PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 7, Pages 1981–1982
S 0002-9939(02)06519-X
Article electronically published on January 25, 2002

POINTWISE DIMENSIONS AND RÉNYI DIMENSIONS

RYSZARD RUDNICKI

(Communicated by David Preiss)

ABSTRACT. We prove that the local lower and upper pointwise dimensions of a probability measure μ are bounded from below by the lower generalized dimension $\underline{D}_\mu(q)$ for $q > 1$ and from above by the upper generalized dimension $\overline{D}_\mu(q)$ for $q < 1$.

Let μ be a probability Borel measure on a metric space (M, ϱ) and let X be the support of μ. For any $q \neq 1$ we define the lower generalized dimensions $\underline{D}_\mu(q)$ by

$$\underline{D}_\mu(q) = \varliminf_{\varepsilon \to 0^+} \frac{\log \int_X \mu(B(x,\varepsilon))^{q-1}\, \mu(dx)}{(q-1)\log \varepsilon},$$

where $B(x,\varepsilon)$ is the open ball with centre $x \in X$ and radius $\varepsilon > 0$. Replacing in the above definition the lower limit by the upper one we define the upper generalized dimensions $\overline{D}_\mu(q)$. The generalized dimensions $\underline{D}_\mu(q)$ and $\overline{D}_\mu(q)$ introduced by Hentschel and Procaccia [2] are decreasing functions of q (see [1]). The generalized dimensions are frequently called Rényi dimensions. They are intensively studied in mathematical and physical literature concerning dynamical systems (see [4] and ref. therein). The lower and upper pointwise dimensions $\underline{d}_\mu(x)$ and $\bar{d}_\mu(x)$ are given by

$$\underline{d}_\mu(x) = \varliminf_{\varepsilon \to 0^+} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}, \quad \bar{d}_\mu(x) = \varlimsup_{\varepsilon \to 0^+} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}.$$

The following theorem was formulated in another but equivalent way in [3] in the case when μ is a Borel measure on $\mathbb{R}^d$ with bounded support. The proof given in [3] is based on the Vitali Covering Lemma and cannot be adapted to the general case.

Theorem. *If $q > 1$, then $\underline{D}_\mu(q) \le \underline{d}_\mu(x)$ for μ-a.e. x. If $q < 1$, then $\overline{D}_\mu(q) \ge \bar{d}_\mu(x)$ for μ-a.e. x.*

Proof. To prove the first part of the Theorem we assume, contrary to our claim, that there exist $q > 1$ and $\lambda_0 > 0$ such that $\underline{D}_\mu(q) > \lambda_0$ and $\mu\{x : \underline{d}_\mu(x) < \lambda_0\} > 0$. Let λ, λ_1 be constants such that $\lambda_0 < \lambda < \lambda_1 < \underline{D}_\mu(q)$. Let $A = \{x : \underline{d}_\mu(x) < \lambda_0\}$

Received by the editors January 15, 2001.
2000 *Mathematics Subject Classification.* Primary 28A80; Secondary 28A78.
Key words and phrases. Pointwise dimension, generalized (Rényi) dimension, measure.
This research was partially supported by the State Committee for Scientific Research (Poland) Grant No. 2 P03A 010 161.

and $\mathcal{E}(x) = \{\varepsilon \in (0,1)\colon\ \mu(B(x,\varepsilon)) > \varepsilon^{\lambda_0}\}$ for $x \in X$. Then from the definition of the set A it follows immediately that $\inf \mathcal{E}(x) = 0$ for $x \in A$. Let $\bar{\varepsilon} \in \mathcal{E}(x)$. Then

$$\mu(B(x,\varepsilon)) > \varepsilon^{\lambda} \quad \text{for} \quad \varepsilon \in \left[\bar{\varepsilon}, \bar{\varepsilon}^{\lambda_0/\lambda}\right]. \tag{1}$$

Set $A(\varepsilon) = \{x\colon\ \mu(B(x,\varepsilon)) > \varepsilon^{\lambda}\}$ and $\varepsilon_n = 2^{-(\lambda/\lambda_0)^n}$ for $n \in \mathbb{N}$. Observe that

$$A \subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A(\varepsilon_n). \tag{2}$$

Indeed, fix an $x \in A$. Then for every $m \in \mathbb{N}$ there exist an $\bar{\varepsilon} \in \mathcal{E}(x)$ and $n \ge m$ such that $\varepsilon_{n+1} \le \bar{\varepsilon} \le \varepsilon_n$. From (1) it follows that $x \in A(\varepsilon_n)$ and condition (2) holds. Since $\mu(A) > 0$, from (2) we obtain

$$\sum_{n=1}^{\infty} \mu(A(\varepsilon_n)) = \infty. \tag{3}$$

Since $\underline{D}_{\mu}(q) > \lambda_1$, there exists $r > 0$ such that

$$\int_X \mu(B(x,\varepsilon))^{q-1}\, \mu(dx) \le \varepsilon^{\lambda_1(q-1)} \quad \text{for} \quad \varepsilon \in (0,r). \tag{4}$$

From (1) and (4) we obtain $\mu(A(\varepsilon))\varepsilon^{\lambda(q-1)} \le \varepsilon^{\lambda_1(q-1)}$ for $\varepsilon \in (0,r)$, which implies that $\mu(A(\varepsilon)) \le \varepsilon^{(\lambda_1-\lambda)(q-1)}$. Since $\sum_{n=1}^{\infty} \varepsilon_n^{(\lambda_1-\lambda)(q-1)} < \infty$, the series (3) is convergent, a contradiction.

In the same manner we can prove the second part of the Theorem. In this case we assume, contrary to our claim, that there are some constants $\lambda_1, \lambda, \lambda_0$ and $q < 1$ such that $\overline{D}_{\mu}(q) < \lambda_1 < \lambda < \lambda_0$ and $\mu\{x\colon\ \bar{d}_{\mu}(x) > \lambda_0\} > 0$. Replacing in the definitions of the sets of $A, \mathcal{E}(x)$, $A(\varepsilon)$ the inequalities by opposite ones we can check that $\mu(B(x,\varepsilon)) < \varepsilon^{\lambda}$ for $\varepsilon \in [\bar{\varepsilon}^{\lambda_0/\lambda}, \bar{\varepsilon}]$. Also, (2) holds if $\varepsilon_n = 2^{-(\lambda_0/\lambda)^n}$ for $n \in \mathbb{N}$. The rest of the proof is exactly the same as the proof of the first part. □

Remark. In the statement of the Theorem inequalities cannot be replaced by equalities. Let $f = \sum_{n=1}^{\infty} 2^{-n+n^2} f_n$, where f_n is the characteristic function of the interval $[2^{-n}, 2^{-n} + 2^{-n^2}]$. If μ is the probability measure on the interval $[0,1]$ with the density f, then $\underline{D}_{\mu}(q) = \overline{D}_{\mu}(q) = 0$ for all $q > 1$ but $\underline{d}_{\mu}(x) = \bar{d}_{\mu}(x) = 1$ for all $x \in \operatorname{supp}\mu$. If $\mu = \sum_{n=1}^{\infty} a_n \delta_{x_n}$, where $a_n = cn^{-1}\log^{-2}(n+1)$ and $\sum_{n=1}^{\infty} a_n = 1$, $x_1 = 1$, $x_n = 1 - \sum_{k=1}^{n-1} a_k$ for $n \ge 2$ and δ_{x_n} is the Dirac measure at x_n, then $\overline{D}_{\mu}(q) \ge \underline{D}_{\mu}(q) \ge 1$ for all $q < 1$ but $\underline{d}_{\mu}(x) = \bar{d}_{\mu}(x) = 0$ for all $x \in \operatorname{supp}\mu$.

References

1. C.D. Cutler, *Some results on the behavior and estimation of the fractal dimensions of distribution on attractors*, J. Statist. Phys. **62** (1991), 651–708. MR **92b:**58137
2. H. Hentschel and I. Procaccia, *The infinite number of generalized dimensions of fractals and strange attractors*, Phys. D, **8** (1983), 435–444. MR **85a:**58064
3. S-N. Ngai, *A dimension result arising from the L^q-spectrum of a measure*, Proc. Amer. Math. Soc. **125** (1997), 2943–2951. MR **97m:**28007
4. Y.B. Pesin, *Dimension Theory in Dynamical Systems. Contemporary views and applications*, University of Chicago Press, Chicago, 1997. MR **99b:**58003

Institute of Mathematics, Polish Academy of Sciences and Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, Poland

E-mail address: `rudnicki@us.edu.pl`