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HYPERBOLIC HYPERSURFACES IN $\mathbb{P}^n$ OF FERMAT-WARING TYPE

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ABSTRACT. In this note we show that there are algebraic families of hyperbolic, Fermat-Waring type hypersurfaces in $\mathbb{P}^n$ of degree $4(n-1)^2$, for all dimensions $n \geq 2$. Moreover, there are hyperbolic Fermat-Waring hypersurfaces in $\mathbb{P}^n$ of degree $4n^2-2n+1$ possessing complete hyperbolic, hyperbolically embedded complements.

Many examples have been given of hyperbolic hypersurfaces in $\mathbb{P}^3$ (e.g., see [ShZa] and the literature therein). Examples of degree 10 hyperbolic surfaces in $\mathbb{P}^3$ were recently found by Shirosaki [Shr2], who also gave examples of hyperbolic hypersurfaces with hyperbolic complements in $\mathbb{P}^3$ and $\mathbb{P}^4$ [Shr1]. Fujimoto [Fu2] then improved Shirosaki's construction to give examples of degree 8.[1] Answering a question posed in [Za3], Masuda and Noguchi [MaNo] constructed the first examples of hyperbolic projective hypersurfaces, including those with complete hyperbolic complements, in any dimension. Improving the degree estimates of [MaNo], Siu and Yeung [SiYe] gave examples of hyperbolic hypersurfaces in $\mathbb{P}^n$ of degree $16(n-1)^2$. (Fujimoto's recent construction [Fu2] provides examples of degree 2^n in $\mathbb{P}^n$.) We remark that it was conjectured in 1970 by S. Kobayashi that generic hypersurfaces in $\mathbb{P}^n$ of (presumably) degree $2n-1$ are hyperbolic (for $n=3$, see [DeEl] and [Mc]).

The following result is an improvement of the example of Siu and Yeung [SiYe]:

Theorem 1. *Let $d \geq (m-1)^2$, $m \geq 2n-1$. Then for generic linear functions $h_1, \ldots, h_m$ on $\mathbb{C}^{n+1}$, the hypersurface*

$$X_{n-1} = \left\{ z \in \mathbb{P}^n : \sum_{j=1}^{m} h_j(z)^d = 0 \right\}$$

is hyperbolic. In particular, there exist algebraic families of hyperbolic hypersurfaces of degree $4(n-1)^2$ in $\mathbb{P}^n$.

Equivalently, X_{n-1} is the intersection of the Fermat hypersurface of degree d in $\mathbb{P}^{m-1}$ with a generic n-plane. The low-dimensional cases $n \leq 4$ of Theorem 1 were given by Shirosaki [Shr1]. Our construction is similar to those of [SiYe] and [Shr1].

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[1]The degree 8 example was also discovered independently by J. Duval (unpublished communication, October 1999).

On the other hand, examples were given in [Za2] of smooth curves of degree 5 in $\mathbb{P}^2$, with hyperbolically embedded, complete hyperbolic complements. Examples of hyperbolically embedded hypersurfaces in $\mathbb{P}^n$ with complete hyperbolic complements were given in [MaNo] and [Kh] for all n, and in [Shr1] for $n \le 4$ (with lower degrees). The following result generalizes the result of Shirosaki [Shr1] to all dimensions.

Theorem 2. *Let $d \ge m^2 - m + 1$, $m \ge 2n$. Then for generic linear functions $h_1, \dots, h_m$ on $\mathbb{C}^{n+1}$, the complement $\mathbb{P}^n \setminus X_{n-1}$ of the hypersurface of Theorem 1 is complete hyperbolic and hyperbolically embedded in $\mathbb{P}^n$. In particular, there exist algebraic families of hyperbolic hypersurfaces of degree $4n^2 - 2n + 1$ in $\mathbb{P}^n$ with hyperbolically embedded, complete hyperbolic complements.*

In particular, Theorem 2 provides algebraic families of curves of degree 13 in $\mathbb{P}^2$, of surfaces of degree 31 in $\mathbb{P}^3$, and so forth, whose complements are complete hyperbolic and hyperbolically embedded in projective space.

We shall use the following notation and lemma in our proofs of Theorems 1 and 2: We let $\mathrm{Gr}_{m,k}$ denote the Grassmannian of complex codimension k subspaces of $\mathbb{C}^m$, and we write

$$Q_{m,k} = 0_k \times \mathbb{C}^{m-k} \in \mathrm{Gr}_{m,k}\,.$$

Furthermore, for

$$\begin{array}{ccccccc} 1 & \le & a & \le & c & \le & m, \\ 1 & \le & b & \le & c & \le & a+b, \end{array}$$

we consider the closed Schubert cells

$$\Gamma_{m,a,b,c} = \{V \in \mathrm{Gr}_{m,a} : \dim(V \cap Q_{m,b}) \ge m - c\}\,.$$

Lemma 3. $\mathrm{codim}_{\mathrm{Gr}_{m,a}} \Gamma_{m,a,b,c} = (m-c)(a+b-c)\,.$

Proof. Let $\mathrm{Mat}_{m,a} = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^a)$, and let $\mathrm{M}_{m,a} \subset \mathrm{Mat}_{m,a}$ denote the surjective homomorphisms. We consider the fiber bundle

$$\begin{array}{ccc} GL(a) & \longrightarrow & \mathrm{M}_{m,a} \\ & & \downarrow \pi \qquad\qquad \pi(A) = \ker A. \\ & & \mathrm{Gr}_{m,a} \end{array}$$

We let

$$\widetilde{\Gamma} := \pi^{-1}(\Gamma) = \{A \in \mathrm{M}_{m,a} : \dim\left(\ker A|_{Q_{m,b}}\right) \ge m - c\}\,,$$

whence

$$\dim \mathrm{M}_{m,a} - \dim \widetilde{\Gamma} = \dim \mathrm{Gr}_{m,a} - \dim \Gamma_{m,a,b,c}\,.$$

Suppose $A \in \mathrm{M}_{m,a}$; i.e., A is an $a \times m$ matrix of rank a. We consider

$$\widetilde{A} = \left(\frac{I_b \quad 0}{A}\right) = \begin{pmatrix} I_b & 0 \\ B & A' \end{pmatrix} \in \mathrm{M}_{m,a+b}\,,$$

where $B \in \mathrm{Mat}_{b,a}$, $A' \in \mathrm{Mat}_{m-b,a}$. Clearly, $\ker \widetilde{A} = \ker A|_{Q_{m,b}}$, and thus

$$A \in \widetilde{\Gamma} \;\Leftrightarrow\; \dim(\ker \widetilde{A}) \ge m - c \;\Leftrightarrow\; \mathrm{rank}\, \widetilde{A} \le c \;\Leftrightarrow\; \mathrm{rank}\, A' \le c - b\,.$$

It is elementary that

$$\mathrm{codim}_{\mathrm{Mat}_{k,l}} \{C \in \mathrm{Mat}_{k,l} : \mathrm{rank}\, C \le r\} = (k-r)(l-r)\,.$$

Therefore,

$$\begin{aligned}\operatorname{codim}_{\mathrm{M}_{m,a}}\widetilde{\Gamma} &= \operatorname{codim}_{\mathrm{Mat}_{m-b,a}}\{A' : \operatorname{rank} A' \le c-b\}\\ &= [(m-b)-(c-b)][a-(c-b)]\\ &= (m-c)(a+b-c)\,.\end{aligned}$$

□

Proof of Theorem 1. Consider the Fermat hypersurface

$$F_d := \left\{(z_1 : \cdots : z_m) \in \mathbb{P}^{m-1} : \sum_{j=1}^{m} z_j^d = 0\right\}$$

of degree d in $\mathbb{P}^{m-1}$. Suppose that $d \ge (m-1)^2,\ m \ge 2n-1$. We must show that $X_d := F_d \cap \mathbb{P}V$ is hyperbolic for a generic $V \in \mathrm{Gr}_{m,m-n-1}$.

Suppose that $f = (f_1, \ldots, f_m) : \mathbb{C} \to X_d$ is a holomorphic curve. By Brody's theorem [Br], it suffices to show that f is constant. We write $J_m = \{1, \ldots, m\}$. Let

$$I_0 = \{j \in J_m : f_j = 0\}\,.$$

(Of course, I_0 may be empty.) We let $I_1, \ldots, I_l$ denote the equivalence classes in $J_m \setminus I_0$ under the equivalence relation

$$j \sim k \iff f_j/f_k = \text{constant}\,.$$

We let $k_\alpha = \operatorname{card} I_\alpha$ and we write

$$I_\alpha = \{i(\alpha,1), \ldots, i(\alpha,k_\alpha)\}\,,$$

for $\alpha = 1, \ldots, l$, and also for $\alpha = 0$ if $k_0 \ge 1$.

The result of Toda [To], Fujimoto [Fu1], and M. Green [Gr, pp. 70-71] says that for $\alpha = 1, \ldots, l$, we have $k_\alpha \ge 2$ and furthermore the constants

$$\mu_{\alpha j} := f_{i(\alpha,j)}/f_{i(\alpha,1)} \in \mathbb{C} \setminus \{0\} \qquad (1 \le \alpha \le l\,,\ 2 \le j \le k_\alpha)$$

satisfy

$$(*) \qquad 1 + \sum_{j=2}^{k_\alpha} \mu_{\alpha j}^d = 0\,.$$

Geometrically, the image $f(\mathbb{C})$ is contained in the projective $(l-1)$-plane $Y_\mu^{\mathcal{I}}$ given by the equations

$$z_{i(\alpha,j)} = \mu_{\alpha j} z_{i(\alpha,1)}\,, \quad 2 \le j \le k_\alpha\,, 1 \le \alpha \le l\,; \qquad z_{i(0,j)} = 0\,, \quad 1 \le j \le k_0\,.$$

Here, $\mathcal{I}$ denotes the partition $\{I_0, I_1, \ldots, I_l\}$ of J_m, and $\mu = \{\mu_{\alpha j}\}$.

Let $\widetilde{Y}_\mu^{\mathcal{I}} \subset \mathbb{C}^m$ be the lift of $Y_\mu^{\mathcal{I}}$. Then $\widetilde{Y}_\mu^{\mathcal{I}} \in \mathrm{Gr}_{m,m-l}$. If $l = 1$, then $Y_\mu^{\mathcal{I}}$ is a point. Otherwise, we consider $Y_\mu^{\mathcal{I}} \cap \mathbb{P}V$ for generic $V \in \mathrm{Gr}_{m,m-n-1}$. Applying Lemma 3 with

$$a = m-n-1,\ b = m-l,\ c = m-2$$

(changing coordinates to make $\widetilde{Y}_\mu^{\mathcal{I}} = Q_{m,m-l}$), we conclude that $Y_\mu^{\mathcal{I}} \cap \mathbb{P}V$ either is a point or is empty, i.e. $\dim(\widetilde{Y}_\mu^{\mathcal{I}} \cap V) < 2$, unless V lies in a subvariety of $\mathrm{Gr}_{m,m-n-1}$ of codimension

$$s = \big[m-(m-2)\big]\big[(m-n-1)+(m-l)-(m-2)\big] = 2(m-n-l+1)\,.$$

But given a partition $\mathcal{I}$, by $(*)$ we see that the μ-moduli space of $\widetilde{Y}^{\mathcal{I}}_\mu$ in $\mathrm{Gr}_{m,m-l}$ has dimension

$$\sum_{\alpha=1}^{l}(k_\alpha - 2) = m - k_0 - 2l \le m - 2l\,.$$

Since $m \ge 2n-1$, we have $s \ge m-2l+1$, and thus, for generic $V \in \mathrm{Gr}_{m,m-n-1}$, $Y^{\mathcal{I}}_\mu \cap \mathbb{P}V$ is at most a point for all $(\mathcal{I},\mu)$. Since $f(\mathbb{C}) \subset Y^{\mathcal{I}}_\mu \cap \mathbb{P}V$ for some $Y^{\mathcal{I}}_\mu$, it follows that f must be constant. □

Proof of Theorem 2. Suppose that $d \ge m^2 - m + 1,\ m \ge 2n$. Since by Theorem 1, X_{n-1} is hyperbolic for generic h_j, it suffices to show that any entire curve $f : \mathbb{C} \to \mathbb{P}^n \backslash X_{n-1}$ is constant (see e.g., [Za2]).

We proceed as in the proof of Theorem 1. Suppose that $f = (f_1, \dots, f_m) : \mathbb{C} \to \mathbb{P}^n \setminus X_{n-1}$ is a holomorphic curve. As before, let $I_0 = \{j \in J_m : f_j = 0\}$, and let $I_1, \dots, I_l$ denote the equivalence classes in $J_m \setminus I_0$ under the equivalence relation

$$j \sim k \iff f_j/f_k = \text{constant}\,.$$

Since $d > m(m-1)$, by [To, Fu1, Gr] we have

$$(**) \qquad k_\alpha \ge 2 \ \text{ and } \ 1 + \sum_{j=2}^{k_\alpha} \mu^d_{\alpha j} = 0 \ \text{ for } \ 2 \le \alpha \le l\,,$$

after permuting the I_α and using our previous notation. (Also, $k_1 \ge 1$, but $1 + \sum_{j=2}^{k_1} \mu^d_{1j} \ne 0$.) The proof of this result proceeds by considering the map $(f_0, \dots, f_m) : \mathbb{C} \to F_d \subset \mathbb{P}^m$, where $f_0 = -\sum_{j=1}^m f_j^d = e^\varphi$. (We let $I_1 = \{j : f_j/f_0 = \text{constant}\} \ne \emptyset$.) The better estimate for d arises from the fact that f_0 has no zeros.

As before, the image $f(\mathbb{C})$ is contained in the projective l-plane $Y^{\mathcal{I}}_\mu$, and $Y^{\mathcal{I}}_\mu \cap \mathbb{P}V$ either is a point or is empty, unless V lies in a subvariety of $\mathrm{Gr}_{m,m-n-1}$ of codimension $s = 2(m-n-l+1)$. But this time, by $(**)$ the μ-moduli space of $\widetilde{Y}^{\mathcal{I}}_\mu$ in $\mathrm{Gr}_{m,m-l}$ has dimension

$$k_1 - 1 + \sum_{\alpha=2}^{l}(k_\alpha - 2) = m - k_0 - 2l + 1 \le m - 2l + 1\,.$$

Since $m \ge 2n$, we have $s > m - 2l + 1$, and hence, for generic $V \in \mathrm{Gr}_{m,m-n-1}$, $Y^{\mathcal{I}}_\mu \cap \mathbb{P}V$ is a point or is empty for all $(\mathcal{I},\mu)$. □

Remark. Note that the algebraic family of degree $d = (m-1)^2$ hyperbolic hypersurfaces in $\mathbb{P}^n$ constructed in Theorem 1 has dimension $(n+1)m - 1$, as does the family of Theorem 2. (Recall that in Theorem 1, $m \ge 2n-1$, whereas in Theorem 2, $m \ge 2n$.) By the stability of hyperbolicity theorems (see [Za2]), in the corresponding projective spaces of degree d hypersurfaces, both families possess open neighborhoods consisting of hyperbolic hypersurfaces, with hyperbolically embedded complements in the second case. We note finally that the best possible lower bound for the degree of a hypersurface in $\mathbb{P}^n$ with hyperbolic complement should be $d = 2n+1$ (see [Za1]), and the degree $2n-3$ hypersurfaces in $\mathbb{P}^n$ are definitely not hyperbolic because they contain projective lines. (In fact, starting with $n = 6$, these lines are the only rational curves on a generic hypersurface of degree $2n-3$ in $\mathbb{P}^n$; see [Pa].)

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References

[Br] Brody R. Compact manifolds and hyperbolicity. Trans. Amer. Math. Soc. 235 (1978), 213–219. MR **57:**10010

[DeEl] Demailly J.-P., El Goul J. Hyperbolicity of generic surfaces of high degree in projective 3-space. Amer. J. Math. 122 (2000), 515–546. MR **2001f:**32045

[Fu1] Fujimoto H. On meromorphic maps into the complex projective space. J. Math. Soc. Japan 26 (1974), 272–288. MR **49:**10924

[Fu2] Fujimoto H. A family of hyperbolic hypersurfaces in the complex projective space. Complex Variables Theory Appl. 43 (2001), 273–283. CMP 2001:10

[Gr] Green M. Some Picard theorems for holomorphic maps to algebraic varieties. Amer. J. Math. 97 (1975), 43–75. MR **51:**3544

[Kh] Ha Huy Khoai. Hyperbolic surfaces in $\mathbf{P}^3(\mathbf{C})$. Proc. Amer. Math. Soc. 125 (1997), 3527–3532. MR **98e:**32046

[MaNo] Masuda K., Noguchi J. A construction of hyperbolic hypersurface of $\mathbb{P}^n(\mathbb{C})$. Math. Ann. 304 (1996), 339–362. MR **97a:**32025

[Mc] McQuillan M. Holomorphic curves on hyperplane sections of 3-folds. Geom. Funct. Anal. 9 (1999), 370–392. MR **2000f:**32035

[Pa] Pacienza G. Rational curves on general projective hypersurfaces. E-print, math.AG/0010037, Oct. 2000, 21pp.

[ShZa] Shiffman B., Zaidenberg M. Two classes of hyperbolic surfaces in $\mathbf{P}^3$. International J. Math. 11 (2000), 65–101. CMP 2000:12

[Shr1] Shirosaki M. Hyperbolic hypersurfaces in the complex projective spaces of low dimensions. Kodai Math. J. 23 (2000), 224–233. MR **2001e:**32038

[Shr2] Shirosaki M. A hyperbolic hypersurface of degree 10. Kodai Math. J. 23 (2000), 376–379. MR **2001i:**32041

[SiYe] Siu Y.-T., Yeung S.-K. Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees. Amer. J. Math. 119 (1997), 1139–1172. MR **98h:**32044

[To] Toda N. On the functional equation $\sum_{i=0}^{p} a_i f_i^{n_i} = 1$. Tôhoku Math. J. (2) 23 (1971), 289–299. MR **45:**551

[Za1] Zaidenberg M. The complement of a generic hypersurface of degree $2n$ in $\mathbf{C}P^n$ is not hyperbolic. Siberian Math. J. 28 (1987), 425–432. MR **88k:**32063

[Za2] Zaidenberg M. Stability of hyperbolic imbeddedness and construction of examples. Math. USSR Sbornik 63 (1989), 351–361. MR **89f:**32047

[Za3] Zaidenberg M. Hyperbolicity in projective spaces. Proc. Conf. "Diophantine Problemes, Hyperbolic Spaces and related topics", RIMS, Kyoto, Japan, 26–31 October 1992; Tokyo, TIT, 1992, 136–156. MR **95a:**32045

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