(a) + \cdots = \frac{1}{1 - F(a)}, \quad G(a,a)m_a = \frac{R(a)}{q_{aa}},$$

and

$$\mathrm{Cap}^{I\setminus D}(\{a\}) = \frac{1}{G(a,a)},$$

we see that

$$\frac{\mathrm{Cap}^{I\setminus D}(\{a\})}{m_a} = q_{aa}(1 - F(a)).$$

Hence, equation (11) can also be derived from Proposition (5.3) in [1].

Example 3.2. Let $B = \{a\} \times D$ $(D \subset I \setminus \{a\})$. Then

$$\mathcal{E}^B(u,v) = \mathcal{E}(u,v) + \sum_{j \in D} u(a)v(j)q_{aj}m_a.$$

Assume that X_t^B is irreducible and K is a one point set $\{a\}$. We then have

$$\begin{aligned} &\lim_{\beta\to\infty} \frac{1}{\beta} \log P_j \left(\int_0^{J_B} I_{\{a\}}(X_s)ds > \beta \right) \\ (12) \qquad &= \quad -\inf\left\{\mathcal{E}^B(H_{\{a\}}u, H_{\{a\}}u) : u^2(a)m_a = 1\right\} \end{aligned}$$

for any $j \in I^B$. By again noting $H_{\{a\}}u(j) = E_j(u(X_{\sigma_{\{a\}}}) : \sigma_{\{a\}} < J_B) = u(a)P_j(\sigma_{\{a\}} < \zeta)$, we see that the right-hand side of (12) is equal to

$$\begin{aligned} &u^2(a)\mathcal{E}(P_j(\sigma_{\{a\}} < \zeta), P_j(\sigma_{\{a\}} < \zeta)) + \sum_{j\in D} u^2(a)q_{aj}m_aP_j(\sigma_{\{a\}} < \zeta) \\ (13) \qquad &= \frac{\mathrm{Cap}(\{a\})}{m_a} + \sum_{j\in D} q_{aj}P_j(\sigma_{\{a\}} < \infty). \end{aligned}$$

Here Cap is 0-order capacity associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$. If X_t is recurrent, then $\mathrm{Cap}(\{a\}) = 0$ and $P_j(\sigma_{\{a\}} < \infty) = 1$. Moreover, since $\int_0^{J_B} I_{\{a\}}(X_s)ds$ is a random variable with the exponential distribution with respect to P_a, we see that $\int_0^{J_B} I_{\{a\}}(X_s)ds$ has the exponential distribution of parameter

$$\sum_{j\in D} q_{aj}.$$

As a result, we rediscover D. Freedman's probabilistic interpretation of the component q_{ij} (Theorem (2) in [5]) through the large deviation theory.

Example 3.3. Let $I = \{0, 1, 2, \ldots\}$ and X be a birth and death process with birth rate ϕ_i and death rate φ_i on I. We denote by

$$s_0 = 0, \quad s_1 = \frac{1}{\phi_0}, \quad s_n = \frac{1}{\phi_0} + \sum_{i=1}^{n-1} \frac{1}{\phi_i m_i}$$

where

$$m_0 = 1, \qquad m_i = \frac{\phi_0\phi_1\cdots\phi_{i-1}}{\varphi_1\varphi_2\cdots\varphi_i}, \quad i = 1, 2, \ldots.$$

Then the corresponding Dirichlet form on $L^2(I;m)$ of X is as follows:

$$\begin{aligned} \mathcal{E}(u,v) &= \sum_{i=0}^{\infty} \left(\frac{u(i+1)-u(i)}{s_{i+1}-s_i}\right)\left(\frac{v(i+1)-v(i)}{s_{i+1}-s_i}\right)(s_{i+1}-s_i) \\ &= \sum_{i=0}^{\infty} (u(i+1)-u(i))(v(i+1)-v(i))\phi_i m_i, \qquad u, v \in \mathcal{F}. \end{aligned}$$

Let $B = I \times \{n\}$ and $K = \{a, a+1, \ldots, a+k-1\}$ $(0 \le a < a+k-1 < n)$. Then the terminal time J_B is the hitting time $\sigma_{\{n\}}$ of $\{n\}$. By observing $P_i\left(\sigma_K < \sigma_{\{n\}}\right) = 1$ for $0 \le i \le a+k-1$ and

$$P_i\left(\sigma_K < \sigma_{\{n\}}\right) = \frac{s_n - s_i}{s_n - s_{a+k-1}} \qquad \text{for} \quad a+k \le i < n,$$

we have

$$\mathcal{E}^B(H_K u, H_K u) = \sum_{i=a}^{a+k-2} (u(i+1) - u(i))^2 \phi_i m_i + \frac{u^2(a+k-1)}{s_n - s_{a+k-1}}. \tag{14}$$

Consequently, the right-hand side of (9) is the extremum problem of (14) with condition

$$\sum_{i=a}^{a+k-1} u^2(i) m_i = 1. \tag{15}$$

For convenience, we put $u(a+i) = x_i$, $i = 0, 1, \ldots, k-1$,

$$G(x_0, \ldots, x_{k-1}) = \sum_{i=1}^{k-1} (x_i - x_{i-1})^2 \phi_i m_i + \frac{x_{k-1}^2}{s_n - s_{a+k-1}}$$

and

$$F(\lambda, x_0, \ldots, x_{k-1}) = G(x_0, \ldots, x_{k-1}) + \lambda \left(\sum_{i=0}^{k-1} x_i^2 m_i - 1 \right).$$

Then by (15)

$$\frac{x_0}{2}\frac{\partial F}{\partial x_0} + \frac{x_1}{2}\frac{\partial F}{\partial x_1} + \cdots + \frac{x_{k-1}}{2}\frac{\partial F}{\partial x_{k-1}} = G(x_0, \ldots, x_{k-1}) + \lambda = 0.$$

Hence we conclude that

$$\begin{aligned} &\lim_{\beta\to\infty} \frac{1}{\beta} \log P_i \left(\int_0^{\sigma_{\{n\}}} I_K(X_s)\, ds > \beta \right) \\ &\quad = \quad -\inf \left\{ \mathcal{E}(H_K u, H_K u) : \sum_{i\in K} u^2(i) m_i = 1 \right\} \\ &\quad = \quad \text{the maximum solution of } \det(\boldsymbol{A}) = 0, \end{aligned} \tag{16}$$

where

$$\boldsymbol{A} = \begin{pmatrix} U_a & -V_a & 0 & \cdots & 0 \\ -V_a & V_a + U_{a+1} & -V_{a+1} & \cdots & 0 \\ 0 & -V_{a+1} & V_{a+1} + U_{a+2} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & 0 & -V_{a+k-2} & \dfrac{1}{S_n - S_{a+k-1}} + U_{a+k-1} \end{pmatrix},$$

$U_i = \phi_i m_i + \lambda m_i$ and $V_i = \phi_i m_i,\ i = a, \ldots, a+k-1$.

4. The large deviation principle for additive functionals

In this section, we further assume that the state space I is finite and the lifetime of $\mathbf{M}$ is infinity, $P_i(\zeta < \infty) = 0$. Let us consider the discontinuous additive functional

$$A_t^{\mu,F} = A_t^{\mu} + \sum_{0<s\le t} F(X_{s-}, X_s),$$

where F is a Borel function on $I \times I$ such that $F(i,j) = F(j,i)$ (symmetry), $F(i,i) = 0$. Then the symmetric bilinear form $\mathcal{E}^{\mu,F}$ on $L^2(I;m)$ generated by the semigroup

$$p_t^{\mu,F} f(i) = E_i \left(\exp \left(A_t^{\mu,F} \right) f(X_t) \right)$$

can be written as

$$\begin{aligned} \mathcal{E}^{\mu,F}(u,v) &= \frac{1}{2} \sum_{i \neq j} (u(j) - u(i))(v(j) - v(i)) q_{ij} m_i \\ &\quad - \sum_i u(i)v(i)\mu_i - \sum_{i \neq j} u(i)v(j) \left(1 - e^{F(i,j)} \right) q_{ij} m_i, \end{aligned} \tag{17}$$

and we have from [7]

$$\lim_{t \to \infty} \frac{1}{t} \log E_i \left(\exp \left(\theta A_t^{\mu,F} \right) \right) = C(\theta) \quad \text{for all } i \in I, \tag{18}$$

where

$$C(\theta) = -\inf \left\{ \mathcal{E}^{\theta\mu,\theta F}(u,u) : \sum_{i \in I} u^2(i) m_i = 1 \right\}.$$

Let us denote by u_θ the function attaining the infimum of the right-hand side above, that is, the first eigenfunction with eigenvalue $-C(\theta)$. Now, let us express $\mathcal{E}^{\theta\mu,\theta F}$ as a power series

$$\mathcal{E}^{\theta\mu,\theta F}(u,u) = \mathcal{E}(u,u) + \theta \mathcal{E}^{(1)}(u,u) + \theta^2 \mathcal{E}^{(2)}(u,u) + \cdots .$$

Then $\mathcal{E}^{(1)}$ is written as

$$\begin{aligned} \mathcal{E}^{(1)}(u,u) &= \sum_{i \in I} u^2(i)\mu_i - \sum_{i \neq j} (u(j) - u(i))^2 F(i,j) q_{ij} m_i \\ &\quad + \sum_i u^2(i) \left(\sum_j F(i,j) q_{ij} m_i \right). \end{aligned} \tag{19}$$

We see from [8] that $C(\theta)$ is differentiable with respect to θ and $C'(\theta) = \mathcal{E}^{(1)}(u_\theta, u_\theta)$. In particular, since $u_0 = 1/\sqrt{m(I)}$ (constant), we see that

$$C'(0) = \frac{\mu(I) + \sum_{i \neq j} F(i,j) q_{ij} m_i}{m(I)}.$$

The Gärtner and Ellis theorem ([3]) tells us that $A_t^{\mu,F}/t$ obeys the large deviation principle with rate function $I(\lambda)(= \sup\{\lambda\theta - C(\theta)\})$; for any Borel set Λ of R^1,

$$-I(\Lambda^\circ) \leq \liminf_{t \to \infty} \frac{1}{t} \log -P_i \left(\frac{A_t^{\mu,F}}{t} \in \Lambda \right) \leq \limsup_{t \to \infty} \frac{1}{t} \log P_i \left(\frac{A_t^{\mu,F}}{t} \in \Lambda \right) \leq -I(\overline{\Lambda}), \tag{20}$$

where Λ° and $\overline{\Lambda}$ are the interior and closure of Λ respectively, and $I(\Lambda)$ stands for $I(\Lambda) = \inf\{I(\lambda) : \lambda \in \Lambda\}$. Therefore, we obtain that $A_t^{\mu,F}/t$ converges to $C'(0)$ exponentially as $t \to \infty$ (cf. [3]); for any $\epsilon > 0$, there exist constants $k_1,\ k_2 > 0$ such that

$$P_i \left(\left| \frac{A_t^{\mu,F}}{t} - C'(0) \right| > \epsilon \right) \leq k_1 e^{-k_2 t}.$$

References

[1] Cinlar, E.: *Introduction to Stochastic Processes*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1975. MR **52:**1809

[2] Dembo, A., Zeitouni, O.: *Large Deviations Techniques and Applications*, Second edition. Applications of Mathematics, 38. Springer-Verlag, New York, 1998. MR **99d:**60030

[3] Ellis, R. S.: Large deviations for a general class of random vectors, Ann. Probab. **12** (1984), 1–12. MR **85e:**60032

[4] Fitzsimmons, P. J.: On the quasi-regularity of semi-Dirichlet forms, preprint.

[5] Freedman, D.: *Approximating Countable Markov Chains*, Holden-Day, 1972. MR **55:**1493

[6] Fukushima, M., Oshima, Y., Takeda, M.: *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics, 19, Walter de Gruyter & Co., Berlin, 1994. MR **96f:**60126

[7] Kim, D.: Asymptotic properties for continuous and jump type's Feynman-Kac functionals, Osaka J. Math. **37** (2000), 147–173. CMP 2000:10

[8] Kato, T.: *A Short Introduction to Perturbation Theory for Linear Operators*, Springer-Verlag, 1982. MR **83m:**47015

[9] Sharpe, M.: *General Theory of Markov Processes*, Academic Press, 1988. MR **89m:**60169

[10] Silverstein, M.: *Symmetric Markov Processes*, Springer Lect. Notes in Math. 426, 1974. MR **52:**6891

[11] Takeda, M.: On a large deviation for symmetric Markov processes with finite lifetime, Stochastics and Stochastic Reports **59** (1996), 143–167. MR **98b:**60057

[12] Takeda, M.: Exponential decay of life times and a theorem of Kac on total occupation times, Potential Analysis **11** (1999), 235–247. MR **2000i:**60084

[13] Ying, J.: Bivariate Revuz measures and the Feyman-Kac formula, Ann. Inst. Henri Poincare, **32** (1995), 251–287. MR **97j:**60139

Department of Mathematics, Pusan National University, Pusan, 609–735, Korea

Current address: Department of Systems and Information, Graduate School of Science and Technology, Kumamoto University, Kurokami, 2-39-1, Kumamoto 860-8555, Japan

E-mail address: `daehong@cs.kumamoto-u.ac.jp`

Mathematical Institute, Tohoku University, Sendai 980–8578, Japan

E-mail address: `takeda@math.tohoku.ac.jp`

Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China

E-mail address: `jying@math.zju.edu.cn`