ON THE REGULARITY OF SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS VIA THE LIOUVILLE PROPERTY

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(Communicated by David S. Tartakoff)

Abstract. We show that any $C^{1,1}$ solution to the uniformly elliptic equation $F(D^2u) = 0$ must belong to $C^{2,\alpha}$, if the equation has the Liouville property.

§1. Introduction

In this paper, we consider the interior regularity of solutions to the following fully nonlinear elliptic equation:

(1) $F(D^2u) = 0$.

We assume that $F$ is uniformly elliptic, i.e., there exist constants $0 < \lambda \leq \Lambda$ such that

(2) $\lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|$, for $M, N \in \mathcal{S}$, $N \geq 0$,

where $\mathcal{S}$ denotes the space of real $n \times n$ symmetric matrices and $\|N\|$ denotes the norm of $N$.

For simplicity, we also assume that $F(0) = 0$.

There have been a number of works concerning equation (1). For instance, see [CC], [GT], [K] and the references cited there. When $F$ is a concave or convex functional, it is well known that the Evans-Krylov estimate

$[D^2u]_{C^0(B_{1/2})} \leq C\|u\|_{C^{1,1}(B_1)}$

holds, and $C^{1,1}$ viscosity solutions of (1) are $C^{2,\alpha}$ for some $\alpha > 0$.

On the contrary, in the case when $F$ is not concave nor convex, $C^{1,1}$ viscosity solutions of (1) may not be in the $C^2$ class. This has recently been shown by Nadirashvili in [N] in which he found a $C^{1,1}$ viscosity solution $u$ to the equation $F(D^2u) = 0$ where $F$ is smooth, uniformly elliptic and $u$ is not $C^2$. Therefore, it would be interesting to know under what condition a $C^{1,1}$ solution of (1) is actually in the $C^2$ class.

It is our purpose in this paper to show that any $C^{1,1}$ viscosity solution of (1) must be $C^{2,\alpha}$ if the elliptic operator $F$ has the Liouville property.

A continuous function $u(x)$ is said to be a viscosity subsolution (resp., supersolution) of (1) in a domain $\Omega$ if for $x_0 \in \Omega$ and $\phi(x) \in C^2$, $u - \phi$ attains the local minimum (maximum) at $x_0$.

Received by the editors September 20, 1999.

2000 Mathematics Subject Classification. Primary 35J60; Secondary 35B65.

Key words and phrases. Fully nonlinear elliptic equation, regularity, Liouville property, VMO.
maximum (resp., minimum) at $x_0$, then $F(D^2\phi(x_0)) \geq 0$ (resp., $\leq 0$). If $u$ is both a subsolution and a supersolution, then we say $u$ is a viscosity solution. We mention that if $u \in C^{1,1}$, then $u$ is a viscosity solution of (1) if and only if $u$ is a strong solution to (1).

Equation (1) or $F$ is said to satisfy the Liouville property if $u \in C^{1,1}_{loc}(\mathbb{R}^n)$ is an entire viscosity solution of (1) with bounded $D^2u$ in $\mathbb{R}^n$, $|D^2u| \leq C$, then $u$ must be a polynomial of degree at most 2.

Let $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

Now we state the main theorem.

**Theorem.** Suppose that $F \in C^1$ satisfies (2) and $F(0) = 0$. Let $u \in C^{1,1}(B_1(0))$ be a viscosity solution of (1) in $B_1(0)$. If equation (1) satisfies the Liouville property, then for any $0 < \alpha < 1$, $u \in C^{2,\alpha}(B_{1/2}(0))$ and $|D^2u|_{C^{\alpha}(B_{1/2}(0))} \leq C$, where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\|u\|_{C^{1,1}(B_1(0))}$, $F$, and the modulus of continuity of $DF$.

§2. The proof of the Theorem

We will use the blow-up technique to prove the Theorem. The tool needed to obtain a subsequence of blow-up solutions converging in $W^{2,2}_{loc}(\mathbb{R}^n)$ is the $W^{2,\delta}$ estimate for nondivergent uniform elliptic equations. For the convenience of our readers, let us give a little more preliminary information.

Recall that $u \in \text{BMO}(\Omega)$ is in VMO($\Omega$) if

$$\eta_u(R, \Omega) = \sup_{0 < r \leq R} \int_{B_r(x_0) \cap \Omega} |u(x) - u_{x_0,r}| \, dx \to 0, \text{ as } R \to 0,$$

where $\int_A f \, dx$ denotes the average of $f$ over $A$ and $u_{x_0,r}$ the average of $u$ over $B_r(x_0) \cap \Omega$. We will call $\eta_u$ the VMO modulus of $u$ in $\Omega$.

Now let us recall the class $S$ of solutions of uniformly elliptic equations. For more details, see [CC]. Let $A_{\lambda, \Lambda}$ denote all symmetric matrices whose eigenvalues belong to $[\lambda, \Lambda]$. Define Pucci extremal operators $M^+(M)$ and $M^-(M)$ by

$$M^+(M) = \sup_{A \in A_{\lambda, \Lambda}} \text{trace}(AM),$$

$$M^-(M) = \inf_{A \in A_{\lambda, \Lambda}} \text{trace}(AM),$$

for $M \in S$. It is easy to check that $M^+$ and $M^-$ are uniformly elliptic operators. A continuous function $u$ is in class $S$ if $M^-(D^2u) \leq 0$ and $M^+(D^2u) \geq 0$ in the viscosity sense.

The following result on precompact sets in $L^p$ is a local variant of Theorem 3.44 in [A].

**Proposition 1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\mathcal{A}$ a bounded subset of $L^p(\Omega)$, $1 \leq p < \infty$. For any domain $D \subset \subset \Omega$, if

$$\sup_{u \in \mathcal{A}} \int_D |u(x + h) - u(x)|^p \, dx \to 0, \text{ as } |h| \to 0,$$

then $\mathcal{A}$ is precompact in $L^p(D)$.

Now let us prove the following lemma.
Lemma 1. Assume that $F$ satisfies (2) and $F(0) = 0$. Then the following two statements are equivalent:

(i) If $u \in C^{1,1}(B_1(0))$ is a viscosity solution of (1) in $B_1(0)$ and $|D^2u| \leq M$ in $B_1(0)$, then $D^2u \in VMO(B_{1/2}(0))$ and $\eta_{D^2u}(R) \leq \eta(R)$, where $\eta_{D^2u}(R)$ is the VMO modulus of $D^2u$ in $B_{1/2}(0)$, $\lim_{R \rightarrow 0^+} \eta(R) = 0$, and $\eta$ depends only on $n$, $\lambda$, $\Lambda$, $F$, and $M$.

(ii) $F$ satisfies the Liouville property.

Proof. (i) implies (ii). Let $u \in C^{1,1}_\text{loc}(\mathbb{R}^n)$ be an entire solution of (1) with $|D^2u| \leq M$ in $\mathbb{R}^n$. Consider

$$v_k(y) = \frac{u(ky) - u(0) - Du(0)ky}{k^2}, \quad k = 1, 2, \ldots$$

Obviously $\|v_k\|_{C^{1,1}(B_1(0))} \leq C_n M$ and

$$F(D^2v_k) = 0 \quad \text{in } B_1(0).$$

Therefore by (i), for $\rho > 0$ we have

$$\int_{B_\rho(0)} |D^2u - (D^2u)_{0,\rho}| \, dx = \int_{B_{\rho/k}(0)} |D^2v_k - (D^2v_k)_{0,\rho/k}| \, dy \leq \eta_{D^2v_k}(\rho/k) \leq \eta(\rho/k) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

This implies that $D^2u = \text{const}$ in $\mathbb{R}^n$ and hence $u$ is a polynomial of degree at most 2.

Suppose that $F$ satisfies the Liouville property. We want to show (i). Let $X_M = \{ u \in C^{1,1}(B_1(0)) : F(D^2u) = 0 \text{ and } |D^2u| \leq M \text{ in } B_1(0) \}$.

To prove that (i) holds, it suffices to show the following claim:

$$\sup_{u \in X_M} \int_{B_r(x_0)} |D^2u - (D^2u)_{x_0,r}|^2 \, dx \rightarrow 0, \quad \text{as } R \rightarrow 0. \quad (3)$$

We will show (3) by contradiction. If (3) is false, then there exist $\varepsilon_0 > 0$, $r_k \rightarrow 0$, $x_k \in B_{1/2}(0)$, $u_k \in X_M$ such that for $k \geq 1$

$$\int_{B_{r_k}(x_k)} |D^2u_k - (D^2u_k)_{x_k,r_k}|^2 \, dx \geq \varepsilon_0.$$ 

Let

$$T_k y = x_k + r_k y, \quad \Omega_k = T_k^{-1}B_1(0);$$

$$v_k(y) = \frac{u_k(x_k + r_k y) - u_k(x_k) - Du_k(x_k)r_k y}{r_k}.$$

It is easy to check that

$$F(D^2v_k) = 0, \quad \text{in } \Omega_k.$$ 

$$\int_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2 \, dy \geq \varepsilon_0. \quad (4)$$

$$\|v_k\|_{C^{1,1}(B_2(0))} \leq C_{n,A} M; \quad \text{if } B_{2Ar_k}(x_k) \subset B_1(0). \quad (5)$$
Now we want to show \( \{D^2v_k\} \) is precompact in \( L^2 \). By \[ \text{CC} \] (see Prop. 5.5)

\[
\triangle_{\tau e}v_k(y) = \frac{v_k(y + \tau e) - v_k(y)}{\tau} \in \mathcal{S}, \quad \text{in } B_{3A/2}(0), \quad |\tau| = 1.
\]

By the \( W^{2,\delta} \) estimate (\( \delta > 0 \)) (see Prop. 7.4, \[ \text{CC} \]) for functions in \( \mathcal{S} \)

\[
\int_{B_{A}(0)} |D^2\triangle_{\tau e}v_k|^{\delta}(y) \, dy \leq C_{A, M, M'} \|D^2v_k\|_{L^\infty(B_{A}(0))} \leq C_{A, M, M'},
\]

where \( \delta \) and \( C_A \) are independent of \( k \).

Therefore, by (5) we have

\[
\int_{B_{A}(0)} |D^2v_k(y + \tau e) - D^2v_k(y)|^2 \, dy \\
\leq C\|D^2v_k\|_{L^\infty(B_{A}(0))}^{2-\delta} \int_{B_{A}(0)} |D^2v_k(y + \tau e) - D^2v_k(y)|^\delta \, dy \\
\leq C\delta.
\]

By Proposition 1, this fact together with

\[
\|D^2v_k\|_{L^2(B_{A}(0))} \leq C_A
\]

implies that \( \{D^2v_k\} \) is precompact in \( L^2(B_{A}(0)) \). Since \( v_k(0) = 0 \), \( Dv_k(0) = 0 \), and \( \|v_k\|_{C^{1, \lambda}(B_{2A}(0))} \leq C_A \), we may assume that by the diagonalizing process

\[
v_k \to v, \quad \text{in } W^{2,2}(\mathbb{R}^n) \cap C^1_{loc}(\mathbb{R}^n),
\]

\[
D^2v_k \to D^2v, \quad \text{a.e. in } \mathbb{R}^n.
\]

Therefore, \( F(D^2v) = 0 \) in \( \mathbb{R}^n \). Since \( \|D^2v_k\|_{L^\infty(B_{A}(0))} \leq \|D^2u_k\|_{L^\infty(B_{1}(0))} \leq M \), \( |D^2v| \leq M \) in \( \mathbb{R}^n \). By the Liouville property, \( v \) must be a polynomial of degree at most 2, and hence \( D^2v = \text{const} \). This contradicts the following:

\[
\lim_{k \to \infty} \int_{B_{1}(0)} |D^2v - (D^2v)_0|^2 \, dy = 0,
\]

Thus, Lemma 1 follows.

**Lemma 2.** Let \( F \in C^1 \) satisfy (2) and \( F(0) = 0 \). If \( u \) is a viscosity solution of (1) in \( B_{1}(0) \) and \( D^2u \in \text{VMO}(B_{1}(0)) \), then for any \( 0 < \alpha < 1 \), \( u \in C^{2, \alpha}(B_{1/2}(0)) \) and

\[
[D^2u]_{C^{2, \alpha}(B_{1/2}(0))} \leq C, \quad \text{where } C \text{ depends on } n, \alpha, \lambda, \Lambda, \text{ the modulus of continuity of } DF, \|u\|_{C^1(B_{1/2}(0))}, \text{ and the VMO modulus of } D^2u.
\]

**Proof.** By differentiating (1), we obtain

\[
a_{ij}(x)D_{ij} \triangle_h u(x) = 0, \quad \text{in } B_{3/4}(0),
\]

where \( \triangle_h u(x) = [u(x + he) - u(x)]/h, \ h < \frac{1}{4}, \ |e| = 1 \), and

\[
a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial M_{ij}}((1 - \theta)D^2u(x) + \theta D^2u(x + he)) \, d\theta.
\]

Let \( u_h(x) = u(x + he) \) and

\[
c_{ij} = \int_0^1 \frac{\partial F}{\partial M_{ij}}((1 - \theta)(D^2u)_{x_0,r} + \theta (D^2u_h)_{x_0,r}) \, d\theta.
\]
Without loss of generality, we can assume that the continuity modulus of $DF$ denoted by $\omega(R)$ is concave. Obviously by Jensen’s inequality
\[
\int_{B_r(x_0)} |a_{ij}(x) - c_{ij}| \, dx \\
\leq \int_{B_r(x_0)} \int_0^1 \omega(\{1 - \theta(1 - \frac{D^2u}{D^2u_h})_{x_0, r}\}) \, d\theta \, dx \\
\leq \omega(\eta D^2u_h(r)) \to 0, \quad \text{as } r \to 0.
\]
Therefore $a_{ij} \in \text{VMO}(B_{3/4}(0))$. Since (2) holds, $\lambda I \leq (a_{ij}) \leq \Lambda' I$ and (6) is uniformly elliptic. By the $L^p$ estimate in [CFL], we obtain
\[
||\Delta_{he} u||_{W^{2, p}(B_{1/2}(0))} \leq C ||\Delta_{he} u||_{L^\infty(B_{3/4}(0))}.
\]
Thus, we finish the proof of Lemma 2.

Theorem follows immediately from Lemma 1 and Lemma 2.

Acknowledgement

The author expresses his gratitude to Professor L. Caffarelli for discussions on several occasions. The author also thanks the referee for some suggestions.

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