

## HYPERBOLIC HYPERSURFACES IN $\mathbb{P}^n$ OF FERMAT-WARING TYPE

BERNARD SHIFFMAN AND MIKHAIL ZAIDENBERG

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ABSTRACT. In this note we show that there are algebraic families of hyperbolic, Fermat-Waring type hypersurfaces in  $\mathbb{P}^n$  of degree  $4(n-1)^2$ , for all dimensions  $n \geq 2$ . Moreover, there are hyperbolic Fermat-Waring hypersurfaces in  $\mathbb{P}^n$  of degree  $4n^2 - 2n + 1$  possessing complete hyperbolic, hyperbolically embedded complements.

Many examples have been given of hyperbolic hypersurfaces in  $\mathbb{P}^3$  (e.g., see [ShZa] and the literature therein). Examples of degree 10 hyperbolic surfaces in  $\mathbb{P}^3$  were recently found by Shirosaki [Shr2], who also gave examples of hyperbolic hypersurfaces with hyperbolic complements in  $\mathbb{P}^3$  and  $\mathbb{P}^4$  [Shr1]. Fujimoto [Fu2] then improved Shirosaki's construction to give examples of degree 8.<sup>1</sup> Answering a question posed in [Za3], Masuda and Noguchi [MaNo] constructed the first examples of hyperbolic projective hypersurfaces, including those with complete hyperbolic complements, in any dimension. Improving the degree estimates of [MaNo], Siu and Yeung [SiYe] gave examples of hyperbolic hypersurfaces in  $\mathbb{P}^n$  of degree  $16(n-1)^2$ . (Fujimoto's recent construction [Fu2] provides examples of degree  $2^n$  in  $\mathbb{P}^n$ .) We remark that it was conjectured in 1970 by S. Kobayashi that generic hypersurfaces in  $\mathbb{P}^n$  of (presumably) degree  $2n-1$  are hyperbolic (for  $n=3$ , see [DeEl] and [Mc]).

The following result is an improvement of the example of Siu and Yeung [SiYe]:

**Theorem 1.** *Let  $d \geq (m-1)^2$ ,  $m \geq 2n-1$ . Then for generic linear functions  $h_1, \dots, h_m$  on  $\mathbb{C}^{n+1}$ , the hypersurface*

$$X_{n-1} = \left\{ z \in \mathbb{P}^n : \sum_{j=1}^m h_j(z)^d = 0 \right\}$$

*is hyperbolic. In particular, there exist algebraic families of hyperbolic hypersurfaces of degree  $4(n-1)^2$  in  $\mathbb{P}^n$ .*

Equivalently,  $X_{n-1}$  is the intersection of the Fermat hypersurface of degree  $d$  in  $\mathbb{P}^{m-1}$  with a generic  $n$ -plane. The low-dimensional cases  $n \leq 4$  of Theorem 1 were given by Shirosaki [Shr1]. Our construction is similar to those of [SiYe] and [Shr1].

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<sup>1</sup>The degree 8 example was also discovered independently by J. Duval (unpublished communication, October 1999).

On the other hand, examples were given in [Za2] of smooth curves of degree 5 in  $\mathbb{P}^2$ , with hyperbolically embedded, complete hyperbolic complements. Examples of hyperbolically embedded hypersurfaces in  $\mathbb{P}^n$  with complete hyperbolic complements were given in [MaNo] and [Kh] for all  $n$ , and in [Shr1] for  $n \leq 4$  (with lower degrees). The following result generalizes the result of Shirosaki [Shr1] to all dimensions.

**Theorem 2.** *Let  $d \geq m^2 - m + 1$ ,  $m \geq 2n$ . Then for generic linear functions  $h_1, \dots, h_m$  on  $\mathbb{C}^{n+1}$ , the complement  $\mathbb{P}^n \setminus X_{n-1}$  of the hypersurface of Theorem 1 is complete hyperbolic and hyperbolically embedded in  $\mathbb{P}^n$ . In particular, there exist algebraic families of hyperbolic hypersurfaces of degree  $4n^2 - 2n + 1$  in  $\mathbb{P}^n$  with hyperbolically embedded, complete hyperbolic complements.*

In particular, Theorem 2 provides algebraic families of curves of degree 13 in  $\mathbb{P}^2$ , of surfaces of degree 31 in  $\mathbb{P}^3$ , and so forth, whose complements are complete hyperbolic and hyperbolically embedded in projective space.

We shall use the following notation and lemma in our proofs of Theorems 1 and 2: We let  $\text{Gr}_{m,k}$  denote the Grassmannian of complex codimension  $k$  subspaces of  $\mathbb{C}^m$ , and we write

$$Q_{m,k} = 0_k \times \mathbb{C}^{m-k} \in \text{Gr}_{m,k}.$$

Furthermore, for

$$\begin{aligned} 1 &\leq a \leq c \leq m, \\ 1 &\leq b \leq c \leq a + b, \end{aligned}$$

we consider the closed Schubert cells

$$\Gamma_{m,a,b,c} = \{V \in \text{Gr}_{m,a} : \dim(V \cap Q_{m,b}) \geq m - c\}.$$

**Lemma 3.**  $\text{codim}_{\text{Gr}_{m,a}} \Gamma_{m,a,b,c} = (m - c)(a + b - c)$ .

*Proof.* Let  $\text{Mat}_{m,a} = \text{Hom}(\mathbb{C}^m, \mathbb{C}^a)$ , and let  $M_{m,a} \subset \text{Mat}_{m,a}$  denote the surjective homomorphisms. We consider the fiber bundle

$$\begin{array}{ccc} GL(a) & \longrightarrow & M_{m,a} \\ & & \downarrow \pi \\ & & \text{Gr}_{m,a} \end{array} \quad \pi(A) = \ker A.$$

We let

$$\tilde{\Gamma} := \pi^{-1}(\Gamma) = \{A \in M_{m,a} : \dim(\ker A|_{Q_{m,b}}) \geq m - c\},$$

whence

$$\dim M_{m,a} - \dim \tilde{\Gamma} = \dim \text{Gr}_{m,a} - \dim \Gamma_{m,a,b,c}.$$

Suppose  $A \in M_{m,a}$ ; i.e.,  $A$  is an  $a \times m$  matrix of rank  $a$ . We consider

$$\tilde{A} = \begin{pmatrix} I_b & 0 \\ A & A' \end{pmatrix} \in M_{m,a+b},$$

where  $B \in \text{Mat}_{b,a}$ ,  $A' \in \text{Mat}_{m-b,a}$ . Clearly,  $\ker \tilde{A} = \ker A|_{Q_{m,b}}$ , and thus

$$A \in \tilde{\Gamma} \Leftrightarrow \dim(\ker \tilde{A}) \geq m - c \Leftrightarrow \text{rank } \tilde{A} \leq c \Leftrightarrow \text{rank } A' \leq c - b.$$

It is elementary that

$$\text{codim}_{\text{Mat}_{k,l}} \{C \in \text{Mat}_{k,l} : \text{rank } C \leq r\} = (k - r)(l - r).$$

Therefore,

$$\begin{aligned} \text{codim}_{\mathbb{M}_{m,a}} \tilde{\Gamma} &= \text{codim}_{\text{Mat}_{m-b,a}} \{A' : \text{rank } A' \leq c - b\} \\ &= [(m - b) - (c - b)][a - (c - b)] \\ &= (m - c)(a + b - c). \end{aligned}$$

□

*Proof of Theorem 1.* Consider the Fermat hypersurface

$$F_d := \left\{ (z_1 : \dots : z_m) \in \mathbb{P}^{m-1} : \sum_{j=1}^m z_j^d = 0 \right\}$$

of degree  $d$  in  $\mathbb{P}^{m-1}$ . Suppose that  $d \geq (m - 1)^2$ ,  $m \geq 2n - 1$ . We must show that  $X_d := F_d \cap \mathbb{P}V$  is hyperbolic for a generic  $V \in \text{Gr}_{m,m-n-1}$ .

Suppose that  $f = (f_1, \dots, f_m) : \mathbb{C} \rightarrow X_d$  is a holomorphic curve. By Brody's theorem [Br], it suffices to show that  $f$  is constant. We write  $J_m = \{1, \dots, m\}$ . Let

$$I_0 = \{j \in J_m : f_j = 0\}.$$

(Of course,  $I_0$  may be empty.) We let  $I_1, \dots, I_l$  denote the equivalence classes in  $J_m \setminus I_0$  under the equivalence relation

$$j \sim k \Leftrightarrow f_j/f_k = \text{constant}.$$

We let  $k_\alpha = \text{card } I_\alpha$  and we write

$$I_\alpha = \{i(\alpha, 1), \dots, i(\alpha, k_\alpha)\},$$

for  $\alpha = 1, \dots, l$ , and also for  $\alpha = 0$  if  $k_0 \geq 1$ .

The result of Toda [To], Fujimoto [Fu1], and M. Green [Gr, pp. 70-71] says that for  $\alpha = 1, \dots, l$ , we have  $k_\alpha \geq 2$  and furthermore the constants

$$\mu_{\alpha j} := f_{i(\alpha,j)} / f_{i(\alpha,1)} \in \mathbb{C} \setminus \{0\} \quad (1 \leq \alpha \leq l, 2 \leq j \leq k_\alpha)$$

satisfy

$$(*) \quad 1 + \sum_{j=2}^{k_\alpha} \mu_{\alpha j}^d = 0.$$

Geometrically, the image  $f(\mathbb{C})$  is contained in the projective  $(l - 1)$ -plane  $Y_\mu^{\mathcal{I}}$  given by the equations

$$z_{i(\alpha,j)} = \mu_{\alpha j} z_{i(\alpha,1)}, \quad 2 \leq j \leq k_\alpha, 1 \leq \alpha \leq l; \quad z_{i(0,j)} = 0, \quad 1 \leq j \leq k_0.$$

Here,  $\mathcal{I}$  denotes the partition  $\{I_0, I_1, \dots, I_l\}$  of  $J_m$ , and  $\mu = \{\mu_{\alpha j}\}$ .

Let  $\tilde{Y}_\mu^{\mathcal{I}} \subset \mathbb{C}^m$  be the lift of  $Y_\mu^{\mathcal{I}}$ . Then  $\tilde{Y}_\mu^{\mathcal{I}} \in \text{Gr}_{m,m-l}$ . If  $l = 1$ , then  $Y_\mu^{\mathcal{I}}$  is a point. Otherwise, we consider  $Y_\mu^{\mathcal{I}} \cap \mathbb{P}V$  for generic  $V \in \text{Gr}_{m,m-n-1}$ . Applying Lemma 3 with

$$a = m - n - 1, \quad b = m - l, \quad c = m - 2$$

(changing coordinates to make  $\tilde{Y}_\mu^{\mathcal{I}} = Q_{m,m-l}$ ), we conclude that  $Y_\mu^{\mathcal{I}} \cap \mathbb{P}V$  either is a point or is empty, i.e.  $\dim(\tilde{Y}_\mu^{\mathcal{I}} \cap V) < 2$ , unless  $V$  lies in a subvariety of  $\text{Gr}_{m,m-n-1}$  of codimension

$$s = [m - (m - 2)][(m - n - 1) + (m - l) - (m - 2)] = 2(m - n - l + 1).$$

But given a partition  $\mathcal{I}$ , by (\*) we see that the  $\mu$ -moduli space of  $\widetilde{Y}_\mu^\mathcal{I}$  in  $\text{Gr}_{m,m-l}$  has dimension

$$\sum_{\alpha=1}^l (k_\alpha - 2) = m - k_0 - 2l \leq m - 2l.$$

Since  $m \geq 2n - 1$ , we have  $s \geq m - 2l + 1$ , and thus, for generic  $V \in \text{Gr}_{m,m-n-1}$ ,  $Y_\mu^\mathcal{I} \cap \mathbb{P}V$  is at most a point for all  $(\mathcal{I}, \mu)$ . Since  $f(\mathbb{C}) \subset Y_\mu^\mathcal{I} \cap \mathbb{P}V$  for some  $Y_\mu^\mathcal{I}$ , it follows that  $f$  must be constant.  $\square$

*Proof of Theorem 2.* Suppose that  $d \geq m^2 - m + 1$ ,  $m \geq 2n$ . Since by Theorem 1,  $X_{n-1}$  is hyperbolic for generic  $h_j$ , it suffices to show that any entire curve  $f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus X_{n-1}$  is constant (see e.g., [Za2]).

We proceed as in the proof of Theorem 1. Suppose that  $f = (f_1, \dots, f_m) : \mathbb{C} \rightarrow \mathbb{P}^n \setminus X_{n-1}$  is a holomorphic curve. As before, let  $I_0 = \{j \in J_m : f_j = 0\}$ , and let  $I_1, \dots, I_l$  denote the equivalence classes in  $J_m \setminus I_0$  under the equivalence relation

$$j \sim k \Leftrightarrow f_j/f_k = \text{constant}.$$

Since  $d > m(m - 1)$ , by [To, Fu1, Gr] we have

$$(**) \quad k_\alpha \geq 2 \quad \text{and} \quad 1 + \sum_{j=2}^{k_\alpha} \mu_{\alpha j}^d = 0 \quad \text{for} \quad 2 \leq \alpha \leq l,$$

after permuting the  $I_\alpha$  and using our previous notation. (Also,  $k_1 \geq 1$ , but  $1 + \sum_{j=2}^{k_1} \mu_{1j}^d \neq 0$ .) The proof of this result proceeds by considering the map  $(f_0, \dots, f_m) : \mathbb{C} \rightarrow F_d \subset \mathbb{P}^m$ , where  $f_0 = -\sum_{j=1}^m f_j^d = e^\varphi$ . (We let  $I_1 = \{j : f_j/f_0 = \text{constant}\} \neq \emptyset$ .) The better estimate for  $d$  arises from the fact that  $f_0$  has no zeros.

As before, the image  $f(\mathbb{C})$  is contained in the projective  $l$ -plane  $Y_\mu^\mathcal{I}$ , and  $Y_\mu^\mathcal{I} \cap \mathbb{P}V$  either is a point or is empty, unless  $V$  lies in a subvariety of  $\text{Gr}_{m,m-n-1}$  of codimension  $s = 2(m - n - l + 1)$ . But this time, by (\*\*) the  $\mu$ -moduli space of  $\widetilde{Y}_\mu^\mathcal{I}$  in  $\text{Gr}_{m,m-l}$  has dimension

$$k_1 - 1 + \sum_{\alpha=2}^l (k_\alpha - 2) = m - k_0 - 2l + 1 \leq m - 2l + 1.$$

Since  $m \geq 2n$ , we have  $s > m - 2l + 1$ , and hence, for generic  $V \in \text{Gr}_{m,m-n-1}$ ,  $Y_\mu^\mathcal{I} \cap \mathbb{P}V$  is a point or is empty for all  $(\mathcal{I}, \mu)$ .  $\square$

*Remark.* Note that the algebraic family of degree  $d = (m - 1)^2$  hyperbolic hypersurfaces in  $\mathbb{P}^n$  constructed in Theorem 1 has dimension  $(n + 1)m - 1$ , as does the family of Theorem 2. (Recall that in Theorem 1,  $m \geq 2n - 1$ , whereas in Theorem 2,  $m \geq 2n$ .) By the stability of hyperbolicity theorems (see [Za2]), in the corresponding projective spaces of degree  $d$  hypersurfaces, both families possess open neighborhoods consisting of hyperbolic hypersurfaces, with hyperbolically embedded complements in the second case. We note finally that the best possible lower bound for the degree of a hypersurface in  $\mathbb{P}^n$  with hyperbolic complement should be  $d = 2n + 1$  (see [Za1]), and the degree  $2n - 3$  hypersurfaces in  $\mathbb{P}^n$  are definitely not hyperbolic because they contain projective lines. (In fact, starting with  $n = 6$ , these lines are the only rational curves on a generic hypersurface of degree  $2n - 3$  in  $\mathbb{P}^n$ ; see [Pa].)

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## REFERENCES

- [Br] Brody R. Compact manifolds and hyperbolicity. *Trans. Amer. Math. Soc.* 235 (1978), 213–219. MR **57**:10010
- [DeEl] Demailly J.-P., El Goul J. Hyperbolicity of generic surfaces of high degree in projective 3-space. *Amer. J. Math.* 122 (2000), 515–546. MR **2001f**:32045
- [Fu1] Fujimoto H. On meromorphic maps into the complex projective space. *J. Math. Soc. Japan* 26 (1974), 272–288. MR **49**:10924
- [Fu2] Fujimoto H. A family of hyperbolic hypersurfaces in the complex projective space. *Complex Variables Theory Appl.* 43 (2001), 273–283. CMP 2001:10
- [Gr] Green M. Some Picard theorems for holomorphic maps to algebraic varieties. *Amer. J. Math.* 97 (1975), 43–75. MR **51**:3544
- [Kh] Ha Huy Khoai. Hyperbolic surfaces in  $\mathbf{P}^3(\mathbf{C})$ . *Proc. Amer. Math. Soc.* 125 (1997), 3527–3532. MR **98e**:32046
- [MaNo] Masuda K., Noguchi J. A construction of hyperbolic hypersurface of  $\mathbb{P}^n(\mathbf{C})$ . *Math. Ann.* 304 (1996), 339–362. MR **97a**:32025
- [Mc] McQuillan M. Holomorphic curves on hyperplane sections of 3-folds. *Geom. Funct. Anal.* 9 (1999), 370–392. MR **2000f**:32035
- [Pa] Pacienza G. Rational curves on general projective hypersurfaces. E-print, math.AG/0010037, Oct. 2000, 21pp.
- [ShZa] Shiffman B., Zaidenberg M. Two classes of hyperbolic surfaces in  $\mathbf{P}^3$ . *International J. Math.* 11 (2000), 65–101. CMP 2000:12
- [Shr1] Shirosaki M. Hyperbolic hypersurfaces in the complex projective spaces of low dimensions. *Kodai Math. J.* 23 (2000), 224–233. MR **2001e**:32038
- [Shr2] Shirosaki M. A hyperbolic hypersurface of degree 10. *Kodai Math. J.* 23 (2000), 376–379. MR **2001i**:32041
- [SiYe] Siu Y.-T., Yeung S.-K. Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees. *Amer. J. Math.* 119 (1997), 1139–1172. MR **98h**:32044
- [To] Toda N. On the functional equation  $\sum_{i=0}^p a_i f_i^{m_i} = 1$ . *Tôhoku Math. J. (2)* 23 (1971), 289–299. MR **45**:551
- [Za1] Zaidenberg M. The complement of a generic hypersurface of degree  $2n$  in  $\mathbf{C}P^n$  is not hyperbolic. *Siberian Math. J.* 28 (1987), 425–432. MR **88k**:32063
- [Za2] Zaidenberg M. Stability of hyperbolic imbeddedness and construction of examples. *Math. USSR Sbornik* 63 (1989), 351–361. MR **89f**:32047
- [Za3] Zaidenberg M. Hyperbolicity in projective spaces. Proc. Conf. “Diophantine Problems, Hyperbolic Spaces and related topics”, RIMS, Kyoto, Japan, 26–31 October 1992; Tokyo, TIT, 1992, 136–156. MR **95a**:32045

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218  
*E-mail address:* `shiffman@math.jhu.edu`

UNIVERSITÉ GRENoble I, INSTITUT FOURIER, UMR 5582 CNRS-UJF, BP 74, 38402 ST. MARTIN D’HÈRES CÉDEX, FRANCE  
*E-mail address:* `zaidenbe@ujf-grenoble.fr`