

## CONVERGENCE OF ZETA FUNCTIONS OF GRAPHS

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ABSTRACT. The  $L^2$ -zeta function of an infinite graph  $Y$  (defined previously in a ball around zero) has an analytic extension. For a tower of finite graphs covered by  $Y$ , the normalized zeta functions of the finite graphs converge to the  $L^2$ -zeta function of  $Y$ .

### INTRODUCTION

Associated to any finite graph  $X$  there is a zeta function  $Z(X, u)$ ,  $u \in \mathbb{C}$ . It is defined as an infinite product but shown (in various different cases) by Ihara, Hashimoto, and Bass [5, 4, 1] to be the reciprocal of a polynomial. Indeed the rationality formula for a  $q + 1$  regular  $X$  states that:

$$(0.1) \quad Z(X, u)^{-1} = (1 - u^2)^{-\chi(X)} \text{Det}(I - \delta u + qu^2).$$

Here  $\delta$  is the adjacency operator of  $X$ .

In [2], an  $L^2$ -zeta function is defined for noncompact graphs with symmetries, using the machinery of von Neumann algebras. A rationality formula similar to (0.1) expresses the relationship between the zeta function and the von Neumann determinant of a Laplace operator. The results of this paper focus on a special case. Let  $Y$  be an infinite graph which covers a finite graph  $B = \pi \backslash Y$ . The  $L^2$ -zeta function  $Z_\pi(Y, u)$  is defined in [2] only in a small neighborhood of zero. The first result of this paper is to extend the  $L^2$ -zeta function to the interior of  $C = \{u \in \mathbb{C} : |u| = q^{-1/2}\} \cup [-1, -\frac{1}{q}] \cup [\frac{1}{q}, 1]$ .

In the second part of the paper, we consider a tower of finite graphs  $B_i$  covered by  $Y$ . Put  $N_i = |B_i|/|B|$ . In Theorem 2.1 we show that the zeta functions for the  $B_i$ , renormalized by taking  $N_i^{\text{th}}$  roots, converge to the  $L^2$ -zeta function for  $Y$ . The argument is inspired by, and uses, work of Lück [7].

In the first section we recall the definitions of the zeta functions of finite and infinite graphs. One of the main results of this paper is Theorem 1.5 in this section. In the second section of this paper we prove the convergence theorem and exhibit interesting examples. Theorem 2.5 generalizes work of Deitmar [3].

### 1. ZETA FUNCTIONS

In this section we recall the definition of the zeta function and related material. We first recall the definition of the zeta function for finite graphs.

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1.1. **Finite graphs.** For a graph  $X$ , let  $VX$  and  $EX$  denote the sets of vertices and edges, respectively, of  $X$ . If each vertex has the same degree, then  $X$  is *regular*.

**Definition 1.1.** Let  $X$  be a finite graph. A closed path in  $X$  is *primitive* if it is not a nontrivial power of another path inside the fundamental group of  $X$ . Let  $P$  be the set of free homotopy classes of primitive closed paths of  $X$ . Then the zeta function of  $X$  is

$$Z(X, u) = \prod_{\gamma \in P} (1 - u^{\ell(\gamma)})^{-1},$$

where  $\ell(\gamma)$  is the minimum length of paths in the class of  $\gamma$ .

Let  $\delta$  be the adjacency operator of the graph  $X$  acting on  $l^2(VX)$ . For  $f \in l^2(VX)$  let  $Qf(x) = q(x)f(x)$  where  $q(x) + 1$  is the degree of the vertex  $x$ . Put  $\Delta(X, u) = I - \delta u + Qu^2$ . The Ihara rationality formula says that  $Z(X, u)^{-1}$  is a polynomial:

$$(1.1) \quad Z(X, u)^{-1} = (1 - u^2)^{-\chi(X)} \text{Det}(\Delta(X, u)).$$

The zeta function satisfies the following functional equation [1, Corollary 3.10]:

$$(1.2) \quad Z(X, (qu)^{-1}) = \left( \frac{1 - u^2}{q^2u^2 - 1} \right)^{-\chi(X)} q^{2e-v} u^{2e} Z(X, u)$$

where  $e = |EX|$ ,  $v = |VX|$  and  $\chi(X)$  is the Euler characteristic of  $X$ .

For more details and examples see [9, 10].

The following proposition is well known. A proof can be found in [6, page 59].

**Proposition 1.2.** *The poles of the zeta function for any finite  $q + 1$  regular graph lie on the set  $C$  (Figure 1) where*

$$C = \{u \in \mathbb{C} : |u| = q^{-1/2}\} \cup [-1, -\frac{1}{q}] \cup [\frac{1}{q}, 1].$$

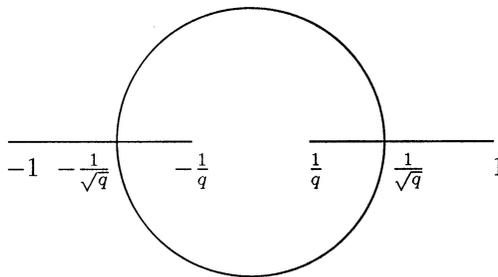


FIGURE 1. The set  $C$

Let  $\Omega$  be the interior of  $C$ . Then:

**Proposition 1.3.** *If  $u \in \Omega$  and  $\lambda \in [-(q+1), q+1]$ , then  $1 - \lambda u + qu^2 \in \mathbb{C} - (-\infty, 0]$ .*

*Proof.* Let  $u = a + bi$ . For  $1 - \lambda u + qu^2$  to be real we must have  $1 - \lambda u + qu^2 = 1 - \lambda \bar{u} + q\bar{u}^2$ . This implies that either  $b = 0$  or  $a = \frac{\lambda}{2q}$ . If  $a = \frac{\lambda}{2q}$ , then  $1 - \lambda u + qu^2 = 1 - q(a^2 + b^2)$  which is negative or zero if  $u$  is on or outside of the circle  $|u| = q^{-1/2}$ . In the case that  $b = 0$  put  $f(a) := 1 - \lambda a + qa^2 = 1 - \lambda u + qu^2$ . If  $f$  has no real root, then  $f$  is always positive. Otherwise  $|\lambda| \geq 2\sqrt{q}$ . Then  $f(-\frac{1}{q})$  and

$f(\frac{1}{q})$  are nonnegative, and  $f'$  is not zero in  $(-\frac{1}{q}, \frac{1}{q})$ . Therefore  $f$  will be positive on  $(-\frac{1}{q}, \frac{1}{q})$ . □

**Corollary 1.4.** *For a  $q + 1$  regular finite graph  $X$ , the polynomial  $\text{Det } \Delta(X, u)$  has an analytic  $N^{\text{th}}$  root on  $\Omega$  for all  $N \in \mathbb{N}$ .*

*Proof.* We know  $\text{Det } \Delta(X, u) = \prod_{\lambda} (1 - \lambda u + qu^2)$  where  $\lambda$  varies over eigenvalues of the adjacency operator of the graph. Then  $\prod_{\lambda} (1 - \lambda u + qu^2)^{\frac{1}{N}}$  is an analytic  $N^{\text{th}}$  root for  $\text{Det } \Delta(X, u)$ . □

**1.2. Infinite graphs.** For  $\pi$  a countable discrete group, the *von Neumann algebra* of  $\pi$  is the algebra  $\mathcal{N}(\pi)$  of bounded  $\pi$ -equivariant operators from  $l^2(\pi)$  to  $l^2(\pi)$ . The *von Neumann trace* of an element  $f \in \mathcal{N}(\pi)$  is defined by

$$\text{Tr}_{\pi} f = \langle f(e), e \rangle$$

for  $e \in \pi$  the unit element. For  $H = \bigoplus_{i=1}^n l^2(\pi)$  and a bounded  $\pi$ -equivariant operator  $f : H \rightarrow H$ , define

$$\text{Tr}_{\pi} f = \sum_{i=1}^n \text{Tr}_{\pi} f_{ii}.$$

The trace as defined is independent of the decomposition of  $H$ . The von Neumann trace extends to bounded  $\pi$ -equivariant operators on Hilbert  $\mathcal{N}(\pi)$ -modules, but they will not be needed.

Now let  $Y = (VY, EY)$  be an infinite graph. Suppose the group  $\pi$  acts freely on  $Y$  with finite quotient  $B$ .

Let  $P$  denote the set of free homotopy classes of primitive closed paths in  $Y$ . For  $\gamma \in P$ ,  $\ell(\gamma)$  is the length of the shortest representative of  $\gamma$ . The group  $\pi_{\gamma}$  is the stabilizer of  $\gamma$  under the action of  $\pi$ . The  *$L^2$ -zeta function* of  $Y$  is the infinite product

$$Z_{\pi}(Y, u) = \prod_{\gamma \in \pi \backslash P} \left(1 - u^{\ell(\gamma)}\right)^{-\frac{1}{|\pi_{\gamma}|}}.$$

This definition comes from [2], but beware that in [2] the notation  $Z_{\pi}(Y, u)$  refers to the reciprocal of the zeta function as defined here.

The adjacency operator  $\delta$  and Laplace operator  $\Delta(Y, u) = I - \delta u + Qu^2$  are  $\pi$ -equivariant operators on  $l^2(VY)$ . Choosing lifts of vertices of  $B$  yields a decomposition  $l^2(VY) = \bigoplus l^2(\pi)$ . Then from [2, Theorem 0.3],

$$(1.3) \quad Z_{\pi}(Y, u)^{-1} = (1 - u^2)^{-\chi^{(2)}(Y)} \text{Det}_{\pi} \Delta(Y, u).$$

In this formula,  $\chi^{(2)}(Y)$  is the  $L^2$ -Euler characteristic of  $Y$ , which in our setting is simply equal to  $\chi(B)$ . The determinant  $\text{Det}_{\pi} \Delta(Y, u)$  is defined via formal power series as  $(\text{Exp} \circ \text{Tr}_{\pi} \circ \text{Log}) \Delta(Y, u)$  and converges for small  $u \in \mathbb{C}$ . More precisely, if  $Y$  is  $q + 1$  regular, then the radius of convergence of  $Z_{\pi}(Y, u)$  is greater than or equal to  $\frac{1}{q}$ .

**Theorem 1.5.** *Let  $Y$  be a  $q + 1$  regular graph. Then  $Z_{\pi}(Y, u)$  has a holomorphic extension to  $\Omega$ .*

*Proof.* By (1.3) it is enough to show that  $\text{Det}_{\pi} \Delta(Y, u)$  has a nonzero holomorphic extension on  $\Omega$ . Let  $g_u(\lambda) = \log(1 - \lambda u + qu^2)$ . Here and in the rest of the paper  $\log$  is the principal branch of the logarithm, defined and analytic on  $\mathbb{C} - (-\infty, 0]$ . Fix

$u \in \Omega$ . Then using Proposition 1.3, there exists an open set  $V_u \supset [-(q + 1), q + 1]$  on which  $g_u$  is analytic. Since  $\delta$  is self-adjoint and  $\|\delta\| \leq q + 1$ , the spectrum  $\sigma(\delta) \subset [-(q + 1), q + 1]$ . By the spectral theorem for self-adjoint operators we can write:

$$(1.4) \quad \delta = \int_{-k^-}^k \lambda \, dE(\lambda).$$

Now

$$(1.5) \quad g_u(\delta) = \int_{-k^-}^k \log(1 - \lambda u + qu^2) \, dE(\lambda)$$

is well defined, and

$$(1.6) \quad \text{Tr}_\pi g_u(\delta) = \int_{-k^-}^k \log(1 - \lambda u + qu^2) \, d(\text{Tr}_\pi E(\lambda))$$

is a holomorphic function of  $u$  on  $\Omega$ . Now for small  $u$ ,

$$(1.7) \quad \text{Det}_\pi(\Delta(Y, u)) = \text{Exp Tr}_\pi \text{Log}(\Delta(Y, u))$$

$$(1.8) \quad = \text{Exp Tr}_\pi \text{Log}(I - \delta u + qu^2)$$

$$(1.9) \quad = \text{Exp Tr}_\pi g_u(\delta).$$

□

*Remark.* Using the functional equation (1.2) it is possible to extend the  $L^2$ -zeta function to the exterior of  $C$ . Defining the  $L^2$ -zeta function on  $C$  itself presents some problems. In examples, the function on the interior of  $C$  and the exterior of  $C$  do not match continuously on  $C$ . However the absolute value of the resulting function is the Fuglede-Kadison determinant, and may extend continuously to  $C$ .

## 2. CONVERGENCE OF ZETA FUNCTIONS FOR TOWERS OF GRAPHS

In this section we prove that the zeta functions for a tower of finite graphs, suitably renormalized, converge to the  $L^2$ -zeta function for an infinite covering graph. The argument uses an idea from [7].

### 2.1. The convergence theorem.

**Theorem 2.1.** *Let  $Y$  be a  $q + 1$  regular graph. Suppose the group  $\pi$  acts freely on  $Y$ , and that  $B = \pi \backslash Y$  is a finite graph. Suppose  $\pi = \pi_1 \supset \pi_2 \supset \dots$  is a tower of finite index normal subgroups and  $\bigcap \pi_i = \{e\}$ . Let  $[\pi : \pi_i] = N_i$  and  $B_i = \pi_i \backslash Y$ . Then for  $u \in \Omega$ , we have*

$$(2.1) \quad \lim_{i \rightarrow \infty} Z(B_i, u)^{\frac{1}{N_i}} = Z_\pi(Y, u).$$

*The convergence is uniform on compact subsets of  $\Omega$ .*

*Remark.* Notice that

$$(2.2) \quad Z(B_i, u)^{-1} = (1 - u^2)^{-\chi(B_i)} \text{Det } \Delta(B_i, u)$$

$$(2.3) \quad = (1 - u^2)^{-N_i \chi(B)} \text{Det } \Delta(B_i, u).$$

So we let

$$(2.4) \quad Z(B_i, u)^{-1/N_i} = (1 - u^2)^{-\chi(B)} (\text{Det } \Delta(B_i, u))^{1/N_i}.$$

The  $N^{\text{th}}$  root in (2.4) is taken in the sense of Corollary 1.4.

*Proof of Theorem 2.1.* From the remark we need to show that

$$(2.5) \quad \lim_{i \rightarrow \infty} (\text{Det } \Delta(B_i, u))^{1/N_i} = \text{Det } \Delta(B, u).$$

Let  $F_i(\lambda) = \frac{1}{N_i} |\{\mu \text{ eigenvalue of } \delta_i, \mu \leq \lambda\}|$ . Let  $F(\lambda) = \text{Tr}_\pi E(\lambda)$ , where  $\{E(\lambda)\}_\lambda$  is the spectral decomposition of  $\delta$  acting on  $l^2(Y)$ . We now set

$$\begin{aligned} \overline{F}(\lambda) &= \limsup_{i \rightarrow \infty} F_i(\lambda); & \underline{F}(\lambda) &= \liminf_{i \rightarrow \infty} F_i(\lambda); \\ \overline{F}^+(\lambda) &= \lim_{\varepsilon \rightarrow 0^+} \overline{F}(\lambda + \varepsilon); & \underline{F}^+(\lambda) &= \lim_{\varepsilon \rightarrow 0^+} \underline{F}(\lambda + \varepsilon). \end{aligned}$$

Then from [7, Theorem 2.3.1], for all  $\lambda \in [-k, k]$ ,

$$F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda).$$

We know

$$(2.6) \quad \log \text{Det } \Delta(Y_i, u) = \int_{-k^-}^k \log(1 - u\lambda + qu^2) dF_i(\lambda).$$

By (1.6)

$$(2.7) \quad \log \text{Det}_\pi \Delta(Y, u) = \int_{-k^-}^k \log(1 - u\lambda + qu^2) dF(\lambda).$$

If  $K \subset \Omega$  is compact, then  $\log(1 - u\lambda + qu^2)$  is bounded uniformly for  $u \in K$  and  $\lambda$  in an open interval containing  $[-k, k]$ . Now integration by parts shows that indeed

$$(2.8) \quad \int_{-k^-}^k \log(1 - u\lambda + qu^2) dF_i(\lambda) \rightarrow \int_{-k^-}^k \log(1 - u\lambda + qu^2) dF(\lambda)$$

as  $i \rightarrow \infty$ . □

### 2.2. Examples.

**Example 2.2.** Let  $Y$  be the graph with vertices  $\mathbb{Z}$  and edges connecting  $n$  to  $n+1$ . The group  $\pi = \mathbb{Z}$  acts on  $Y$  with quotient having one vertex and one edge. Let  $\pi_n = n\mathbb{Z} \subset \mathbb{Z}$ , so  $B_n = \pi_n \backslash Y$  is an  $n$ -cycle. As  $B_n$  has only two primitive loops and  $Y$  has none, we have  $Z(B_n, u) = (1 - u^n)^{-2}$  and  $Z_\pi(Y, u) = 1$ . The graphs are 2-regular, so  $\Omega$  is the (open) unit disk. For  $u \in \Omega$ ,

$$\lim_{n \rightarrow \infty} (1 - u^n)^{2/n} = 1.$$

Notice that the functional equation (1.2) gives  $Z_\pi(Y, u) = u^{-2}$  outside of the unit disk.

**Example 2.3.** Let  $\pi = \mathbb{Z} \times \mathbb{Z}$ , and let  $Y$  be the Cayley graph of  $\pi$  with generators  $(1, 0)$  and  $(0, 1)$ .  $Y$  is an infinite grid. We take for  $\{\pi_i\}$  any nested sequence of  $\mathbb{Z} \times \mathbb{Z}$ 's with  $\bigcap \pi_i = \{e\}$ . You can think of the quotient graphs  $B_i$  as finite grids wrapped around tori. Then Theorem 2.1 gives that

$$(2.9) \quad \lim_{i \rightarrow \infty} \frac{1}{i} \text{Log } Z(B_i, u) = \text{Log } Z_\pi(Y, u)$$

$$(2.10) \quad = \sum_{M=1}^{\infty} \left[ \frac{1}{M} + \sum_{d=0}^M \frac{(-3)^{M-d}}{M+d} \binom{M+d}{M-d} \binom{2d}{d}^2 \right] u^{2M}$$

as computed in [2].

**Example 2.4.** Let  $Y$  be the  $q+1$  regular tree and  $B = \pi \backslash Y$  finite. Since  $Y$  has no closed loops, the  $L^2$ -zeta function of  $Y$  is the constant function 1. So if  $B_n = \pi_n \backslash Y$  is a tower of finite graphs covering  $B$ ,

$$\lim_{n \rightarrow \infty} Z(B_n, u)^{1/[\pi: \pi_n]} = 1$$

for  $u \in \Omega$ . This result is contained in [3] for  $|u|$  small.

**Theorem 2.5.** *Let  $Y$  and  $B$  be as in the previous example. We have:*

$$Z(B, u) = \frac{\text{Det}_\pi \Delta(Y, u)}{\text{Det} \Delta(B, u)}$$

for  $u \in \Omega$ .

For small  $u$ , the above theorem is the main result of [3].

*Proof.* From [2, Theorem 0.3] we know that

$$Z_\pi(Y, u)^{-1} = (1 - u^2)^{-\chi^{(2)}(Y)} \text{Det}_\pi \Delta(Y, u).$$

As in the previous example,  $Z_\pi(Y, u) = 1$ . Now we have

$$Z(B, u) = \frac{Z(B, u)}{Z_\pi(Y, u)} = \frac{(1 - u^2)^{-\chi^{(2)}(Y)} \text{Det}_\pi \Delta(Y, u)}{(1 - u^2)^{-\chi(B)} \text{Det} \Delta(B, u)}.$$

As  $\chi(B) = \chi^{(2)}(Y)$ , the theorem follows.  $\square$

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