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STONE'S DECOMPOSITION OF THE RENEWAL MEASURE VIA BANACH-ALGEBRAIC TECHNIQUES

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ABSTRACT. A Banach-algebraic approach to Stone's decomposition of the renewal measure is discussed. Estimates of the rate of convergence in a key renewal theorem are given.

1. INTRODUCTION

Let F be a probability distribution on $\mathbf{R}$ with positive mean $\mu = \int_{\mathbf{R}} x\, F(dx)$, and let $H = \sum_{n=0}^{\infty} F^{n*}$ be the corresponding renewal measure; here $F^{1*} := F$, $F^{(n+1)*} := F * F^{n*}$, $n \geq 1$, and $F^{0*} := \delta$, the atomic measure of unit mass at the origin. Suppose F is *spread-out*, i.e., for some $m \geq 1$, F^{m*} has a nonzero absolutely continuous component. Stone [14] showed that then there exists a decomposition $H = H_1 + H_2$, where H_2 is a finite measure and H_1 is absolutely continuous with bounded continuous density $h(x)$ such that $\lim_{x\to\infty} h(x) = \mu^{-1}$ and $\lim_{x\to-\infty} h(x) = 0$. Under additional assumptions, $h(x)$ has further nice properties [14, Theorem].

The measures H_1 and H_2 are constructed as follows. There exists an integer $p \geq 1$ such that $F^{p*} = F_1 + F_2$, where $F_1 \neq 0$ has a continuous density with compact support. Put $Q = \sum_{n=0}^{\infty} F_2^{pn*}$. Then $H_1 := F_1 * Q * H$ and $H_2 := Q * \sum_{n=0}^{p-1} F^{n*}$ yield the desired decomposition [14] (see also [1, Theorem 2.6.2]).

In the present paper, we obtain Stone's decomposition $H = H_1 + H_2$ by using Banach-algebraic techniques, which will allow us to extract detailed information about the asymptotic properties of the terms H_1 and H_2. We will show that, under suitable hypotheses, H_1 and H_2 belong to specific Banach algebras of measures. In this connection, it is appropriate to mention the paper by R. Grübel [4] as a major contribution to renewal theory based on Banach algebra techniques.

2. PRELIMINARIES

Our discussion will rely on Banach algebras of measures with submultiplicative weights.

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Definition 2.1. A function $\varphi(x)$, $x \in \mathbf{R}$, is called *submultiplicative* if $\varphi(x)$ is a finite, positive, Borel-measurable function with the following properties:

$$\varphi(0) = 1, \qquad \varphi(x+y) \le \varphi(x)\varphi(y) \qquad \forall x, y \in \mathbf{R}.$$

We give some examples of such functions on $[0,\infty)$: $\varphi(x) = (1+x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\alpha)$ with $c > 0$ and $\alpha \in (0,1)$; $\varphi(x) = \exp(rx)$ with $r \in \mathbf{R}$. Moreover, if $R(x)$, $x \in \mathbf{R}_+$, is a positive, ultimately nondecreasing regularly varying function at infinity with a nonnegative exponent α (i.e., $R(tx)/R(x) \to t^\alpha$ for $t > 0$ as $x \to \infty$ [3, Section VIII.8]), then there exist a nondecreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0,\infty)$ such that $c_1 R(x) \le \varphi(x) \le c_2 R(x)$ for all $x \ge x_0$, where c_1 and c_2 are some positive constants [9, Proposition]. The product of a finite number of submultiplicative functions is again a submultiplicative function.

It is well known [5, Section 7.6] that

$$\begin{aligned}(2.1) \qquad -\infty < r_1 := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} &= \sup_{x<0} \frac{\log\varphi(x)}{x} \\ &\le \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_2 < \infty\end{aligned}$$

and $M(h) := \sup_{|x|\le h} \varphi(x) < \infty \ \forall h > 0$.

Consider the collection $S(\varphi)$ of all complex-valued measures κ such that $\|\kappa\|_\varphi := \int_{\mathbf{R}} \varphi(x)\,|\kappa|(dx) < \infty$; here $|\kappa|$ stands for the total variation of κ. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ is defined as their convolution $\nu * \kappa$ [5, Section 4.16]. The unit element of $S(\varphi)$ is the measure δ. Define the Laplace transform of a measure κ as $\widehat{\kappa}(s) := \int_{\mathbf{R}} \exp(sx)\,\kappa(dx)$. Then relation (2.1) implies that the Laplace transform of any $\kappa \in S(\varphi)$ converges absolutely with respect to $|\kappa|$ for all s in the strip $\Pi(r_1, r_2) := \{s \in \mathbf{C} : r_1 \le \Re s \le r_2\}$.

Let ν be a finite complex-valued measure. Denote by $T\nu$ the σ-finite measure with the density $v(x;\nu) := \nu((x,\infty))$ for $x \ge 0$ and $v(x;\nu) := -\nu((-\infty,x])$ for $x < 0$. In the case $\int_{\mathbf{R}} |x|\,|\nu|(dx) < \infty$, $T\nu$ is a finite measure whose Laplace transform is given by $(T\nu)^\wedge(s) = [\hat\nu(s) - \hat\nu(0)]/s$, $\Re s = 0$, the value $(T\nu)^\wedge(0)$ being defined by continuity as $\int_{\mathbf{R}} x\,\nu(dx) < \infty$.

The absolutely continuous part of any distribution F will be denoted by F_c, and its singular component by F_s, i.e., $F_s = F - F_c$.

3. An abstract theorem

Let $S(r_1, r_2)$ be the Banach algebra $S(\varphi)$ with $\varphi(x) = \max(e^{r_1 x}, e^{r_2 x})$, where $r_1 \le 0 \le r_2$. In this section, we shall consider Banach algebras $\mathcal{A}$ of measures such that (i) $\mathcal{A} \subset S(r_1, r_2)$ and (ii) each homomorphism $\mathcal{A} \mapsto \mathbf{C}$ is the restriction to $\mathcal{A}$ of some homomorphism $S(r_1, r_2) \mapsto \mathbf{C}$. Property (ii) can be restated as follows: Each maximal ideal M of $\mathcal{A}$ is of the form $M_1 \cap \mathcal{A}$, where M_1 is a maximal ideal of $S(r_1, r_2)$. It follows from the general theory of Banach algebras that if $\nu \in \mathcal{A}$ is invertible in $S(r_1, r_2)$, then $\nu^{-1} \in \mathcal{A}$.

In Sections 4 and 5 we apply the results of this section to the special case $\mathcal{A} = S(\varphi)$, where $\varphi(x)$ is an *arbitrary* submultiplicative function.

In what follows, F will denote a spread-out probability distribution with finite mean $\mu > 0$ such that $F \in S(r_1, r_2)$, $r_1 \le 0 \le r_2$; $(F^{m*})_s^\wedge(r_i) < 1$, $i = 1, 2$, for some integer $m \geq 1$; and $\hat{F}(s) \neq 1\ \forall s \in \Pi(r_1, r_2) \setminus \{0\}$. Let L be the restriction of Lebesgue measure to $[0, \infty)$.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra having properties* (i) *and* (ii). *Suppose F, $TF \in \mathcal{A}$. Then the renewal measure $H = \sum_{n=0}^{\infty} F^{n*}$ admits a Stone-type decomposition $H = H_1 + H_2$, where $H_2 \in \mathcal{A}$ and $H_1 = L/\mu + rTH_2$ for some $r > r_2$. If, in addition, $T^2F \in \mathcal{A}$, then $H_1 - L/\mu \in \mathcal{A}$.*

Proof. Choose $r > r_2$. The function

$$v(s) := \frac{(s-r)[1-\hat{F}(s)]}{s}, \qquad s \in \Pi(r_1, r_2),$$

is the Laplace transform of the measure $V := \delta - F + rTF \in \mathcal{A}$, the value $v(0)$ being defined by continuity as $r\mu$. By Lemma 2 of [10] with obvious changes, there exists $W := V^{-1} \in S(r_1, r_2)$, and hence $W \in \mathcal{A}$. We have $\hat{W}(s) = 1/v(s)$ and

$$\begin{aligned}(3.1)\quad \hat{H}(s) := \frac{1}{1-\hat{F}(s)} &= \frac{(s-r)\hat{W}(s)}{s} = \hat{W}(s) - \frac{r\hat{W}(0)}{s} - \frac{r[\hat{W}(s)-\hat{W}(0)]}{s} \\ &= -\frac{1}{\mu s} + \hat{W}(s) - r(TW)^{\wedge}(s), \qquad \Re s = 0, s \neq 0.\end{aligned}$$

Put $H_2 := W$ and $H_1 := L/\mu - rTW$. The desired decomposition follows from (3.1) [10, Lemmas 3 and 4]. (Note that in the case $F([0,\infty)) = 1$, relation (3.1) is the Laplace-transform version of the decomposition $H = H_1 + H_2$.)

Finally, let $T^2F \in \mathcal{A}$. Then

$$\begin{aligned}(TW)^{\wedge}(s) &= \frac{\hat{W}(s)-\hat{W}(0)}{s} = -\frac{v(s)-v(0)}{s} \cdot \frac{1}{v(s)v(0)} \\ &= -\hat{W}(s)(TV)^{\wedge}(s)/v(0) = \hat{W}(s)[(TF)^{\wedge}(s) - r(T^2F)^{\wedge}(s)]/v(0).\end{aligned}$$

This means that $TW = W*(TF - rT^2F)/v(0) \in \mathcal{A}$, and hence the second assertion of the theorem follows. □

4. Submultiplicative case

This section deals with submultiplicative moments of the measures $H_1 - L/\mu$ and H_2 of the decomposition given by Theorem 3.1.

Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function such that $r_1 \leq 0 \leq r_2$. By Theorem 1 of [7], $\mathcal{A} := S(\varphi)$ satisfies properties (i) and (ii) of the preceding section, and hence Theorem 3.1 applies. We note some nuances. Relation $TF \in S(\varphi)$ implies $F \in S(\varphi)$. Actually,

$$\begin{aligned}\int_0^{\infty} \varphi(x) F((x,\infty))\, dx &\geq \sum_{k=0}^{\infty} \inf_{x \in [k,k+1)} \varphi(x) F((k+1,k+2]) \\ &\geq \frac{1}{M(1)} \sum_{k=0}^{\infty} \int_{k+1}^{k+2} \varphi(x)\, F(dx) = \frac{1}{M(1)} \int_1^{\infty} \varphi(x)\, F(dx).\end{aligned}$$

Since, obviously, $\int_0^1 \varphi(x)\, F(dx) < \infty$, we have $\int_0^{\infty} \varphi(x)\, F(dx) < \infty$. Similarly, $\int_{-\infty}^0 \varphi(x)\, F(dx) < \infty$. Therefore, instead of the hypotheses F, $TF \in S(\varphi)$ in Theorem 3.1, we may assume only $TF \in S(\varphi)$. Similarly, the set of conditions F, TF, $T^2F \in S(\varphi)$ may be replaced by $T^2F \in S(\varphi)$. Suppose now that $\varphi(x)/\exp(r_1 x)$

is nonincreasing on $(-\infty, 0)$ and $\varphi(x)/\exp(r_2 x)$ is nondecreasing on $[0,\infty)$. Theorem 3 of [11] implies that if $r_1 = 0 = r_2$ and $\int_{\mathbf{R}}(1+|x|)^k\varphi(x)\,F(dx) < \infty$ for some integer $k \geq 1$, or if $r_1 < 0 = r_2$ and $\int_0^\infty (1+x)^k\varphi(x)\,F(dx) < \infty$, or if $r_1 = 0 < r_2$ and $\int_{-\infty}^0 (1+|x|)^k\varphi(x)\,F(dx) < \infty$, then $T^kF \in S(\varphi)$. If $r_1 < 0 < r_2$, then $F \in S(\varphi) \Rightarrow T^kF \in S(\varphi)\ \forall k \geq 1$ [11, Theorem 2]. Suppose now that $r_1 = 0 = r_2$. Then, instead of the hypotheses F, $TF \in S(\varphi)$ in Theorem 3.1, we may assume only $F \in S(\varphi_1)$, where $\varphi_1(x) := (1+|x|)\varphi(x)$. Similarly, the set of conditions F, TF, $T^2F \in S(\varphi)$ may be replaced by $F \in S(\varphi_2)$, where $\varphi_2(x) := (1+|x|)^2\varphi(x)$. In the latter case, H_2 will be in $S(\varphi_1)$. Suppose $r_1 < 0 < r_2$. Then the set of conditions F, TF, $T^2F \in S(\varphi)$ may be replaced by $F \in S(\varphi)$. The intermediary cases $r_1 < 0 = r_2$ and $r_1 = 0 < r_2$ are dealt with in a similar way.

Corollary 4.1. *Let $\varphi(x)$ be a submultiplicative function such that $r_1 \leq 0 \leq r_2$. Suppose that $\varphi(x)$ is nonincreasing on $(-\infty,0)$ and nondecreasing on $[0,\infty)$. Assume $TF \in S(\varphi)$. Fix any $h > 0$. Then*

$$|H - L/\mu|((x,x+h]) = o(1/\varphi(x)) \qquad \text{as } |x| \to \infty.$$

Proof. For the sake of definiteness, we consider the case $x \to \infty$. By Theorem 1, $H_2 \in S(\varphi)$, and hence

$$\begin{aligned}\varphi(x)|H - L/\mu|((x,x+h]) &\leq \int_x^{x+h} \varphi(y)\,|H_2|(dy) + r\int_x^{x+h} \varphi(y)\,|H_2((y,\infty))|\,dy \\ &\leq \int_x^\infty \varphi(y)\,|H_2|(dy) + rhM(h)\varphi(x)|H_2|((x,\infty)) \\ &\leq [1 + rhM(h)]\int_x^\infty \varphi(y)\,|H_2|(dy) = o(1) \qquad \text{as } x \to \infty.\end{aligned}$$

□

Remark 4.2. In the case $r_1 = r_2 = 0$, the assertion of Corollary 4.1 was obtained in [6, Corollary 3] (see also [1, Theorem 2.6.4 (b)], where $\varphi(x) \equiv 1$).

5. Convergence rates in a key renewal theorem

Consider the renewal equation

$$X(t) = g(t) + \int_{\mathbf{R}} X(t-y)\,F(dy) =: g(t) + X * F(t), \tag{5.1}$$

where $g \in L_1(\mathbf{R})$ and F is a spread-out probability distribution on $\mathbf{R}$ with positive mean μ. The function $X(t) := g * H(t) + c$ is clearly a solution to (5.1); here c is any constant. So the asymptotic properties of the solution to (5.1) are those of the convolution $g * H(t)$, which, under various assumptions, have been studied by several authors [12, 13, 8, 2, 1]. Some properties of $X(t)$ were later rediscovered in a slightly more general setting [15, Theorem 5.1]. Usually, assertions about the asymptotic behavior of $g * H(t)$ are called *key renewal theorems*. As pointed out in [2] and carried out in [1], Stone's decomposition allows us to obtain an elegant proof of a key renewal theorem, which in simplified form can be stated as follows:

$$g * H(t) \to \begin{cases} \mu^{-1}\int_{\mathbf{R}} g(x)\,dx & \text{as } t \to \infty, \\ 0 & \text{as } t \to -\infty, \end{cases} \tag{5.2}$$

provided that $g(x)$ is bounded and $\lim_{|x|\to\infty} g(x) = 0$.

In this section, we shall obtain submultiplicative rates of convergence in (5.2) by means of the Stone-type decomposition of Theorem 3.1 with $\mathcal{A} = S(\varphi)$.

Theorem 5.1. *Let $\varphi(x)$ be a submultiplicative function such that $r_1 \le 0 \le r_2$, and let $g(x)$, $x \in \mathbf{R}$, be a Borel-measurable function such that (a) $g \in L_1(\mathbf{R})$, (b) $g \cdot \varphi \in L_\infty(\mathbf{R})$, (c) $g(x)\varphi(x) \to 0$ as $|x| \to \infty$ outside a set of Lebesgue measure zero, and (d) $\varphi(t)\int_t^\infty |g(x)|\,dx \to 0$ as $t \to \infty$ and $\varphi(t)\int_{-\infty}^t |g(x)|\,dx \to 0$ as $t \to -\infty$. Suppose $T^2F \in S(\varphi)$. Then, as t approaches $\pm\infty$ outside a set of Lebesgue measure zero,*

$$\sup_{\alpha:|\alpha|\le|g|} \left| \alpha * H(t) - \mu^{-1} \int_{\mathbf{R}} \alpha(x)\,dx \right| = o\left(\frac{1}{\varphi(t)}\right)$$

and $\sup_{\alpha:|\alpha|\le|g|} |\alpha * H(t)| = o(1/\varphi(t))$, the $\alpha(x)$ being Borel-measurable functions on $\mathbf{R}$.

Proof. By Theorem 3.1 with $\mathcal{A} = S(\varphi)$, both $H_1 - L/\mu$ and H_2 are elements of $S(\varphi)$. Choose $\tilde{g} \in L_1(\mathbf{R})$ such that $\tilde{g} = g$ a.e., $\sup_{x\in\mathbf{R}}|\tilde{g}(x)|\varphi(x) < \infty$, and $\tilde{g}(x)\varphi(x) \to 0$ as $|x| \to \infty$ in the usual sense. It suffices to put $\tilde{g}(x) = 0$ on $\{x \in \mathbf{R} : |g(x)|\varphi(x) > \|g\cdot\varphi\|_\infty\}$ and on a set, say B, of Lebesgue measure zero such that $\lim_{x\notin B,|x|\to\infty} g(x)\varphi(x) = 0$; otherwise, $\tilde{g}(x) := g(x)$. By Fubini's theorem, the sets $A_1 := \{x : |\tilde{g}|*|H_1 - L/\mu|(x) \ne |g|*|H_1 - L/\mu|(x)\}$ and $A_2 := \{x : |\tilde{g}|*|H_2|(x) \ne |g| * |H_2|(x)\}$ are both of Lebesgue measure zero. Set $A := A_1 \cup A_2$. We have

$$\varphi(t)|\tilde{g}| * |H_2|(t) \le \int_{\mathbf{R}} |\tilde{g}(t-x)|\varphi(t-x)\varphi(x)|H_2|(dx).$$

By dominated convergence, it follows from the hypotheses of the theorem that the right-hand side tends to zero as $t \to \infty$, and so does the left-hand side. By the same reasons, $\lim_{t\to\infty} \varphi(t)|\tilde{g}| * |H_1 - L/\mu|(t) = 0$. Hence both $\varphi(t)|g| * |H_1 - L/\mu|(t)$ and $\varphi(t)|g| * |H_2|(t)$ tend to zero as $t \to \infty$, remaining outside the set A of Lebesgue measure zero. The first assertion of the theorem now follows from the obvious inequality

$$\left| \alpha * H(t) - \mu^{-1} \int_{\mathbf{R}} \alpha(x)\,dx \right| \le |g| * |H_1 - L/\mu|(t) + |g| * |H_2|(t) + \mu^{-1} \int_t^\infty |g(x)|\,dx$$

and condition (d). The case $t \to -\infty$ is dealt with in a similar way. ☐

References

1. G. Alsmeyer, *Erneuerungstheorie*, B.G. Teubner, Stuttgart, 1991. MR **92f:**60148
2. E. Arjas, E. Nummelin, R. L. Tweedie, *Uniform limit theorems for non-singular renewal and Markov renewal processes.* J. Appl. Probab. **15** (1978), 112–125. MR **57:**7798
3. W. Feller, *An Introduction to Probability Theory and Its Applications II*, Wiley, New York, 1966. MR **35:**1048
4. R. Grübel, *On subordinated distributions and generalized renewal measures.* Ann. Probab. **15** (1987), 394–415. MR **88d:**60045
5. E. Hille, R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, Providence, RI, 1957. MR **19:**664d
6. B. A. Rogozin, *Asymptotics of renewal functions.* Theory Probab. Appl. **21** (1976), 669–686. MR **54:**8911
7. B. A. Rogozin, M. S. Sgibnev, *Banach algebras of measures on the line.* Siberian Math. J. **21** (1980), 265–273. MR **81e:**43003

8. M. Schäl, *Über Lösungen einer Erneuerungsgleichung.* Abh. Math. Sem. Univ. Hamburg **36** (1971), 89–98. MR **48:**12659
9. M. S. Sgibnev, *Submultiplicative moments of the supremum of a random walk with negative drift.* Statist. Probab. Lett. **32** (1997), 377–383. MR **99e:**60158
10. M. S. Sgibnev, *Exact asymptotic behaviour in a renewal theorem for convolution equivalent distributions with exponential tails.* SUT Journal of Mathematics **35** (1999), 247–262. MR **2000m:**60102
11. M. S. Sgibnev, *An asymptotic expansion for the distribution of the supremum of a random walk.* Studia Math. **140** (2000), 41–55. MR **2001g:**60109
12. W. L. Smith, *Regenerative stochastic processes.* Proc. R. Soc. London A **232** (1955), 6–31. MR **17:**502b
13. W. L. Smith, *Remarks on the paper 'Regenerative stochastic processes'.* Proc. R. Soc. London A **256** (1960), 496–501. MR **22:**6025
14. C. Stone, *On absolutely continuous components and renewal theory.* Ann. Math. Statist. **37** (1966), 271–275. MR **33:**4981
15. N. B. Yengibarian, *Renewal equation on the whole line.* Stochastic Process. Appl. **85** (2000), 237–247. MR **2001d:**60097b

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