QUADRATIC INITIAL IDEALS OF ROOT SYSTEMS

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ABSTRACT. Let \( \Phi \subset \mathbb{Z}^n \) be one of the root systems \( \mathcal{A}_{n-1}, \mathcal{B}_n, \mathcal{C}_n \) and \( \mathcal{D}_n \) and write \( \Phi^{(+) \downarrow} \) for the set of positive roots of \( \Phi \) together with the origin of \( \mathbb{R}^n \). Let \( K[t, t^{-1}, s] \) denote the Laurent polynomial ring \( K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s] \) over a field \( K \) and write \( R_K[\Phi^{(+) \downarrow}] \) for the affine semigroup ring which is generated by those monomials \( t^a \) with \( a \in \Phi^{(+) \downarrow} \), where \( t^a = t_1^{a_1} \cdots t_n^{a_n} \) if \( a = (a_1, \ldots, a_n) \). Let \( K[\Phi^{(+) \downarrow}] = K[[x_a; a \in \Phi^{(+) \downarrow}]] \) denote the polynomial ring over \( K \) and write \( I_{\Phi^{(+) \downarrow}} \subset K[\Phi^{(+) \downarrow}] \) for the toric ideal of \( \Phi^{(+) \downarrow} \). Thus \( I_{\Phi^{(+) \downarrow}} \) is the kernel of the surjective homomorphism \( \pi : K[\Phi^{(+) \downarrow}] \to R_K[\Phi^{(+) \downarrow}] \) defined by setting \( \pi(x_a) = t^a \) for all \( a \in \Phi^{(+) \downarrow} \). In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov discovered a quadratic initial ideal of the toric ideal \( I_{\Phi^{(+) \downarrow}} \) of \( \mathcal{A}_{n-1}^{(+) \downarrow} \).

The purpose of the present paper is to show the existence of a reverse lexicographic (squarefree) quadratic initial ideal of the toric ideal of each of \( \mathcal{B}_n^{(+) \downarrow}, \mathcal{C}_n^{(+) \downarrow} \) and \( \mathcal{D}_n^{(+) \downarrow} \). It then follows that the convex polytope of the convex hull of each of \( \mathcal{B}_n^{(+) \downarrow}, \mathcal{C}_n^{(+) \downarrow} \) and \( \mathcal{D}_n^{(+) \downarrow} \) possesses a regular unimodular triangulation arising from a flag complex, and that each of the affine semigroup rings \( R_K[\mathcal{B}_n^{(+) \downarrow}], R_K[\mathcal{C}_n^{(+) \downarrow}] \) and \( R_K[\mathcal{D}_n^{(+) \downarrow}] \) is Koszul.

INTRODUCTION

A configuration in \( \mathbb{R}^n \) is a finite set \( \mathcal{A} \subset \mathbb{Z}^n \). Let \( K[t, t^{-1}, s] \) denote the Laurent polynomial ring \( K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s] \) over a field \( K \). We associate a configuration \( \mathcal{A} \subset \mathbb{Z}^n \) with the homogeneous semigroup ring \( R_K[\mathcal{A}] = K[\{t^a; a \in \mathcal{A}\}] \), the subalgebra of \( K[t, t^{-1}, s] \) generated by all monomials \( t^a \) with \( a \in \mathcal{A} \), where \( t^a = t_1^{a_1} \cdots t_n^{a_n} \) if \( a = (a_1, \ldots, a_n) \). Let \( K[\mathcal{A}] = K[\{x_a; a \in \mathcal{A}\}] \) be the polynomial ring in the variables \( x_a \) with \( a \in \mathcal{A} \) over \( K \). The toric ideal \( I_{\mathcal{A}} \) of \( \mathcal{A} \) is the kernel of the surjective homomorphism \( \pi : K[\mathcal{A}] \to R_K[\mathcal{A}] \) defined by setting \( \pi(x_a) = t^a \) for all \( a \in \mathcal{A} \). It is known [10] Lemma 4.1 that the toric ideal \( I_{\mathcal{A}} \) is generated by the binomials \( u - v \), where \( u \) and \( v \) are monomials of \( K[\mathcal{A}] \), with \( \pi(u) = \pi(v) \).

Before discussing the details of the present paper, we recall fundamental material on initial ideals of toric ideals. Let \( \mathcal{M}(K[\mathcal{A}]) \) denote the set of monomials belonging to \( K[\mathcal{A}] \). Thus, in particular, \( 1 \in \mathcal{M}(K[\mathcal{A}]) \). Fix a monomial order \( < \) on \( K[\mathcal{A}] \); thus < is a total order on \( \mathcal{M}(K[\mathcal{A}]) \) such that (i) \( 1 < u \) if \( 1 \neq u \in \mathcal{M}(K[\mathcal{A}]) \) and (ii) for \( u, v, w \in \mathcal{M}(K[\mathcal{A}]) \), if \( u < v \) then \( uv < vw \). The initial monomial \( \text{in}_<(f) \) of \( 0 \neq f \in I_{\mathcal{A}} \) with respect to < is the biggest monomial appearing in \( f \) with respect to <. The initial ideal of \( I_{\mathcal{A}} \) with respect to < is the ideal \( \text{in}_<(I_{\mathcal{A}}) \) of \( K[\mathcal{A}] \) generated

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by all initial monomials \( \text{in}_<(f) \) with \( 0 \neq f \in I_\mathcal{A} \). One of the most fundamental facts on the initial ideal \( \text{in}_<(I_\mathcal{A}) \) is that \( \{ \text{in}_<(u) \colon u \in \mathcal{M}(K[\mathcal{A}]), u \notin \text{in}_<(I_\mathcal{A}) \} \) is a \( K \)-basis of \( \mathcal{R}_K[\mathcal{A}] \). An initial ideal \( \text{in}_<(I_\mathcal{A}) \) is called quadratic (resp. squarefree) if \( \text{in}_<(I_\mathcal{A}) \) is generated by quadratic (resp. squarefree) monomials. Let, in general, \( \mathcal{G} \) be a finite subset of \( I_\mathcal{A} \) and write \( \text{in}_<(\mathcal{G}) \) for the ideal \( (\text{in}_<(g) : g \in \mathcal{G}) \) of \( K[\mathcal{A}] \). A finite set \( \mathcal{G} \) of \( I_\mathcal{A} \) is said to be a \( \text{Gröbner basis} \) of \( I_\mathcal{A} \) with respect to \( < \) if \( \text{in}_<(\mathcal{G}) = \text{in}_<(I_\mathcal{A}) \).

Dickson’s Lemma [3] p. 69], which says that any nonempty subset of \( \mathcal{M}(K[\mathcal{A}]) \) (in particular, \( \text{in}_<(I_\mathcal{A}) \cap \mathcal{M}(K[\mathcal{A}]) \)) has only finitely many minimal elements in the partial order by divisibility, guarantees that a \( \text{Gröbner basis} \) of \( I_\mathcal{A} \) with respect to \( < \) always exists. Moreover, if \( \mathcal{G} \) is a \( \text{Gröbner basis} \) of \( I_\mathcal{A} \), then \( I_\mathcal{A} \) is generated by \( \mathcal{G} \).

Even though the following fact (0.1) on \( \text{Gröbner bases} \) is simple and well-known (e.g., [1] Lemma 1.1) and, in fact, can be easily proved, this technique will play important roles throughout the present paper.

(0.1) A finite set \( \mathcal{G} \) of \( I_\mathcal{A} \) is a \( \text{Gröbner basis} \) of \( I_\mathcal{A} \) with respect to \( < \) if and only if \( \{ \text{in}_<(u) : u \in \mathcal{M}(K[\mathcal{A}]), u \notin \text{in}_<(\mathcal{G}) \} \) is linearly independent over \( K \); in other words, if and only if \( \text{in}_<(u) \neq \text{in}_<(v) \) for all \( u \notin \text{in}_<(\mathcal{G}) \) and \( v \notin \text{in}_<(\mathcal{G}) \) with \( u \neq v \).

Consult, e.g., [4] and [5] for detailed information about \( \text{Gröbner bases} \) and related topics on commutative algebra, algebraic geometry and combinatorics.

Let \( \Phi \subset \mathbb{Z}^n \) be one of the classical irreducible root systems \( \Phi_{n-1} \), \( \Phi_n \), \( \Phi_n \) and \( \Phi_n \) (8 pp. 64–65) and write \( \Phi^{(+)} \) for the configuration consisting of the origin of \( \mathbb{R}^n \) together with all positive roots of \( \Phi \). More explicitly,

\[
\begin{align*}
A^{(+)}_{n-1} &= \{0\} \cup \{e_i - e_j : 1 \leq i < j \leq n\}, \\
B^{(+)}_n &= A^{(+)}_{n-1} \cup \{e_1, \ldots, e_n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\}, \\
C^{(+)}_n &= A^{(+)}_{n-1} \cup \{2e_1, \ldots, 2e_n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\}, \\
D^{(+)}_n &= A^{(+)}_{n-1} \cup \{e_i + e_j : 1 \leq i < j \leq n\}.
\end{align*}
\]

Here \( e_i \) is the \( i \)-th unit coordinate vector of \( \mathbb{R}^n \) and \( O \) is the origin of \( \mathbb{R}^n \).

In the study of combinatorics of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [7] discovered a squarefree quadratic initial ideal of the toric ideal of the configuration \( A^{(+)}_{n-1} \). Let

\[
K[\mathcal{A}^{(+)}_{n-1}] = K[\{x\} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]
\]

denote the polynomial ring in \( n(n-1)/2 + 1 \) variables over \( K \). The affine semigroup ring \( \mathcal{R}_K[\mathcal{A}^{(+)}_{n-1}] \) is the subalgebra of \( K[t, t^{-1}, s] \) generated by \( s \) together with the \( n(n-1)/2 \) monomials \( t_it_j^{-1}s \) with \( 1 \leq i < j \leq n \). The toric ideal \( I_{A^{(+)}_{n-1}} \) is the kernel of the surjective homomorphism \( \pi : K[\mathcal{A}^{(+)}_{n-1}] \to \mathcal{R}_K[\mathcal{A}^{(+)}_{n-1}] \) defined by setting \( \pi(x) = s \) and \( \pi(f_{i,j}) = t_it_j^{-1}s \). Write \( <_{\text{rev}} \) for the reverse lexicographic order on \( K[\mathcal{A}^{(+)}_{n-1}] \) induced by the ordering of the variables satisfying (i) \( x < f_{i,j} \) for all \( 1 \leq i < j \leq n \) and (ii) \( f_{i,j} < f_{i',j'} \) if either (a) \( i < i' \), or (b) \( i = i' \) and \( j > j' \). Then [7] Theorem 6.3 guarantees the following:

(0.2) The set of binomials

\[
\begin{align*}
f_{i,k}f_{j,k} - f_{i,k}f_{j,k}, & \quad i < j < k < \ell, \\
f_{i,j}f_{j,k} - xf_{i,k}, & \quad i < j < k,
\end{align*}
\]

is a \( \text{Gröbner basis} \) of \( I_{A^{(+)}_{n-1}} \) with respect to \( <_{\text{rev}} \).
Since \( f_{i,j,k} \) generated by the squarefree quadratic monomials \( f_{i,k,j,\ell} \) with \( i < j < k < \ell \), it follows that \( \text{in}_{<_{rev}}(I_{A_{n-1}^{(+)}}) \) is generated by the squarefree quadratic monomials \( f_{i,k,j,\ell} \) with \( i < j < k < \ell \) and \( f_{i,j,k} \) with \( i < j < k \). (In [7] Theorem 6.3), instead of using the notion of Gröbner bases and initial ideals, Gelfand, Graev and Postnikov state and prove their result in terms of triangulations of the convex hull of the configuration \( A_{n-1}^{(+)} \). A simple and quick proof of the above fact (0.2) is obtained by applying the technique (0.1).

Consult, e.g., (Case I) with \( r = p = 0 \) in the proof of Theorem [1.1].

It turns out that the question whether the toric ideal of each of the configurations \( B_{n}^{(+)}, C_{n}^{(+)}, D_{n}^{(+)}, \) possesses a squarefree quadratic initial ideal is reasonable. In fact, our goal of the present paper is to show the existence of a reverse lexicographic ordering of the variables. In fact, the toric ideal of the configuration consisting of all positive roots of the root system \( \Phi \subset \mathbb{Z}^{n} \), where \( \Phi \) is one of the root systems \( A_{n-1}, B_{n}, C_{n}, D_{n} \), is not generated by quadratic binomials if \( n \geq 6 \).

1. The root system \( B_{n} \)

First of all, the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of the configuration \( B_{n}^{(+)}. \) will be proved. Let

\[
K[B_{n}^{(+)}) = K[x] \cup \{y_{i}\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}
\]

be the polynomial ring in \( n^{2} + 1 \) variables over \( K \). The affine semigroup ring \( \mathcal{R}[K[B_{n}^{(+)}) \) is the subalgebra of \( K[t, t^{-1}, s] \) generated by the \( n^{2} + 1 \) monomials

\[
s, \ t_{i}s (1 \leq i \leq n), \ t_{i}t_{j}s (1 \leq i < j \leq n), \ t_{i}t_{j}^{-1}s (1 \leq i < j \leq n).
\]

The toric ideal \( I_{B_{n}^{(+)}} \) is the kernel of the surjective homomorphism \( \pi : K[B_{n}^{(+)}) \rightarrow \mathcal{R}[K[B_{n}^{(+)}) \) defined by setting

\[
\pi(x) = s, \ \pi(y_{i}) = t_{i}s, \ \pi(e_{i,j}) = t_{i}t_{j}s, \ \pi(f_{i,j}) = t_{i}t_{j}^{-1}s.
\]

We now introduce the reverse lexicographic order \( <_{rev} \) on \( K[B_{n}^{(+)}) \) induced by the ordering of the variables

\[
y_{1} < y_{2} < \cdots < y_{n} < f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3} < \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n} < e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3} < \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-1,n} \cdot x.
\]

To simplify the notation below, we understand \( e_{j,i} = e_{i,j} \) in case of \( i < j \).
Theorem 1.1. Under the above conditions, the set of the binomials
\[ e_{i,j} e_{k,\ell} - e_{i,\ell} e_{j,k}, \quad i < j < k < \ell, \]
\[ e_{i,k} e_{j,\ell} - e_{i,\ell} e_{j,k}, \quad i < j < k < \ell, \]
\[ f_{i,k} f_{j,\ell} - f_{i,\ell} f_{j,k}, \quad i < j < k < \ell, \]
\[ f_{i,j} f_{j,k} - x f_{i,k}, \quad i < j < k, \]
\[ e_{i,j} f_{k,\ell} - e_{i,\ell} f_{j,k}, \quad i < j < k < \ell, \]
\[ e_{i,\ell} f_{j,k} - f_{i,k} e_{j,\ell}, \quad i < j < k < \ell, \]
\[ f_{i,j} e_{j,k} - y f_{i,k}, \quad i < j, \quad j \neq k, \]
\[ e_{i,j} y - y e_{j,k}, \quad i < j < k, \]
\[ e_{i,k} y - y e_{j,k}, \quad i < j < k, \]
\[ f_{i,k} y - y f_{j,k}, \quad i < j < k, \]
\[ f_{i,j} y - y f_{j,k}, \quad i < j, \]
\[ x e_{i,j} - y e_{j,k}, \quad i < j, \]
\[ f_{i,j} y - y f_{j,k}, \quad i < j < k. \]
is a Gröbner basis of \( I_{B_n^+} \) with respect to the reverse lexicographic order \( <_{rev} \).

Proof. Each binomial \( g = u - v \) listed above belongs to \( I_{B_n^+} \) with \( u \) of its initial monomial \( \text{in}_{<_{rev}} (g) \). Let \( \mathcal{G} \) denote the set of binomials listed above and \( \text{in}_{<_{rev}} (\mathcal{G}) = (\text{in}_{<_{rev}} (g) : g \in \mathcal{G}) \). To see why the finite set \( \mathcal{G} \) is a Gröbner basis of \( I_{B_n^+} \), we rely on the technique (0.1), and our work is to show that
\[ \{ \pi(u) : u \in \mathcal{M}(K[B_n^+]), u \not\in \text{in}_{<_{rev}} (\mathcal{G}) \} \]
is linearly independent over \( K \).

Let
\[ u = x^{a_1} y_{k_1} \cdots y_{k_r} e_{a_1,b_1} \cdots e_{a_p,b_p} f_{i_1,j_1} \cdots f_{i_q,j_q}, \]
\[ u' = x^{a_1'} y_{k_1'} \cdots y_{k_r'} e_{a_1',b_1'} \cdots e_{a_p',b_p'} f_{i_1',j_1'} \cdots f_{i_q',j_q'} \]
belong to \( \mathcal{M}(K[B_n^+]) \) with \( u \not\in \text{in}_{<_{rev}} (\mathcal{G}) \) and \( u' \not\in \text{in}_{<_{rev}} (\mathcal{G}) \), where
\[ y_{k_i} \leq_{rev} \cdots \leq_{rev} y_{k_r}, \quad e_{a_1,b_1} \leq_{rev} \cdots \leq_{rev} e_{a_p,b_p}, \quad f_{i_1,j_1} \leq_{rev} \cdots \leq_{rev} f_{i_q,j_q}, \]
\[ y_{k_i'} \leq_{rev} \cdots \leq_{rev} y_{k_r'}, \quad e_{a_1',b_1'} \leq_{rev} \cdots \leq_{rev} e_{a_p',b_p'}, \quad f_{i_1',j_1'} \leq_{rev} \cdots \leq_{rev} f_{i_q',j_q'}. \]
In what follows, it will be proved that if \( \pi(u) = \pi(u') \), then at least one variable of \( K[B_n^+] \) appears in both \( u \) and \( u' \).

To begin with, the equalities \( \alpha = \alpha' \), \( r = r' \), \( p = p' \) and \( q = q' \) will be proved. Since \( f_{i,j} f_{j,k} \in \text{in}_{<_{rev}} (\mathcal{G}) \) if \( i < j < k \), \( f_{i,j} e_{j,k} \in \text{in}_{<_{rev}} (\mathcal{G}) \) if \( i < j \) with \( j \neq k \) and \( f_{i,j} y \in \text{in}_{<_{rev}} (\mathcal{G}) \) if \( i < j \), one has \( q = q' \), \( \alpha + r + p = \alpha' + r' + p' \) and \( r + 2p = r' + 2p' \). If \( \alpha = \alpha' = 0 \), then \( p = p' \) and \( r = r' \). If \( \alpha \geq \alpha' > 0 \), then \( p = p' = 0 \) since \( x e_{i,j} \in \text{in}_{<_{rev}} (\mathcal{G}) \) for all \( i < j \); thus \( r = r' \) and \( \alpha = \alpha' \). If \( \alpha = 0 \) and \( \alpha' > 0 \), then \( r + p = \alpha' + r' \) and \( r + 2p = r' + 2p' \); thus \( \alpha' + p = 0 \), a contradiction.

(Case 1) Let \( \alpha = \alpha' = 0 \) and \( q = q' > 0 \).

We will prove \( f_{i_1,j_1} = f_{i_1,j_1}' \). Let \( i_q < i_q' \). One has \( i_q' \leq i_q < j_q = j_q' \) for some \( 1 \leq q' \leq q \). Since \( f_{i,j} y \in \text{in}_{<_{rev}} (\mathcal{G}) \) if \( i < j < k \), there is no \( k_{j'} \), with \( i_q \leq k_{j'} < j_q \). Thus \( p = p' > 0 \) and \( \alpha'_1 \leq i_q \). Note, in particular, that \( i_q' = i_q \) if \( p = p' = 0 \).
(i) Let $a_1 < i_q$. Since $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$ and $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, one has $b_1 = i_q$. Since $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, it follows that $a_\xi \leq b_\xi = i_q$ for all $1 \leq \xi \leq p$. If $a_\xi < i_q$, then $b_\xi = i_q$. Hence, for each $1 \leq \xi \leq p$, one has either $a_\xi = i_q$ or $b_\xi = i_q$. Thus the total number of the variable $t_{i_q}$ appearing in $\pi(u)$ is at least $p + 1$. However, since $k_{\mu'} = i_q$ for no $1 \leq \mu \leq r$, the total number of the variable $t_{i_q}$ appearing in $\pi(u')$ is at most $p$, a contradiction.

(ii) Let $i_q \leq a_1$. If either $r = r' > 0$ or $r = r' > 0$ with $k_{\mu'} < i_q$, then the total number of variables $t_3$ with $\delta \geq i_q$ appearing in $\pi(u')$ (resp. $\pi(u)$) is at least $2p + 1$ (resp. at most $2p$).

If $r = r' > 0$ with $(i'_q < i_q \leq k_{\mu'}$, then $(j'_q = i_q$). Since $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k$ and $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k$, it follows from $a_1 < i_q$ that one has $b_1 = k'_{\mu'}$. In addition, if $k_{\mu'} < k'_{\mu'}$, then $k_{\mu'} \leq i'_q$ since $k_{\mu'} \leq a_1 < i'_q$. Again, since $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, there is no $1 \leq \xi \leq p$ with $(b'_1 = k'_{\mu'} < a'_\xi$. Thus, for each $1 \leq \xi \leq p$, one has either $a'_\xi = k'_{\mu'}$ or $b'_\xi = k'_{\mu'}$. Hence the total number of variables $t_3$ with $k_{\mu'} \neq \delta \geq i_q$ appearing in $\pi(u')$ is at most $p$. Since either $i_q < a_\xi \neq k'_{\mu'}$ or $i_q \leq b_\xi \neq k'_{\mu'}$ for each $1 \leq \xi \leq p$, the total number of variables $t_3$ with $k_{\mu'} \neq \delta \geq i_q$ appearing in $\pi(u)$ is at least $p + 1$.

This completes the proof of $i_q = i'_q$. Let $i'_q < j_q$. Since $f_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, there is no $1 \leq \eta \leq q$ with $i_q < i'_q = i'_q < j'_q < j_q$. Hence $j_q = j'_q$, as desired.

(Case II) Let $\alpha = \alpha' = 0$, $r = r' > 0$, $p = p' > 0$ and $q = q' = 0$.

If $k_1 \leq a_1$ and $k_{\mu'} \leq a_1$, then $k_1 = k_{\mu'}$. If $a_1 < k_1$, then $b_1 = k_{\mu'}$ for all $1 \leq \mu \leq r$. One has $a_\xi \leq b_1$ for all $1 \leq \xi \leq p$ and, moreover, $b_\xi = k_1$ if $a_\xi < b_1 = k_1$. Hence the total number of the variable $t_{k_1}$ appearing in $\pi(u)$ is $r + p$. It then follows that $k_{\mu'} = k_1$ for all $1 \leq \mu' \leq r$ and that each $a_{\mu'} < b_{\mu'}$ satisfies either $a_{\mu'} = k_1$ or $b_{\mu'} = k_1$.

(Case III) Let $\alpha = \alpha' = 0$, $r = r' = 0$, $p = p' > 0$ and $q = q' = 0$.

Even though the desired result follows from [7, Theorem 1.4], we give its quick proof here for the sake of completeness. Let $a_\eta < a_{\mu'}$. Since $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, each $b_{\xi}$ satisfies $a_\eta \leq b_{\xi}$. In addition, each $a'_{\xi}$ satisfies $a'_{\xi} < a_{\mu'}$. Hence the total number of variables $t_3$ with $\delta \geq a_\eta$ appearing in $\pi(u')$ (resp. $\pi(u)$) is at least $p + 1$ (resp. at most $p$), a contradiction. Thus $a_{\mu'} = a_\eta$. Let $b_{\mu'} < b_{\mu'}$. Since $e_i, j, k, \ell \in \mathfrak{n}_{< \mathfrak{q}}(G)$ if $i < j < k < \ell$, there is no $b_\xi$ with $b_\xi = b_{\mu'}$. Thus $b_{\mu'} = b_{\mu'}$, as required.

2. The root system $C_n$

Second, the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of the configuration $C_n^{(+)\+}$ will be proved. Let

$$K[C_n^{(+)\+}] = K[x_{1 \leq i \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

be the polynomial ring in $n^2 + 1$ variables over $K$. The affine semigroup ring $K[C_n^{(+)\+}]$ is the subalgebra of $K[t, t^{-1}, s]$ generated by the $n^2 + 1$ monomials

$$s, \quad t t_j s (1 \leq i \leq j \leq n), \quad t t_j^{-1} s (1 \leq i < j \leq n).$$

The toric ideal $I_{C_n^{(+)\+}}$ is the kernel of the surjective homomorphism $\pi : K[C_n^{(+)\+}] \rightarrow K[C_n^{(+)\+}]$ defined by setting

$$\pi(x) = s, \quad \pi(e_{i,j}) = t t_j s, \quad \pi(f_{i,j}) = t t_j^{-1} s.$$
We now introduce the reverse lexicographic order \(<_{\text{rev}}^{c}\) on \(K[C_n^{(+)\ast}]\) induced by the ordering of the variables
\[
x < f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3}
\]
\[
< \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n}
\]
\[
e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3} < e_{2,2}
\]
\[
< \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-2,n-2} < e_{n-1,n} < e_{n-1,n-1} < e_{n,n}.
\]
To simplify the notation below, we understand \(e_{j,i} = e_{i,j}\) in case of \(i < j\).

**Theorem 2.1.** The set of the binomials
\[
e_{i,j}e_{k,l} - e_{i,l}e_{k,j}, \quad i \leq j < k \leq l,
\]
\[
e_{i,k}e_{j,l} - e_{i,l}e_{j,k}, \quad i < j < k < l,
\]
\[
e_{i,j}e_{k,l} - e_{i,k}e_{j,l}, \quad i < j < k,
\]
\[
f_{i,j}f_{k,l} - f_{i,l}f_{j,k}, \quad i < j < k < l,
\]
\[
f_{i,j}f_{k,l} - xf_{i,k}, \quad i < j < k,
\]
\[
e_{i,j}f_{k,l} - f_{i,j}e_{k,l}, \quad i < j < k < l,
\]
\[
e_{i,k}f_{j,l} - f_{i,j}e_{k,l}, \quad i < j < k < l,
\]
\[
e_{i,j}f_{k,l} - f_{i,k}e_{j,l}, \quad i < j < k < l,
\]
\[
f_{i,j}e_{k,l} - xe_{i,k}, \quad i < j,
\]
is a Gröbner basis of \(I_{C_n^{(+)\ast}}\) with respect to the reverse lexicographic order \(<_{\text{rev}}^{c}\).

**Proof.** Each binomial \(g = u - v\) listed above belongs to \(I_{C_n^{(+)\ast}}\) with \(u \in in_{<_{\text{rev}}^{c}}(g)\). Let \(G\) denote the set of binomials listed above and \(in_{<_{\text{rev}}^{c}}(G) = (in_{<_{\text{rev}}^{c}}(g) : g \in G)\). Our proof will proceed along the proof of Theorem \([\,]\) and the goal is to show that
\[
\{ \pi(u) : u \in M(K[C_n^{(+)\ast}]), u \notin in_{<_{\text{rev}}^{c}}(G) \}
\]
is linearly independent over \(K\).

Let
\[
u = x^\alpha e_{a_1,b_1} \cdots e_{a_p,b_p} f_{i_1,j_1} \cdots f_{i_q,j_q},
\]
\[
u' = x'^\alpha e_{a'_1,b'_1} \cdots e_{a'_{p'},b'_{p'}} f_{i'_1,j'_1} \cdots f_{i'_{q'},j'_{q'}}
\]
belong to \(M(K[C_n^{(+)\ast}])\) with \(u \notin in_{<_{\text{rev}}^{c}}(G)\) and \(u' \notin in_{<_{\text{rev}}^{c}}(G)\), where
\[
e_{a_1,b_1} \leq_{\text{rev}} e_{a_2,b_2} \cdots \leq_{\text{rev}} e_{a_p,b_p}, \quad f_{i_1,j_1} \leq_{\text{rev}} \cdots \leq_{\text{rev}} f_{i_q,j_q},
\]
\[
e_{a'_1,b'_1} \leq_{\text{rev}} e_{a'_{p'},b'_{p'}}, \quad f_{i'_1,j'_1} \leq_{\text{rev}} \cdots \leq_{\text{rev}} f_{i'_{q'},j'_{q'}}.
\]
In what follows, it will be proved that if \(\pi(u) = \pi(u')\), then at least one variable of \(K[C_n^{(+)\ast}]\) appears in both \(u\) and \(u'\). Since \(f_{i,j}f_{j,k} \in in_{<_{\text{rev}}^{c}}(G)\) if \(i < j < k\) and \(f_{i,j}e_{j,k} \in in_{<_{\text{rev}}^{c}}(G)\) if \(i < j\), one has \(\alpha = \alpha', p = p'\) and \(q = q'\).

**(Case I)** Let \(\alpha = \alpha' = 0\) and \(q = q' > 0\).

We claim \(f_{i_q,j_q} = f_{i'_q,j'_q}\). Let \(i_q < i_q'.\) Since the monomials \(e_{i,j}f_{k,l} (i \leq j < k < \ell), e_{i,k}f_{j,l} (i < j < k < \ell), e_{i,j}f_{j,k} (i < j < k)\) and \(e_{i,k}f_{j,l} (i < j < k < \ell)\) belong to \(in_{<_{\text{rev}}^{c}}(G)\), there is no \(a_\ell\) with \(a_\ell < i_q'.\) Hence the total number of variables \(t_\delta\) with \(\delta \geq i_q'\) appearing in \(\pi(u)\) (resp. \(\pi(u')\)) is at least \(2p + 1\) (resp. at most \(2p\)), a
contradiction. Thus $i' = i_q$. Since $f_{i,j,k} \in \text{in}_{<\text{rev}} \mathcal{G}$ if $i < j < k$, it follows that $j_q = j'_q$, as desired.

(\textbf{Case II}) Let $\alpha = \alpha' = 0$ and $q = q' = 0$.

A slight modification of Case III in the proof of Theorem 1.1 is valid. Let $a'_p < a_p$. Since $e_{i,j}e_{k,l} \in \text{in}_{<\text{rev}} \mathcal{G}$ if $i < j < k < \ell$, each $b_k$ satisfies $a_p \leq b_k$. In addition, each $a'_{i,j,k}$ satisfies $a'_{i,j,k} \leq a'_p < a_p$. Hence the total number of variables $t_{ij} = \delta \geq a_p$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least $p + 1$ (resp. at most $p$), a contradiction. Thus $a'_p = a_p$. Let $b'_p < b_p$. Since $e_{i,j}e_{k,l} \in \text{in}_{<\text{rev}} \mathcal{G}$ if $i < j < k < \ell$ and $e_{i,j}e_{k,l} \in \text{in}_{<\text{rev}} \mathcal{G}$ if $i < j < k$, each $b_k$ satisfies $b_p \leq b_k$. Moreover, each $a'_{i,j,k}$ satisfies $a'_{i,j,k} \leq a'_p < b'_p < b_p$. Hence the total number of variables $t_{ij} = \delta \geq b_p$ appearing in $\pi(u)$ (resp. $\pi(u')$) is at least $p$ (resp. at most $p - 1$), a contradiction. Thus $b'_p = b_p$, as required.

\section{3. The root system $D_n$}

Finally, the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of the configuration $D_n^{(+)}$ will be proved. Let

$$K[D_n^{(+)}] = K[\{x\} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

be the polynomial ring in $n^2 - n + 1$ variables over $K$. The affine semigroup ring $\mathcal{R}_K[D_n^{(+)}]$ is the subalgebra of $K[t, t^{-1}, s]$ generated by the $n^2 - n + 1$ monomials

$$s, \ t_it_js \ (1 \leq i < j \leq n), \ t_it_j^{-1}s \ (1 \leq i < j \leq n).$$

The toric ideal $I_{D_n^{(+)}}$ is the kernel of the surjective homomorphism $\pi : K[D_n^{(+)}] \to \mathcal{R}_K[D_n^{(+)}]$ defined by setting

$$\pi(x) = s, \ \pi(e_{i,j}) = t_it_js, \ \pi(f_{i,j}) = t_it_j^{-1}s.$$

We now introduce the reverse lexicographic order $<_{\text{rev}}$ on $K[D_n^{(+)}]$ induced by the ordering of the variables

$$f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3}$$

$$< \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n}$$

$$< e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3}$$

$$< \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-1,n} < x.$$

\textbf{Theorem 3.1.} The set of the binomials

$$e_{i,j}e_{k,l} - e_{i,j}e_{k,l}, \quad i < j < k < \ell,$$

$$e_{i,j}e_{k,l} - e_{i,j}e_{k,l}, \quad i < j < k < \ell,$$

$$f_{i,j}f_{j,k} - f_{i,j}f_{j,k}, \quad i < j < k < \ell,$$

$$f_{i,j}f_{j,k} - x_i, \quad i < j < k,$$

$$e_{i,j}f_{i,j} - f_{i,j}e_{i,j}, \quad i < j < k < \ell,$$

$$e_{i,j}f_{i,j} - f_{i,j}e_{i,j}, \quad i < j < k < \ell,$$

$$e_{i,j}f_{i,j} - e_{i,j}e_{j,k}, \quad i < j < k < \ell,$$

$$e_{i,j}f_{i,j} - e_{i,j}e_{j,k}, \quad i < j < k < \ell,$$

$$e_{i,j}f_{i,j} - f_{i,j}e_{j,k}, \quad i < j < k < \ell,$$

$$e_{i,j}f_{i,j} - f_{i,j}e_{j,k}, \quad i < j < k < \ell,$$

$$f_{i,j}e_{j,k} - f_{i,j}n_e_{j,n}, \quad i < j < k < n,$$

$$f_{i,j}e_{j,k} - f_{i,j}n_e_{j,n}, \quad i < j < k < n,$$

$$f_{i,j}e_{j,k} - f_{i,j}n_e_{j,n}, \quad i < j < k < n,$$

$$f_{i,j}e_{j,k} - f_{i,j}n_e_{j,n}, \quad i < j < k < n,$$
Proof. Each binomial \( g = u - v \) listed above belongs to \( I_{D_n^+} \) with \( u = \in_{<d_{rev}}(g) \).
\( G \) for the set of binomials listed above and \( \in_{<d_{rev}}(G) = (\in_{<d_{rev}}(g) : g \in G) \).
Again, our work is to show that
\[
\{ \pi(u); u \in M(K[D_n^+]) \}, \quad u \notin \in_{<d_{rev}}(G)
\]
is linearly independent over \( K \).

Let
\[
u = \mu \in M(K[D_n^+]) \quad (\mu \notin \in_{<d_{rev}}(G))
\]
\[
u' = \mu' \in M(K[D_n^+]) \quad (\mu' \notin \in_{<d_{rev}}(G))
\]
belong to \( M(K[D_n^+]) \) with \( u \notin \in_{<d_{rev}}(G) \) and \( u' \notin \in_{<d_{rev}}(G) \), where
\[
f_i,j;1 \leq d_{rev} \quad \cdots \quad d_{rev} f_i,j;\ell, \quad e_{a_1, b_1} \leq d_{rev} \quad \cdots \quad d_{rev} e_{a_p, b_p},
\]
Write \( p_1 \) (resp. \( p'_1 \)) for the number of \( e_{a, b} \) (resp. \( e_{a', b'} \)) with \( b = n \) (resp. \( b' = n \)) and write \( p_2 \) (resp. \( p'_2 \)) for the number of \( e_{a, b} \) appearing in \( u \) (resp. \( u' \)). Write \( q_1 \) (resp. \( q'_1 \)) for the number of \( f_i,j;1 \) (resp. \( f_i,j';\ell' \)) with \( j = n \) (resp. \( j' = n \)) and write \( q_2 \) (resp. \( q'_2 \)) for the number of \( f_i,j;\ell \) (resp. \( f_i,j';\ell' \)) with \( j = n - 1 \) (resp. \( j' = n - 1 \)). Let
\[
r = \min\{p_1, q_1\} + \min\{p_2, q_2\}, \quad r' = \min\{p'_1, q'_1\} + \min\{p'_2, q'_2\}.
\]
If \( \pi(u) = \pi(u') \), then
\[
p = p', \quad \alpha + q = \alpha' + q', \quad \alpha + r = \alpha' + r', \quad p_1 - q_1 = p'_1 - q'_1.
\]
Now, in what follows, it will be proved that a contradiction arises if \( \pi(u) = \pi(u') \) and if none of the variables of \( K[D_n^+] \) appears in both \( u \) and \( u' \).

**Case I** Let \( p = 0 \).
Since \( f_i,j;1 \leq d_{rev} \) (resp. \( f_i,j';\ell \leq d_{rev} \)) if \( i < j < k \), one has \( q = q' \). Since \( f_i,j;1 \leq d_{rev} \) (resp. \( f_i,j';\ell \leq d_{rev} \)) if \( i < j < k \), it follows that \( j = j' \), as required.

**Case II** Let \( p > 0 \), \( \alpha > 0 \) and \( \alpha' = 0 \).
Since \( xe_{i,j} \leq d_{rev} \) (resp. \( xe_{i,j}' \leq d_{rev} \)) if \( i < j < k \), the monomial \( u \) is of the form
\[
u = x^n f_{i_1,j_1} \cdots f_{i_q,j_q} e_{a_{n-1,n}}^p.
\]
Since \( p > 0 \), one has \( p'_2 = 0 \). Let \( q'_1 \leq p'_1 \). Then \( r \geq q_1 \) and \( r' = q'_1 \).

**Case III** Let \( p > 0 \), \( \alpha = \alpha' = 0 \) and \( q = 0 \).
Since \( e_{i,j} \leq d_{rev} \) (resp. \( e_{i,j}' \leq d_{rev} \)) if \( i < j < k \), the required argument coincides with Case III in the proof of Theorem II.

**Case IV** Let \( \alpha = \alpha' = 0 \), \( p_2 > 0 \) and \( q_2 = 0 \).
Since \( p_2 > 0 \), one has \( p'_2 = 0 \). Since \( e_{i,j} \leq d_{rev} \) (resp. \( e_{i,j}' \leq d_{rev} \)) if \( i < j < k \), the required argument coincides with Case III in the proof of Theorem II.
If \( q \) is a complex, i.e., every minimal nonface of \( \Gamma \) in \( \mathbb{Q} \), the convex hull of \( \Gamma \), a contradiction arises. Let \( q_1 > q'_1 \). Then \( r' = p'_1 \) and \( r > p \), a contradiction.

(Case V) Let \( a = a' = 0, p > 0, q > 0, p_2q_2 = 0 \) and \( p_1^2q_2^2 = 0 \).

Since \( r = r' \) and \( p_1 - q_1 = p'_1 - q'_1 \), one has \( p_1 = p'_1 \) and \( q_1 = q'_1 \). Let \( \pi(u)^+ = \prod_{j=1}^{p_1} t_{a_j} t_{b_j} \prod_{j=1}^{q_1} t_{v_j} \), \( \pi(u)^- = \prod_{j=q_1+1}^{p_1} t_{v_j} \), \( \pi(u) = \prod_{j=1}^{p_1} t_{a_j} t_{v_j} \) and \( \pi(u')^- = \prod_{j=1}^{q_1} t_{v_j} \). Then \( \pi(u)^+ = \pi(u')^+ \) and \( \pi(u)^- = \pi(u')^- \).

(i) Let \( i < j < k < \ell \). Then \( i, j, k, \ell \) and \( i, j, k, \ell \) belong to \( \text{in}_{<\text{rev}}(\mathbb{G}) \) if \( i < j < k < \ell \), it follows that if \( a < t \) then \( b = i \). In particular, \( b_1 = i \). Since \( e_{i,j} t_{k,\ell} \in \text{in}_{<\text{rev}}(\mathbb{G}) \) if \( i < j < k < \ell \), one has either \( a < i \) or \( b = i \). Hence the variable \( t_{i,j} \) appears in \( \pi(u)^+ \) at least \( p + 1 \) times. However, \( t_{i,j} \) appears in \( \pi(u)^+ \) at most \( p \) times, a contradiction.

(ii) Let \( i < j < k < \ell \) and \( a < i \). Then each \( a \) satisfies \( a \geq i \). Hence the variables \( t_{\delta} \) with \( \delta \geq i \) appear in \( \pi(u)^+ \) at least \( 2p + 1 \) times. However, \( t_{\delta} \) with \( \delta \geq i \) appear in \( \pi(u)^+ \) at most \( 2p \) times, a contradiction.

(iii) Let \( i < j < k < \ell \). Since \( f_{i,k} t_{j,\ell} \in \text{in}_{<\text{rev}}(\mathbb{G}) \) if \( i < j < k < \ell \), it follows that \( t_{i,j}^{-1} \) cannot appear in \( \pi(u)^- \), a contradiction.

**Appendix**

We briefly discuss the combinatorial and algebraic significance of the existence of (squarefree) quadratic initial ideals of toric ideals. We work with the same notation \( \mathcal{A} \subset \mathbb{Z}^n, \mathcal{R}_K[\mathcal{A}], K[\mathcal{A}] \) and \( I_{\mathcal{A}} \) as in the Introduction.

First, a fundamental question in commutative algebra is to determine whether \( \mathcal{R}_K[\mathcal{A}] \) is Koszul [2]. Even though it is difficult to prove that \( \mathcal{R}_K[\mathcal{A}] \) is Koszul, the hierarchy (i) \( \rightarrow \) (ii) \( \rightarrow \) (iii) is known, e.g., [3], among the following properties:

(i) \( I_{\mathcal{A}} \) possesses a quadratic initial ideal;

(ii) \( \mathcal{R}_K[\mathcal{A}] \) is Koszul;

(iii) \( I_{\mathcal{A}} \) is generated by quadratic binomials.

Thus, in particular, each of the affine semigroup rings \( \mathcal{R}_K[\mathcal{A}^{(+)\mathbb{N}}], \mathcal{R}_K[\mathcal{B}^{(+)\mathbb{N}}], \mathcal{R}_K[\mathcal{C}^{(+)\mathbb{N}}] \) and \( \mathcal{R}_K[\mathcal{D}^{(+)\mathbb{N}}] \) is Koszul.

Second, to construct a triangulation of the convex polytope \( \text{conv}(\mathcal{A}) \subset \mathbb{R}^n \), the convex hull of \( \mathcal{A} \), is one of the most traditional topics in discrete geometry and combinatorics. Let \( < \) be any monomial order on \( K[\mathcal{A}] \) and \( \sqrt{\text{in}_{<}(I_{\mathcal{A}})} \) the radical of the initial ideal \( \text{in}_{<}(I_{\mathcal{A}}) \). Write \( \Delta(\text{in}_{<}(I_{\mathcal{A}})) \) for the (abstract) simplicial complex on the vertex set \( \mathcal{A} \) whose faces are those subsets \( \mathcal{A}' \subset \mathcal{A} \) with \( \prod_{a \in \mathcal{A}'} x_a \notin \sqrt{\text{in}_{<}(I_{\mathcal{A}})} \).

It is known [10] Theorem 8.3 that \( |\Delta(\text{in}_{<}(I_{\mathcal{A}}))| = \{ \text{conv}(\mathcal{A}') : \mathcal{A}' \in \Delta(\text{in}_{<}(I_{\mathcal{A}})) \} \) is a triangulation of \( \text{conv}(\mathcal{A}) \). Such a triangulation is called regular. Moreover, \( |\Delta(\text{in}_{<}(I_{\mathcal{A}}))| \) is unimodular (i.e., the normalized volume of each simplex \( \text{conv}(\mathcal{A}') \) with \( \mathcal{A}' \in \Delta(\text{in}_{<}(I_{\mathcal{A}})) \) is equal to 1) if and only if \( \text{in}_{<}(I_{\mathcal{A}}) \) is squarefree.

When \( \text{in}_{<}(I_{\mathcal{A}}) \) is squarefree and quadratic, the simplicial complex \( \Delta(\text{in}_{<}(I_{\mathcal{A}})) \) is a flag complex, i.e., every minimal nonface of \( \Delta(\text{in}_{<}(I_{\mathcal{A}})) \) is a two-element subset of \( \mathcal{A} \). Thus the combinatorics on \( |\Delta(\text{in}_{<}(I_{\mathcal{A}}))| \) can be discussed in terms of the skeleton \( \Delta(\text{in}_{<}(I_{\mathcal{A}}))^{(1)} \) of \( \Delta(\text{in}_{<}(I_{\mathcal{A}})) \), the finite graph on the vertex set \( \mathcal{A} \) whose edge set consists of all two-element faces of \( \Delta(\text{in}_{<}(I_{\mathcal{A}})) \). For example, the normalized volume of \( \text{conv}(\mathcal{A}) \) coincides with the number of maximal complete subgraphs of \( \Delta(\text{in}_{<}(I_{\mathcal{A}}))^{(1)} \). Note that the normalized volume of the convex polytope
conv\((A_{n-1})\) is computed in [7, Theorem 6.4] and the normalized volume of each of the convex polytopes conv\((B_n^{(+)}\), conv\((C_n^{(+)}\) and conv\((D_n^{(+)}\) is computed in Fong [6].

References


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