A CONTINUUM WHOSE HYPERSPACE OF SUBCONTINUA IS NOT $g$-CONTRACTIBLE

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Abstract. A topological space $Y$ is said to be $g$-contractible provided that there exists a continuous onto function $f : Y \to Y$ such that $f$ is homotopic to a constant function. Answering a question by Sam B. Nadler, Jr., in this paper we construct a metric continuum $Z$ such that its hyperspace of subcontinua $C(Z)$ is not $g$-contractible.

1. Introduction

A continuum is a compact connected metric space. A map is a continuous function. A topological space $Y$ is said to be $g$-contractible provided that there exists an onto map $f : Y \to Y$ such that $f$ is homotopic to a constant map. For a continuum $X$, $C(X)$ (resp., $2^X$) denotes the hyperspace of subcontinua (resp., nonempty closed subsets) of $X$, with the Hausdorff metric.

Clearly, every contractible space is $g$-contractible. A simple closed curve is an easy example of a $g$-contractible and non-contractible continuum. In fact, the Hahn-Mazurkiewicz Theorem (see [7, Theorem 8.14]) implies that any locally connected continuum is $g$-contractible. The notion of $g$-contractibility was introduced by D. P. Bellamy in [1]. In [5], S. B. Nadler, Jr., studied $g$-contractibility in hyperspaces. He proved that, for any continuum $X$, $C(X)$ and $2^X$ is $g$-contractible ([5, 3.9] or [6, Theorem 4.10]), and if $X$ is a continuum such that $X$ contains an open subset with uncountably many components, then $C(X)$ is $g$-contractible ([5, 3.12] or [6, Theorem 4.12]). Nadler also asked if $C(X)$ is $g$-contractible for any continuum $X$ ([5, 3.10] or [6, Question 4.11]). In this paper we answer Nadler’s question in the negative by constructing an example of a continuum $Z$ such that $C(Z)$ is not $g$-contractible.

2. Auxiliary results

Lemma 1. If $C(X)$ is $g$-contractible, then there exists an onto map $f : C(X) \to C(X)$ and there exists a map $\psi : C(X) \times [0, 1] \to C(X)$ such that:
- $\psi(A, 0) = f(A)$ and $\psi(A, 1) = X$ for each $A \in C(X)$,
- if $A \in C(X)$ and $0 \leq s \leq t \leq 1$, then $\psi(A, s) \subset \psi(A, t)$.

Proof. Suppose that $C(X)$ is $g$-contractible. Then there exist:
- an onto map $f : C(X) \to C(X),

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A; then \( n \) be as in Lemma 1. From Lemma 3, it follows that \( C \) such that \( A \setminus B \).

Thus, if \( A \) is an element of \( \mathcal{C} \) that is connected im kleinen at \( p \):

Clearly, \( F \) is a map such that \( F(A,0) = f(A) \) and \( F(A,1) = X \) for each \( A \in C(X) \).

Now, let \( \psi : C(X) \times [0,1] \to C(X) \) be given by

\[
\psi(a,t) = \bigcup \{ F(A,s) : s \in [0,t] \}.
\]

It is easy to show that \( \psi \) has the required properties. \( \square \)

The proof of the following lemma is similar to the proof of Theorem (2) of \([2]\).

**Lemma 2.** Let \( f : X \to Y \) be an onto map between continua. Let \( q \in Y \). If \( X \) is connected im kleinen at each point of \( f^{-1}(q) \), then \( Y \) is connected im kleinen at \( q \).

The following result is an easy consequence of Theorem 2 of \([3]\).

**Lemma 3.** Let \( X \) be a continuum and let \( p \in X \) be a point such that \( X \) is connected im kleinen at \( p \). Then \( C(X) \) is connected im kleinen at each element \( A \) that satisfies \( p \in A \).

**3. The example**

The example is constructed in the euclidean plane \( \mathbb{R}^2 \). For each subset \( A \) of \( \mathbb{R}^2 \), let \(-A\) denote the set \(-A = \{-p \in \mathbb{R}^2 : p \in A\}. \) Let \( \theta = (0,0) \in \mathbb{R}^2. \) Let \( X_0 = \{0\} \times [-1,1]. \) For each \( n \geq 1, \) let \( X_n = \{\frac{1}{n}\} \times [0,\frac{1}{n}]. \) For each \( n \geq 1, \) let \( L_n \) be a homeomorphic copy of the real line such that: (a) \( L_n \subset (\frac{1}{n+1}, \frac{1}{n}) \times [0,1], \) (b) \( L_n \cap \{[0,1]\} \neq \emptyset, \) and (c) \( \text{cl}_{2}(L_n) = L_n \cup X_n \cup X_{n+1}. \) Let \( Y_0 = X_0 \cup (\bigcup \{X_n : n \geq 1\}) \cup (\bigcup \{L_n : n \geq 1\}). \)

Finally, put \( X = \overline{(-Y_0)} \cup Y_0. \)

Clearly, \( X \) is a continuum (shown on the next page).

**Proof.** Now we prove that \( C(X) \) is not \( g \)-contractible. Suppose, to the contrary, that \( C(X) \) is \( g \)-contractible. Let \( f : C(X) \to C(X) \) and \( \psi : C(X) \times [0,1] \to C(X) \) be as in Lemma 1. From Lemma 3, it follows that \( C(X) \) is connected im kleinen at any element \( A \in C(X) \) such that \( A \cap \bigcup \{L_n : n \geq 1\} \cup \bigcup \{-L_n : n \geq 1\} \neq \emptyset. \)

Thus, if \( A \) is an element of \( C(X) \) such that \( C(X) \) is not connected im kleinen at \( A, \) then \( A \) is contained at some arc \( X_n \) or at some arc \( -X_n \) \( (n \geq 0). \)

Now we prove the following.

If \( A \) is a locally connected subcontinuum of \( C(X) \), then \( A \) intersects finitely many sets of the form \( C(X_n) \) (and finitely many sets of the form \( C(-X_n) \)).

Suppose, to the contrary, that there is a sequence of integers \( n_1 < n_2 < ... \) such that \( A \cap C(X_{n_k}) \neq \emptyset \) for each \( k \geq 1. \) Fix an element \( B_k \in A \cap C(X_{n_k}). \) Then \( B_k \to \{\theta\}. \) Thus \( \{\theta\} \in A. \) Since \( A \) is locally connected, there exists a subcontinuum \( B \) of \( A \) such that \( \{\theta\} \in B. \) \( B_k \in B \) for some \( k \) and \( H(\{\theta\}, B) = \frac{1}{4} \) for each \( B \in B, \) where \( H \) is the Hausdorff metric for \( C(X). \) Let \( B_0 = \bigcup B. \) From \([6]\) Lemma 1.43],
Next we show that there exists a sequence of subcontinua \( \{A_n\}_{n=1}^\infty \) of \( X \) such that \( A_n \to \{\theta\} \), \( f(A_n) \to \{\theta\} \) and \( f(A_n) \subset \bigcup \{X_m : m \geq 1\} \) for each \( n \).

Each continuum of one of the forms \( X_i \) or \( -X_i \) is an arc, thus the hyperspaces \( C(X_i) \) and \( C(-X_i) \) are 2-cells ([3] Example 0.54). Therefore, the sets \( f[C(X_i)] \) and \( f[C(-X_i)] \) are locally connected (see [7] Proposition 8.16). Given \( n \geq 1 \), by the previous claim, there exists a positive integer \( k_n \) such that, \( n \leq k_n \) and \( f[C(X_0) \cup C(X_1) \cup C(-X_1) \cup \ldots \cup C(X_n) \cup C(-X_n)] \cap C(X_{k_n}) = \emptyset \). Fix a point \( p \in X_{k_n} \); then it is easy to check that \( C(X) \) is not connected im kleinen at \( \{p\} \). According to Lemma 2, there exists an element \( A_n \in C(X) \) such that \( C(X) \) is not connected im kleinen at \( A_n \) and \( f(A_n) = p \). Thus, \( A_n \) is contained in some set of the form \( X_r \) or in some set of the form \( -X_r \) for some \( r \geq 0 \). By the choice of \( k_n \), \( r \geq n \). This implies that \( H(\{\theta\}, A_n) < \frac{1}{n} \). Since \( k_n \geq n \) and \( p \in X_{k_n} \), \( H(\{\theta\}, f(A_n)) = H(\{\theta\}, \{p\}) \leq \frac{1}{n} \). This completes the construction of \( A_n \) and proves the claim.

In a similar way, it is possible to construct a sequence of subcontinua \( \{B_n\}_{n=1}^\infty \) of \( X \) such that \( B_n \to \{\theta\} \), \( f(B_n) \to \{\theta\} \) and \( f(B_n) \subset \bigcup \{-X_m : m \geq 1\} \) for each \( n \).

We are ready to obtain the final contradiction.

Since \( f \) is continuous, \( f(A_n) \to f(\{\theta\}) \). Thus, \( f(\{\theta\}) = \{\theta\} \). We know that \( \psi(\{\theta\}, 0) = f(\{\theta\}) = \{\theta\} \) and \( \psi(\{\theta\}, 1) = X \); then the number \( t_0 = \max \{t \in [0, 1] : \psi(\{\theta\}, t) = \{\theta\}\} \) is in the interval \([0, 1]\). Thus, \( \psi(\{\theta\}, t_0) = \{\theta\} \). From the continuity of the map \( \psi \), there exists a number \( s \in (t_0, 1) \) such that \( H(\{\theta\}, \psi(\{\theta\}, s)) < \frac{1}{8} \). Hence, \( \{\theta\} \) is properly contained in \( \psi(\{\theta\}, s) \).
On the other hand, \( \lim_{n \to \infty} \psi(B_n, s) = \psi(\{\theta\}, s) = \lim_{n \to \infty} \psi(A_n, s) \), so there exists \( R \geq 1 \) such that, for each \( r \geq R \), \( \max\{H(\psi(\{\theta\}, s), \psi(A_r, s)), H(\psi(\{\theta\}, s), \psi(B_r, s))\} < \frac{1}{8} \). Given \( r \geq R \), \( \psi(A_r, s) \) is a subcontinuum of \( X \) such that it contains the set \( \psi(A_r, 0) = f(A_r) \subset \bigcup_{m \geq 1} X_m \) and the diameter of \( \psi(A_r, s) \) is less than \( \frac{1}{2} \). From the construction of \( X \), it follows that \( \psi(A_r, s) \) is contained in \([0, 1] \times [0, 1]\). It follows that \( \psi(\{\theta\}, s) \subset [0, 1] \times [0, 1] \). Using \( B_r \) instead of \( A_r \), it follows that \( \psi(\{\theta\}, s) \subset [-1, 0] \times [-1, 0] \). Hence, \( \psi(\{\theta\}, s) = \{\theta\} \). This is a contradiction that completes the proof that \( C(X) \) is not \( g \)-contractible.

References