

BRANCHED COVERINGS AND NONZERO DEGREE MAPS BETWEEN SEIFERT MANIFOLDS

HONG HUANG

(Communicated by Ronald A. Fintushel)

ABSTRACT. In this note we give a necessary and sufficient condition for the existence of a fiber preserving branched covering between two closed, orientable Seifert manifolds (for sufficiency we need the additional assumption that the genus of the base orbifold of the target manifold ≥ 1). Combining this with two theorems of Rong we get a necessary and sufficient condition for the existence of a nonzero degree map between two such manifolds.

0. INTRODUCTION

In [7] Yongwu Rong proved:

Theorem. *Let $f : M \rightarrow N$ be a nonzero degree map between closed P^2 -irreducible Seifert manifolds of infinite π_1 . Then $f \simeq pg\pi$, where π is a composition of finitely many vertical pinches, g is a fiber preserving branched covering, and p is a covering. Furthermore, p can be chosen to be the identity map unless N is a Euclidean manifold.*

Using this theorem Rong [8] gave a necessary and sufficient condition for the existence of a degree one map between two aspherical, closed, orientable Seifert manifolds (see Theorem 3.2 of [8]), and proved as a corollary: If M is an aspherical, closed, orientable Seifert manifold, then there are only finitely many aspherical Seifert manifolds N such that there exists a degree one map $f : M \rightarrow N$ (see Corollary 4.1 of [8]).

In this note we try to generalize the results above in [8] to the case of arbitrary degree (but for sufficiency we need the additional condition that the orbifold genus of the target manifold ≥ 1). In Section 1 we give two general facts about branched coverings of Seifert manifolds, which will not be used in the later sections. Then we give a necessary and sufficient condition that there exist a fiber preserving branched covering between two closed, orientable Seifert manifolds (for sufficiency we need the additional assumption that the orbifold genus of the target manifold ≥ 1) (see Theorem 2.2 in Section 2). One of the key ingredients in the proof is a theorem of Husemoller (see Theorem 4 of [4] and Proposition 3.3 of [2]) which guarantees the existence of the branched covering of a closed, orientable surface with nonpositive Euler characteristic with any given branch data with even total branching. As a

Received by the editors October 27, 2000 and, in revised form, February 21, 2001.
2000 *Mathematics Subject Classification.* Primary 57N10; Secondary 57M12, 55M25.
Key words and phrases. Seifert manifold, branched covering, nonzero degree map.

consequence, we get an elementary proof of the naturality of the Euler numbers of Seifert manifolds [6]. Combining Theorem 2.2 with Rong's theorems above we get a necessary and sufficient condition that there exist a nonzero degree map between two such manifolds (see Theorem 3.0 in Section 3). As a consequence, we show that for a given aspherical, closed, oriented Seifert manifold M and a nonzero integer d , there are only finitely many such Seifert manifolds N such that there is a degree d map $f : M \rightarrow N$ (see Corollary 3.1).

1. TWO GENERAL FACTS ABOUT BRANCHED COVERINGS OF SEIFERT MANIFOLDS

In this section we generalize two well-known facts about coverings of Seifert manifolds to the case of branched coverings, which will not be used in the later sections.

Proposition 1.0. *A finite sheeted branched covering space of a Seifert manifold branched over fibers admits a Seifert fibration.*

The proof is easy and omitted.

Remark. If we remove the condition “branched over fibers”, then the result is obviously not true.

Conversely, we have

Proposition 1.1. *Let S be a compact, connected, irreducible, orientable 3-manifold. Suppose that some finite sheeted branched covering space S' of S admits a Seifert fibration such that the inverse image of the downstairs branch set consists of some fibers. Then S itself admits a Seifert fibration such that the downstairs branch set consists of some fibers.*

Proof. Suppose we have a finite sheeted branched covering $p : S' \rightarrow S$, where S' is a Seifert manifold and the inverse image of the downstairs branch set consists of some fibers. Let $p| : S' - p^{-1}(N(B_p)) \rightarrow S - N(B_p)$ be the corresponding unbranched covering, where B_p is the downstairs branch set and $N(B_p)$ is an open regular neighborhood of B_p such that $p^{-1}(N(B_p))$ is fibered. By [5, Theorem 6.3] and the positive solution of the Seifert fibered space conjecture by Gabai, Casson and Jungreis [1], [3], $S - N(B_p)$ admits a Seifert fibration, which can extend to $N(B_p)$ such that B_p consists of some fibers as is easily seen.

Note that for any nonzero degree map $f : M \rightarrow N$ between two aspherical, closed Seifert manifolds, we have $f \simeq f_1 p$, where $p : M \rightarrow M_1$ is a fiber preserving branched covering (in fact p is the natural projection from M to some M/Z_n , where Z_n is in the S^1 action on M), and $f_{1*}(h) = h$, where h represents regular fibers of both M, N . So we shall first consider nonzero degree map $f : M \rightarrow N$ of fiber degree 1 in the next section.

2. FIBER PRESERVING BRANCHED COVERINGS BETWEEN SEIFERT MANIFOLDS

From now on, “Seifert manifold” will always mean an oriented closed connected 3-manifold admitting a fixed point free action of S^1 . As is well known, such a manifold is equivariantly classified by its “Seifert invariants” [10]. We shall use unnormalized Seifert invariants (see [6]). The unnormalized Seifert invariant of a Seifert

manifold M is denoted by $I(M) = \{g; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ or $\{g; r_1, \dots, r_m\}$ where g is the orbifold genus of M , (α_i, β_i) (α_i, β_i coprime and $\alpha_i \geq 1$) are the Seifert invariants of the maybe-singular fibers, and $r_i = \beta_i/\alpha_i$.

The unnormalized Seifert invariant of a Seifert manifold M is not unique. We can transform it while not changing the isomorphic type of the Seifert fibration of M by the following moves which we call *equivalent moves* (see [6], [8]):

- (1) permute the r_i .
- (2) add 0 to or delete 0 from the list r_1, \dots, r_n .
- (3) replace r_i, r_j by $r_i + 1, r_j - 1$.

As preparation, let's see when there exists a k -fold fiber preserving branched covering of fiber degree 1 between two Seifert fibered solid tori V_1 and V_2 such that upstairs and downstairs branch sets are their central fibers.

Let the Seifert invariant of V_2 be (α_2, β_2) under a (cross section, fiber) basis (c_2, h_2) , i.e. the meridian μ_2 of V_2 can be represented by $c_2^{\alpha_2} h_2^{\beta_2}$. If such a branched covering $f: V_1 \rightarrow V_2$ exists, then c_2 lifts to a cross section c_1 of the fibration of V_2 restricted to ∂V_2 for fiber degree one. Thus we have $f|_*(c_1) = c_2^k$, where $f|$ is the restriction of f to ∂V_1 .

By the definition of branched covering we can write $f|_*(\mu_1) = \mu_2^s$, where s is some nonzero integer. So $f|_*(c_1^{\alpha_1} h_1^{\beta_1}) = (c_2^{\alpha_2} h_2^{\beta_2})^s$, where we set the Seifert invariant of V_1 under the (section, fiber) basis (c_1, h_1) to be (α_1, β_1) . Thus $c_2^{k\alpha_1} h_2^{\beta_1} = c_2^{s\alpha_2} h_2^{s\beta_2}$, i.e. $k\alpha_1 = s\alpha_2$, $\beta_1 = s\beta_2$. So $\beta_1/\alpha_1 = k\beta_2/\alpha_2$.

Conversely, if V_i has Seifert invariant (α_i, β_i) , and α_i, β_i ($i = 1, 2$), and k satisfy the relation above, then clearly there exists a branched covering $V_1 \rightarrow V_2$ satisfying the given condition (by a direct construction; cf. [10]).

Definition 2.0. Let $R = \{\beta_1/\alpha_1, \dots, \beta_s/\alpha_s\}$, $R' = \{\beta'_1/\alpha'_1, \dots, \beta'_k/\alpha'_k\}$ be two unordered sets of rational numbers (repetitions allowed). Let d be a positive integer. We say $R \geq_d R'$ if R can be split into a "disjoint" union $R = R_1 \cup \dots \cup R_k$ such that for any i ($1 \leq i \leq k$) and any $\beta_j/\alpha_j \in R_i$, $\beta_j/\alpha_j = a_j \beta'_i/\alpha'_i$ (where a_j is some positive integer), and $\sum a_j = d$, where the summation is over all j such that $\beta_j/\alpha_j \in R_i$.

Lemma 2.1. Let M, N be closed oriented Seifert manifolds. Suppose $I(N) = \{g'; R'\}$, where $g' \geq 1$ and $R' = \{\beta'_1/\alpha'_1, \dots, \beta'_n/\alpha'_n\}$ ($n \geq 1$). Then there is a d -fold fiber preserving branched covering $M \rightarrow N$ with fiber degree one if and only if M has Seifert invariant $\{g; R\}$, where $R \geq_d R'$ and $nd - |R| \leq 2d(1 - g') - 2(1 - g)$. Here $|R|$ is the number of elements in the multiset R (counted with multiplicity).

Proof. Suppose there is such a branched covering $f: M \rightarrow N$. Recall how we get the Seifert invariant of N as in [6]. Let O_1, \dots, O_n be a nonempty collection of fibers in N , including all singular fibers. Let T_1, \dots, T_n be disjoint Seifert fibered tubular neighborhoods of O_1, \dots, O_n and $N_0 = N - \text{int}(T_1 \cup \dots \cup T_n)$. Take the section c_2 of the bundle $N_0 \rightarrow N_0/S^1$ which leads to the Seifert invariant $\{g'; R'\}$ of N . Let $M_0 = M - f^{-1}(\text{int}(T_1 \cup \dots \cup T_n))$. Then c_2 lifts to a section c_1 of the bundle $M_0 \rightarrow M_0/S^1$ for fiber degree one. Let $\{g; R\}$ be the Seifert invariant of M under the (section, fiber) frame (c_1, h_1) , where h_1 stands for a regular fiber of M . Considering T_i and its inverse image under f we get the relation $R \geq_d R'$ by the local computation before Definition 2.0. The inequality $nd - |R| \leq 2d(1 - g') - 2(1 - g)$ follows from the Riemann-Hurwitz formula since $nd - |R|$ is a part of the total branching of the induced branched covering $\bar{f}: M/S^1 \rightarrow N/S^1$.

Conversely suppose the Seifert invariants of M, N satisfy the relation in the lemma we'll go to construct a branched covering $f : M \rightarrow N$ as required. Let

$$k = 2d(1 - g') - 2(1 - g) - (nd - |R|).$$

Let $\mathcal{D} = \{A_1, \dots, A_n, A_{n+1}, \dots, A_{n+k}\}$, where A_i ($i = 1, \dots, n$) are the partitions of d given by the relation $R \geq_d R'$ (i.e. each A_i consists of the a_j 's (depending on i) in Definition 2.0), $A_{n+j} = [2, 1, \dots, 1]$ ($d-2$ times 1) for $j = 1, \dots, k$. By Theorem 4 of [4] or Proposition 3.3 of [2] there is a d -fold branched covering $\bar{f} : F_g \rightarrow F_{g'}$ with branch data \mathcal{D} , where F_g is the closed, orientable surface of genus g . Based on this and the local constructions (as what we have done in the preparation before Definition 2.0) decided by the relation $R \geq_d R'$, a branched covering $M \rightarrow N$ as required can be constructed: We have the induced covering between surfaces with boundary $\bar{f}| : F_g - \bar{f}^{-1}(\text{int}(D_1 \cup \dots \cup D_{n+k})) \rightarrow F_{g'} - \text{int}(D_1 \cup \dots \cup D_{n+k})$. Then clearly

$$\begin{aligned} \bar{f}| \times id : (F_g - \bar{f}^{-1}(\text{int}(D_1 \cup \dots \cup D_{n+k}))) \\ \times S^1 \rightarrow (F_{g'} - \text{int}(D_1 \cup \dots \cup D_{n+k})) \times S^1 \end{aligned}$$

extends to a d -fold fiber preserving branched covering $f : M \rightarrow N$ with fiber degree one.

Remark. In the proof of the necessity of the lemma above we do not need the condition $g' \geq 1$. But for sufficiency we do need $g' \geq 1$ as shown in the following example.

Example. Let $I(M) = \{0; R\} = \{0; 2/13, 2/13, 4/17, 4/17, 9/19, 3/19\}$, $I(N) = \{0; R'\} = \{0; 1/13, 2/17, 3/19\}$. Then $R \geq_4 R'$ and $nd - |R| = 6 = 2d(1 - g') - 2(1 - g)$. However there is no 4-fold fiber preserving branched covering $M \rightarrow N$ with fiber degree one, for $\{[2, 2], [2, 2], [3, 1]\}$ cannot arise as the branch data of a connected branched covering of S^2 (see Proposition 5.3 of [2]).

In the case of arbitrary fiber degree we have the following:

Theorem 2.2. *Let M, N be closed, oriented Seifert manifolds with Seifert invariants*

$$\begin{aligned} I(M) = \{g; R\} &= \{g; \beta_1/\alpha_1, \dots, \beta_m/\alpha_m\}, \\ I(N) = \{g'; R'\} &= \{g'; \beta'_1/\alpha'_1, \dots, \beta'_n/\alpha'_n\}, \end{aligned}$$

where $g' \geq 1$. Let d, d_1 be positive integers and $d_1|d$. Then there is a d -fold fiber preserving branched covering $M \rightarrow N$ with fiber degree d_1 if and only if $d_1R := \{d_1\beta_1/\alpha_1, \dots, d_1\beta_m/\alpha_m\}$ can be equivalently moved to a set $\bar{R} \geq_{d_2} R'$ (where $d_2 = d/d_1$) and $nd_2 - |\bar{R}| \leq 2d_2(1 - g') - 2(1 - g)$.

Proof. First we prove necessity. If there exists such a branched covering $f : M \rightarrow N$, then f factors through $M/(Z/d_1)$ (where Z/d_1 is inside the S^1 action on M). By a fact due to Seifert (see [10] or Lemma 1.3 of [6]) $M/(Z/d_1)$ has Seifert invariant $\{g; d_1\beta_1/\alpha_1, \dots, d_1\beta_m/\alpha_m\}$. Now the necessity clearly follows from the lemma above.

Sufficiency can be proved by a direct construction as in the lemma above.

Remark. Again in the proof of the necessity $g' \geq 1$ is not used.

As an application we have the following.

Corollary 2.3 ([6]). *Let M, N be aspherical, closed, oriented Seifert manifolds. Suppose there is a degree d ($d \neq 0$) fiber preserving map $f : M \rightarrow N$ with fiber degree d_1 . Then $e(M) = (d_2/d_1)e(N)$, where $e(M)$ (resp. $e(N)$) is the Euler number of the Seifert fibration of M (resp. N), and $d_2 = d/d_1$.*

Proof. By the theorem of Rong cited at the beginning of this note, $f \simeq g\pi$, where $\pi : M \rightarrow \hat{M}$ is a composition of finitely many of vertical pinches, and $g : \hat{M} \rightarrow N$ is a fiber preserving branched covering. By Corollary 3.3 of [8] $e(M) = e(\hat{M})$. We can write $g = g_2g_1$, where $g_1 : \hat{M} \rightarrow \overline{M} := \hat{M}/(Z/d_1)$ is the natural projection, and $g_2 : \overline{M} \rightarrow N$ is a d_2 -fold fiber preserving branched covering of fiber degree one. By the fact due to Seifert cited above, $e(\hat{M}) = (1/d_1)e(\overline{M})$. By Theorem 2.2 and Definition 2.0, $e(\overline{M}) = d_2e(N)$. So $e(M) = (d_2/d_1)e(N)$.

Remark. The proof of the fact above sketched in [6] used more algebraic topology, while our proof here is quite elementary.

Corollary 2.4 ([8]). *Let M, N be closed, oriented Seifert manifolds, where $\chi(O_N) \leq 0$. Suppose there is a d -fold fiber preserving branched covering $f : M \rightarrow N$ with orbifold degree d_2 . Then $|\chi(O_M)| \geq d_2|\chi(O_N)|$, where O_M (resp. O_N) is the base orbifold of M (resp. N) and $\chi(O_M)$ (resp. $\chi(O_N)$) is its Euler characteristic. The equality holds if and only if f is a covering.*

Proof. It follows from a direct computation using the Theorem.

Let $\overline{M} = M/(Z/d_1)$, where $d_1 = d/d_2$. We have

$$\begin{aligned} & -\chi(O_{\overline{M}}) + d_2\chi(O_N) \\ = & -\chi(|O_{\overline{M}}|) + \sum_{i=1}^{|\overline{R}|} (1 - 1/\overline{\alpha}_i) + d_2\chi(|O_N|) - d_2 \sum_{j=1}^n (1 - 1/\alpha'_j) \\ = & d_2\chi(|O_N|) - \chi(|O_{\overline{M}}|) - (nd_2 - |\overline{R}|) + \sum_{j=1}^n (d_2/\alpha'_j - \sum_{\beta_i/\overline{\alpha}_i \in \overline{R}_j} 1/\overline{\alpha}_i) \\ \geq & d_2\chi(|O_N|) - \chi(|O_{\overline{M}}|) - (nd_2 - |\overline{R}|) \quad (\text{using the relation } \overline{R} \geq_{d_2} R') \\ \geq & 0. \end{aligned}$$

The last two equalities hold if and only if $\overline{M} \rightarrow N$ is a covering.

On the other hand, $-\chi(O_M) \geq -\chi(O_{\overline{M}})$ by the fact due to Seifert cited above; the equality holds if and only if $\overline{f} : M \rightarrow \overline{M}$ is a covering.

Combining these two things the proof is complete.

Remark. It is indicated in [8] that the inequality in Corollary 2.4 can be proved by a covering space argument. This in turn depends on the fact that every good, compact 2-dimensional orbifold without boundary is finitely covered by a manifold (see for example [9]). But the proof of this fact is not very easy. We also note that the inequality has independent meaning, i.e. it can be applied directly to 2-orbifold branched coverings.

We note that however the condition “ $nd - |R| \leq 2d(1 - g') - 2(1 - g)$ ” in Lemma 2.1 cannot be replaced by $|\chi(O_M)| \geq d|\chi(O_N)|$ as shown in the following example.

Example. Let $I(M) = \{5; R\} = \{5; 2/5, 3/5, 2/7, 3/7, 2/11, 3/11\}$, $I(N) = \{1; R'\} = \{1; 1/5, 1/7, 1/11\}$, $d = 5$. Then $R \geq_5 R'$ and $|\chi(O_M)| \geq 5|\chi(O_N)|$. But it is easy to see that R cannot be equivalently moved to a set R_1 such that $R_1 \geq_5 R'$ and $15 - |R_1| \leq 10(1 - 1) - 2(1 - 5) = 8$.

Similarly using Theorem 4 of [4] (or Proposition 3.3 of [2]) we have the following.

Proposition 2.5. *Let N be a closed, orientable Seifert manifold with Seifert invariant $I(N) = \{g'; (\alpha'_1, \beta'_1), \dots, (\alpha'_n, \beta'_n)\}$, where $g' \geq 1$. Suppose d, d_2 are two positive integers and $d_2|d$. Let $\mathcal{D} = \{A_1, \dots, A_n, A_{n+1}, \dots, A_{n+k}\}$ be a collection of partitions of d_2 (with repetitions allowed). Then there is a closed, oriented Seifert manifold M and a d -fold fiber preserving branched covering $M \rightarrow N$ with orbifold degree d_2 and orbifold branch data \mathcal{D} (the corresponding branch set in N/S^1 is the n cone points and k regular points) if and only if the total branching $v(\mathcal{D}) := (n+k)d_2 - \sum_{l=1}^{n+k} |A_l|$ is even, where $|A_l|$ denotes the number of components of A_l (abusing notation).*

Remark. A cone point of the orbifold O_N of N may be a downstairs branch point of the induced map $\overline{f}: O_M \rightarrow O_N$ while the corresponding singular fiber is not in the downstairs branch set of $f: M \rightarrow N$, and vice versa.

3. APPLICATIONS TO NONZERO DEGREE MAPS BETWEEN SEIFERT MANIFOLDS

Now we can generalize Theorem 3.2 of [8] to the case of arbitrary degree (but for sufficiency we need the additional assumption that the orbifold genus of the target manifold ≥ 1).

Theorem 3.0. *Let M, N be aspherical, closed, oriented Seifert manifolds with Seifert invariants $I(M) = \{g; R\}$, $I(N) = \{g'; R'\}$, where $g' \geq 1$. Let d be a positive integer. Suppose N admits no Euclidean geometry. Then there is a map $M \rightarrow N$ with degree d if and only if there is a set $\{\hat{g}; \hat{R}\}$ such that $g \geq \hat{g}$ and $R \geq \hat{R}$ (which is defined in [8]), and $\{\hat{g}; \hat{R}\}$ and $\{g'; R'\}$ satisfy the relation in Theorem 2.2 (satisfied by $\{g; R\}$ and $\{g'; R'\}$) for some integer $d_1|d$.*

Proof. First we prove necessity. If there is a map $f: M \rightarrow N$ of degree d , then by the theorem of Rong cited at the beginning of this note we have $f \simeq g\pi$, where $\pi: M \rightarrow \hat{M}$ is a composition of finitely many vertical pinches, and g is a fiber preserving branched covering. By Theorem 3.2 of [8] \hat{M} has Seifert invariant $\{\hat{g}; \hat{R}\}$ which satisfies $g \geq \hat{g}$ and $R \geq \hat{R}$. Now the necessity follows from Theorem 2.2.

Sufficiency follows immediately from Theorem 3.2 of [8] and Theorem 2.2.

Now we have the following generalization of Corollary 4.1 of [8].

Corollary 3.1. *Let M be an aspherical, closed, oriented Seifert manifold. Then given a nonzero integer d , there are only finitely many aspherical, closed, oriented Seifert manifolds N such that there is a map $f: M \rightarrow N$ with degree d .*

Proof. We may assume $d > 0$. Suppose N is an aspherical, closed, oriented Seifert manifold such that there is a map $f: M \rightarrow N$ with degree d . Note that there are only 10 closed Euclidean 3-manifolds, so we can assume that N admits no Euclidean geometry. Then the Seifert invariants of M and N have the relation as described in Theorem 3.0. By Corollary 4.1 of [8] there are only finitely many aspherical, closed, oriented Seifert manifolds \hat{M} such that there is a degree one map $M \rightarrow \hat{M}$. So we need only show that given an aspherical, closed, oriented Seifert manifold \hat{M} , there are only finitely many aspherical, closed, oriented Seifert manifolds N such that there is a d -fold fiber preserving branched covering $\hat{M} \rightarrow N$. By Corollary 2.4 $-\chi(O_{\hat{M}}) \geq -d_2\chi(O_N)$, where d_2 is the orbifold degree. Let $\overline{M} = \hat{M}/(Z/d_1)$,

where $d_1 = d/d_2$. From Theorem 2.2 and Definition 2.0 we have $\bar{\alpha}_j \leq \alpha'_i \leq d_2\bar{\alpha}_j$ for any i , where $\bar{\beta}_j/\bar{\alpha}_j \in \bar{R}_i$. There are only finitely many 2-orbifolds O_N which satisfy these two conditions. Also $e(N) = (d_1/d_2)e(M)$. So there are only finitely many such N . \square

Corollary 3.2. *Let M, N be aspherical, closed, oriented Seifert manifolds with given Seifert invariants, where the orbifold genus of $N \geq 1$. Then given a nonzero integer d there is an algorithm to decide whether there is a map $f : M \rightarrow N$ of degree d . Furthermore, if $e(N) \neq 0$ and $\chi(O_N) \neq 0$ the set $\{d : \text{there is a map } f : M \rightarrow N \text{ of degree } d\}$ is finite and can be decided.*

Proof. For simplicity we assume $e(N) \neq 0$ and $\chi(O_N) \neq 0$. If there is a map $f : M \rightarrow N$ of degree d , say, then $e(M) \neq 0$ and $|d| \leq |e(N)|\chi(O_M)^2/(|e(M)|\chi(O_N)^2)$ by Corollaries 2.3 and 2.4. We can find the finitely many Seifert manifolds \hat{M} such that there is a degree one map $M \rightarrow \hat{M}$ by the proof of Corollary 4.1 of [8]. For each \hat{M} let $\bar{M} = \hat{M}/(Z/d_1)$ (where d_1 is the divisor of d such that $e(M) = (d_2/d_1)e(N)$ and $|\chi(O_M)| \geq d_2|\chi(O_N)|$ (where $d_2 = d/d_1$)). By Lemma 2.1 and the finiteness of the number of partitions of d_2 we can decide whether there is a d_2 -fold fiber preserving branched covering $\bar{M} \rightarrow N$ of fiber degree one.

ACKNOWLEDGEMENTS

I am very grateful to Professor Shicheng Wang for introducing me to this field. I would also like to thank the referee for his helpful comments.

REFERENCES

- [1] A. Casson, D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, *Invent. Math.* 118 (1994), 441-456. MR **96f**:57011
- [2] A. L. Edmonds, R. S. Kulkarni and R. E. Stong, Realizability of branched coverings of surfaces, *Trans. Amer. Math. Soc.* 282 (1984), No. 2, 773-790. MR **85k**:57005
- [3] D. Gabai, Convergence groups are Fuchsian groups. *Ann. of Math.* 136 (1992), 447-510. MR **93m**:20065
- [4] D. Husemoller, Ramified coverings of Riemann surfaces, *Duke Math. J.* 29 (1962), 167-174. MR **25**:188
- [5] W. Jaco, P. Shalen, Seifert Fibered Spaces in 3-manifolds, *Memoirs of Amer. Math. Soc.* 21 (1979) (no. 220). MR **81c**:57010
- [6] W. D. Neumann, F. Raymond, Seifert manifolds, plumbing, μ -invariant and orientation reversing maps, *Lecture Notes in Mathematics* 664, 162-195. MR **80e**:57008
- [7] Y. Rong, Maps between Seifert fibered spaces of infinite π_1 , *Pacific J. Math.* 160 (1993), No. 1, 143-154. MR **94e**:55026
- [8] Y. Rong, Degree one maps of Seifert manifolds and a note on Seifert volume, *Topol. Appl.* 64 (1995), 191-200. MR **96c**:57034
- [9] P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* 15 (1983), 401-487. MR **84m**:57009
- [10] H. Seifert, Topologie dreidimensionaler gefaserner Raume, *Acta Math.* 60 (1933), 147-238.

SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, PEOPLE'S REPUBLIC OF CHINA

Current address: Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

E-mail address: `hhuang01@263.net`