

## $L^1$ CONVERGENCE OF THE RECONSTRUCTION FORMULA FOR THE POTENTIAL FUNCTION

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(Communicated by Carmen C. Chicone)

ABSTRACT. It is known that the potential function of the Sturm-Liouville problem can be reconstructed from the nodal data by a pointwise limit. We show that this convergence is in fact  $L^1$ .

### 1. INTRODUCTION

Consider the Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y$$

such that

$$\begin{cases} y(0) \cos \alpha + y'(0) \sin \alpha = 0, \\ y(1) \cos \beta + y'(1) \sin \beta = 0, \end{cases}$$

where  $q \in L^1(0, 1)$ , and  $\alpha, \beta \in [0, \pi)$ .

In 1988, J.R. McLaughlin [8] showed that knowledge of the nodal set of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. This is the so-called inverse nodal problem [1]. C.F. Yang [13] showed that this uniqueness result is valid for any  $q \in L^1(0, 1)$ . Independently C.L. Shen [10] had similar ideas for the density function  $\rho$  of the string equation  $y'' + \lambda\rho y = 0$ . See also [1, 11, 12]. The uniqueness result holds for the situation that  $\rho$  is only assumed to be of bounded variation [3].

The inverse nodal problem in two dimension is much more difficult. Hald and McLaughlin [2] proved the uniqueness result for the potential function of the Schrödinger operator defined on a rectangular domain in  $\mathbf{R}^2$ . Some results for density function of string equation on a membrane can be found in [9, 7].

In this note we concern ourselves with the reconstruction of the potential function of the one-dimensional Sturm-Liouville problem. Law-Shen-Yang [4], improving a result of X.F. Yang [13], gave a reconstruction formula for the potential function  $q$ .

Let  $\lambda_n$  be the  $n$ -th eigenvalue,  $s_n = \sqrt{\lambda_n}$  and  $x_i^{(n)}$  be the  $i$ -th nodal point of the  $n$ -th eigenfunction  $y_n$ . In other words,  $y_n(x_i^{(n)}) = 0$ ,  $i = 1, 2, \dots, n-1$ . Let  $I_i^{(n)} = (x_i^{(n)}, x_{i+1}^{(n)})$ , and  $l_i^{(n)}$  be the nodal length where  $l_i^{(n)} = |I_i^{(n)}| = x_{i+1}^{(n)} - x_i^{(n)}$ .

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Received by the editors November 21, 2000 and, in revised form, February 20, 2001.

2000 *Mathematics Subject Classification*. Primary 34A55; Secondary 34B24.

*Key words and phrases*. Inverse nodal problem, potential function, reconstruction formula.

Define  $x_0^{(n)} = 0, x_n^{(n)} = 1$ . We also define the function  $j_n(x)$  to be the largest index  $j$  such that  $0 \leq x_j^{(n)} \leq x$ . Thus,  $j = j_n(x)$  if and only if  $x \in [x_j^{(n)}, x_{j+1}^{(n)})$ .

**Theorem 1.1** ([4]). *The potential function  $q \in L^1(0, 1)$  satisfies*

$$q(x) = \lim_{n \rightarrow \infty} 2s_n^2 \left( \frac{s_n l_j^{(n)}}{\pi} - 1 \right)$$

for a.e.  $x \in (0, 1)$ , with  $j = j_n(x)$ .

Note that Theorem 1.1, with the asymptotic expression for  $s_n$  (see [5, Lemma 2] and [6, Lemma 2.2]) implies that  $q(x) = \lim_{n \rightarrow \infty} F_n(x)$ , where  $F_n$  is determined only by the nodal data and the constant  $\int_0^1 q$ :

(a) If  $\alpha = \beta = 0$  or  $\alpha, \beta > 0$ , then

$$F_n(x) = 2n^2 \pi^2 \left\{ n l_j^{(n)} - 1 + \frac{l_j^{(n)}}{n \pi^2} \left( \frac{1}{2} \int_0^1 q - \operatorname{scot} \alpha + \operatorname{scot} \beta \right) \right\}.$$

(b) If  $\alpha = 0 < \beta$  or  $\beta = 0 < \alpha$ , then

$$F_n(x) = 2(n - \frac{1}{2})^2 \pi^2 \left\{ (n - \frac{1}{2}) l_j^{(n)} - 1 + \frac{l_j^{(n)}}{n \pi^2} \left( \frac{1}{2} \int_0^1 q - \operatorname{scot} \alpha + \operatorname{scot} \beta \right) \right\}.$$

Here,  $\operatorname{scot} \gamma = 0$  if  $\gamma = 0$ ;  $\operatorname{scot} \gamma = \cot \gamma$  otherwise.

Since  $q$  is in  $L^1$ , one may ask whether the convergence is in fact  $L^1$ . The answer is affirmative.

**Theorem 1.2.**  *$F_n$  converges to  $q$  in  $L^1$ .*

## 2. PROOF

**Lemma 2.1** ([5]). *Let  $q \in L^1$ . Then*

$$(2.1) \quad l_i^{(n)} = \frac{\pi}{s_n} + \frac{1}{2s_n^2} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} (1 + \alpha_0 \cos(2s_n y)) q(y) dy + o\left(\frac{1}{s_n^3}\right),$$

where  $\alpha_0 = 1$  if  $\alpha > 0$ ,  $\alpha_0 = -1$  if  $\alpha = 0$ .

In the above lemma, the order estimate is independent of  $i$ . As a result,

$$\frac{s_n l_i^{(n)}}{\pi} = 1 + O\left(\frac{1}{s_n}\right) = 1 + O\left(\frac{1}{n}\right).$$

**Lemma 2.2.** *Suppose the sequence  $f_k \in C[0, 1]$  converges to  $f$  in  $L^1$ . For any  $\epsilon > 0$ , then with  $j = j_n(x)$ ,*

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (f_k(y) - f(y)) dy \right\|_1 < \epsilon$$

for all sufficiently large  $n$  and  $k$ .

*Proof.* By Lemma 2.1 and the observation that the integral  $\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (f_k(y) - f(y)) dy$  is constant on any nodal interval  $I_j^{(n)}$ ,

$$\begin{aligned} & \int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (f_k(y) - f(y)) dy \right| dx \\ &= \sum_{i=0}^{n-1} \frac{s_n l_i^{(n)}}{\pi} \left| \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} (f_k(y) - f(y)) dy \right| \\ &\leq \sum_{i=0}^{n-1} \left(1 + O\left(\frac{1}{n}\right)\right) \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} |f_k(y) - f(y)| dy \\ &= \left(1 + O\left(\frac{1}{n}\right)\right) \int_0^1 |f_k(y) - f(y)| dy, \end{aligned}$$

and hence converges to 0 as  $k \rightarrow \infty$ . □

**Lemma 2.3.** *Suppose  $q \in L^1(0, 1)$ . Then as  $n \rightarrow \infty$ , with  $j = j_n(x)$ ,*

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y) dy - q(x) \right\|_1 \rightarrow 0.$$

*Proof.* We first show that the result holds if  $q$  is continuous on  $[0, 1]$ . Let  $M = \max_{x \in (0, 1)} |q(x)|$  and  $x_0 \in [x, y]$ . By intermediate value theorem, there exists  $\xi \in (x, y)$  such that

$$\left| \frac{1}{y-x} \int_x^y q - q(x_0) \right| = |q(\xi) - q(x_0)|.$$

Due to the uniform continuity of  $q$  there is a  $\delta > 0$  such that the difference is small whenever  $|y - x| < \delta$ . Hence, given  $\epsilon > 0$ , when  $n$  is large enough such that  $l_j^{(n)} < \delta$  and  $|\frac{s_n l_j^{(n)}}{\pi} - 1| < \min\{\frac{\epsilon}{M}, 1\}$ , with  $j = j_n(x)$ , we have

$$\begin{aligned} & \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y) dy - q(x) \right| \\ &\leq \left| \frac{s_n(x_{j+1}^{(n)} - x_j^{(n)})}{\pi} \left[ \frac{1}{x_{j+1}^{(n)} - x_j^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right] \right| + |q(x)| \left| \frac{s_n l_j^{(n)}}{\pi} - 1 \right| \\ &\leq 2\epsilon. \end{aligned}$$

Therefore, if  $q \in C[0, 1]$ , then  $\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y) dy - q(x) \right\|_1$  can be arbitrarily small for all large  $n$ .

Since  $C[0, 1]$  is dense in  $L^1(0, 1)$ , for any  $q \in L^1(0, 1)$  there exists a sequence  $q_k \in C[0, 1]$  that converges to  $q$  in  $L^1(0, 1)$ . Now

$$\begin{aligned} & \int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y) dy - q(x) \right| dx \\ & \leq \int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (q(y) - q_k(y)) dy \right| dx \\ & \quad + \int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_k(y) dy - q_k(x) \right| dx \\ & \quad + \int_0^1 |q_k(x) - q(x)| dx . \end{aligned}$$

For any  $\epsilon > 0$ , fix  $k$  large enough such that the first and last terms are smaller than  $\epsilon$ . Then for all  $n$  large enough, the second term is smaller than  $2\epsilon$  by above. Hence, as  $n \rightarrow \infty$ ,

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y) dy - q(x) \right\|_1 \rightarrow 0 .$$

□

*Proof of Theorem 1.2.* Since

$$\left| F_n(x) - 2s_n^2 \left( \frac{s_n l_j^{(n)}}{\pi} - 1 \right) \right| = O\left(\frac{1}{n}\right),$$

by an asymptotic estimate of  $s_n$ , it suffices to show that as  $n \rightarrow \infty$ ,

$$\left\| 2s_n^2 \left( \frac{s_n l_j^{(n)}}{\pi} - 1 \right) - q \right\|_1 \rightarrow 0.$$

By (2.1), we have

$$2s_n^2 \left( \frac{s_n l_j^{(n)}}{\pi} - 1 \right) = \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q + \frac{\alpha_0 s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \cos(2s_n y) q(y) dy + o(1) .$$

Hence we only need to prove that as  $n \rightarrow \infty$ ,

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q \right\|_1 \rightarrow 0,$$

and

$$(2.2) \quad \left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \cos(2s_n y) q(y) dy \right\|_1 \rightarrow 0 .$$

The first limit holds because of Lemma 2.3. By the proof of [4, Theorem 3.2], the sequence of functions

$$h_n(x) = \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \cos(2s_n y) q(y) dy$$

converges to 0 for a.e.  $x \in (0, 1)$ . Furthermore,

$$|h_n(x)| \leq \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(y)| dy = g_n(x)$$

and

$$\begin{aligned} \int_0^1 g_n(x) dx &= \sum_{i=0}^{n-1} \frac{s_n l_i^{(n)}}{\pi} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} |q(y)| dy \\ &= \left(1 + O\left(\frac{1}{n}\right)\right) \|q\|_1. \end{aligned}$$

Thus we may apply the Lebesgue dominated convergence theorem to show that (2.2) is valid. The proof is complete.  $\square$

#### ACKNOWLEDGEMENT

We thank Ngai-Ming Mok of the University of Hong Kong for suggesting the problem. The authors are partially supported by National Science Council, Taiwan, Republic of China.

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