TWISTED HIGHER MOMENTS OF KLOOSTERMAN SUMS

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Abstract. Let \( \chi \) be a nontrivial Dirichlet character modulo an odd prime \( p \).
Write
\[
S(a) = \sum_{x=1}^{p-1} e\left(\frac{x + ax^{-1}}{p}\right) = 2\sqrt{p}\cos\theta(a).
\]
We shall prove
\[
\frac{1}{p} \sum_{a=1}^{p-1} \chi(a) S(a)^2 = \chi(-1)g(\chi)^2 J(\chi, \chi^2)
\]
and, for complex \( \chi \),
\[
\left| \frac{1}{p} \sum_{a=1}^{p-1} \chi(a) \frac{\sin(k+1)\theta(a)}{\sin\theta(a)} \right| \leq c(k)\sqrt{p}, \quad k > 0,
\]
where \( c(k) \) is a constant depending only on \( k \).

1. Introduction

Let \( p \) be an odd prime. Write
\[
S(a) = \sum_{x=1}^{p-1} e\left(\frac{x + ax^{-1}}{p}\right).
\]
It is a Kloosterman sum. It is known that (\( \mathbb{I} \))
\[
\frac{1}{p} \sum_{a=1}^{p-1} S(a) = 1,
\]
\[
\frac{1}{p} \sum_{a=1}^{p-1} S(a)^2 = p^2 - p - 1,
\]
\[
\frac{1}{p} \sum_{a=1}^{p-1} S(a)^3 = \left(\frac{3}{p}\right)p^2 + 2p + 1
\]

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By A. Weil’s result ([W]), we may write
\[ \cos \theta(a) = \frac{1}{2\sqrt{p}} S(a). \]

Katz ([K]) proved the following equidistribution result:
\[ \left| \sum_{a=1}^{p-1} \sin(k+1)\theta(a) \sin \frac{\theta(a)}{\sin \theta(a)} \right| \leq \frac{k+1}{2} \sqrt{p}, \quad k \geq 1. \]

We may expect, for any Dirichlet character \( \chi \) to the modulus \( p \), that
\[ \left| \sum_{a=1}^{p-1} \chi(a) \sin(k+1)\theta(a) \sin \frac{\theta(a)}{\sin \theta(a)} \right| \leq \frac{k+1}{2} \sqrt{p}, \quad k \geq 1. \]

D. H. and E. Lehmer ([L]) found empirically in 1952 and proved in 1959 that
\[ \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) \sin 4\theta(a) = 2\sqrt{p} (\frac{-1}{p}) (A^2/p - 1) \]
if \( p = A^2 + 3B^2, 3|(A + 1) \) and vanishes if \( 3|(p + 1) \).

We shall prove

**Theorem 1.** If \( \chi \) is a nonquadratic Dirichlet character to the modulus \( p \), then
\[ \left| \sum_{a=1}^{p-1} \chi(a) \sin(k+1)\theta(a) \sin \frac{\theta(a)}{\sin \theta(a)} \right| \leq c(k) \sqrt{p}, \quad k > 0, \]
where \( c(k) \) is a constant depending only on \( k \).

If \( k = 2 \), we can prove more.

**Theorem 2.**
\[ \sum_{a=1}^{p-1} \chi(a) \frac{\sin 3\theta(a)}{\sin \theta(a)} = \frac{1}{p} \chi(-1) g(\chi)^2 J(\chi, \chi^2), \]
where \( \chi \) is a nontrivial Dirichlet character to the modulus \( p \),
\[ g(\chi) = \sum_{x=1}^{p-1} \chi(x) e\left( \frac{x}{p} \right) \]
is a Gauss sum and
\[ J(\chi_1, \chi_2) = \sum_{x=1}^{p-1} \chi_1(x) \chi_2(1 - x) \]
is a Jacobi sum.
Another form of Theorem 2 is
\[
\sum_{a=1}^{p-1} \chi(a)S(a)^2 = \chi(-1)g(\chi)^2 J(\chi, \chi^2),
\]
where \(\chi\) is a nontrivial Dirichlet character to the modulus \(p\). It is equivalent to
\[
(p - 1)S(a)^2 = p^2 - p - 1 + \sum_{\chi} \chi(-1)g(\chi)^2 J(\chi, \chi^2)\overline{\chi}(a),
\]
where \(\chi\) runs over all nontrivial Dirichlet characters to the modulus \(p\). So
\[
(p - 1)S(a)^2 = p^2 - p - 1 + \sum_{\chi} \chi(-1)g(\chi)^2 J(\chi, \chi^2)\overline{\chi}^2(a),
\]
where \(\chi\) runs over all nontrivial Dirichlet characters to the modulus \(p\). It implies
\[
\sum_{a=1}^{p-1} |\chi(a)S(a^2)|^2 \leq 2p^{3/2},
\]
where \(\chi\) is a nontrivial Dirichlet character to the modulus \(p\). That is a little sharper than
\[
\sum_{a=1}^{p-1} |\chi(a)S(a^2)|^2 \leq 4p^{3/2},
\]
which was proved for quadratic Dirichlet character \(\chi\) to the modulus \(p\), and conjectured for general nontrivial Dirichlet character \(\chi\) in [C] by Conrey and Iwaniec.

Remark. Theorems 1 and 2 generalize to the finite field case.

2. THE TWISTED SQUARE MOMENT

We now prove Theorem 2. Opening \(S(a)^2\), we get
\[
\sum_{a=1}^{p-1} \chi(a)S(a)^2 = \sum_{a=1}^{p-1} \chi(a) \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{x + y}{p}\right) e\left(\frac{a(x^{-1} + y^{-1})}{p}\right).
\]
Summing over \(a\) first and applying
\[
\sum_{a=1}^{p-1} \chi(a)e\left(\frac{ax}{p}\right) = \overline{\chi}(x)g(\chi),
\]
we get
\[
\sum_{a=1}^{p-1} \chi(a)S(a)^2 = g(\chi) \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{x + y}{p}\right) \overline{\chi}(x^{-1} + y^{-1}).
\]
A change of variable yields
\[
\sum_{a=1}^{p-1} \chi(a)S(a)^2 = g(\chi) \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{x + xy}{p}\right) \overline{\chi}(x^{-1} + x^{-1}y^{-1}).
\]
Summing over \(x\) first and applying
\[
\sum_{a=1}^{p-1} \chi(a)e\left(\frac{ax}{p}\right) = \overline{\chi}(x)g(\chi)
\]
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once more, we get
\[
\sum_{a=1}^{p-1} \chi(a)S(a)^2 = g(\chi)^2 \sum_{y=1}^{p-1} \chi(1 + y^{-1})\chi(1 + y)
\]
\[
= g(\chi)^2 \sum_{y=1}^{p-1} \chi(y)\chi^2(1 + y) = \chi(-1)g(\chi)^2 J(\chi, \chi^2).
\]

3. The twisted higher moments

We now prove Theorem 1. For \( q = p^m \), write
\[
S(a; q) = \sum_x e\left(\frac{tr(x + ax^{-1})}{p}\right) = 2\sqrt{q} \cos(\theta(\psi; q)),
\]
where \( x \) runs over all nonzero elements in \( \mathbb{F}_q \), and \( tr \) is the trace map from \( \mathbb{F}_q \) to \( \mathbb{F}_p \). Write
\[
T(\psi; q) = \sum_a \psi(a) \frac{\sin(k + 1)\theta(a; q)}{\sin\theta(a; q)},
\]
where \( a \) runs over all nonzero elements in \( \mathbb{F}_q \), and \( \psi \) is a multiplicative character of \( \mathbb{F}_q \). Write
\[
L(t) = \exp\left(\sum_{m \geq 1} T(\chi_m; p^m)t^m/m\right),
\]
where \( \chi_m \) is the lift of \( \chi \) from \( \mathbb{F}_p \) to \( \mathbb{F}_{p^m} \). According to B. Dwork ([Dw]), \( L(t) \) is a rational function
\[
L(t) = \prod_{v \in I}(1 - \alpha_v(\chi)t) \prod_{v \in J}(1 - \alpha_v(\chi)t)^{-1}.
\]
Equivalently,
\[
T(\chi_m; p^m) = -\sum_{v \in I} \alpha_v(\chi)^m + \sum_{v \in J} \alpha_v(\chi)^m.
\]
According to P. Deligne ([D]),
\[
|\alpha_v(\chi)| = p^{c_v/2}
\]
and the total number of \( \alpha_v(\chi) \) is bounded by a number \( c(k) \) depending only on \( k \).
We conclude that
\[
|\alpha_v(\chi)| \leq \sqrt{p},
\]
from which Theorem 2 follows. Otherwise, according to E. Bombieri’s arguments ([B]),
\[
|T(\chi_m; p^m)| > (1 - \varepsilon)p^m
\]
for infinitely many \( m \). Then
\[
|T(\chi_m; p^m)|^2 + |T(\chi_m; p^m)|^2 > 2(1 - \varepsilon)^2 p^{2m}
\]
for infinitely many \( m \), contradicting the following lemma.
Lemma 3.

\[ \frac{1}{q-1} \sum_{\psi} |T(\psi; q)|^2 = q + O(\sqrt{q}), \]

where \( \psi \) runs over all multiplicative characters of \( \mathbb{F}_q \).

Indeed, we have

\[ \frac{1}{q-1} \sum_{\psi} |T(\psi; q)|^2 = \sum_a \left( \frac{\sin(k+1)\theta(a; q)}{\sin \theta(a; q)} \right)^2. \]

As

\[ \left( \frac{\sin(k+1)\theta}{\sin \theta} \right)^2 = 1 + \sum_{1 \leq i \leq 2k} \frac{\sin(l+1)\theta}{\sin \theta}, \]

we have

\[ \frac{1}{q-1} \sum_{\psi} |T(\psi; q)|^2 = q - 1 + \sum_{1 \leq i \leq 2k} c_l \sum_a \frac{\sin(l+1)\theta(a; q)}{\sin \theta(a; q)}. \]

The lemma now follows from N. Katz’s equidistribution result ([K])

\[ \left| \sum_a \frac{\sin(l+1)\theta(a; q)}{\sin \theta(a; q)} \right| \leq \frac{l+1}{2} \sqrt{q}. \]

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Added in proof

From Theorem 1 one can deduce that, for every odd integer \( m \),

\[ \left| \sum_{a=1}^{p-1} \frac{\sin(k+1)\theta(a^m)}{\sin \theta(a^m)} \right| \leq (m, p-1)c(k)\sqrt{p}, \quad k > 0, \]

where \( c(k) \) is the constant in Theorem 1. That is, for every fixed odd integer \( m \), the angles \( \theta(a^m) \), for \( 1 \leq a \leq p-1 \), are equidistributed with respect to the Sato-Tate measure as \( p \) goes to infinity.

References


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