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BOUNDEDNESS OF THE BERGMAN TYPE OPERATORS ON MIXED NORM SPACES

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ABSTRACT. Conditions sufficient for boundedness of the Bergman type operators on certain mixed norm spaces $L_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$ of functions on the unit ball of C^n are given, and this is used to solve Gleason's problem for the mixed norm spaces $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$.

1. INTRODUCTION

For $z = (z_1, z_2, \cdots, z_n)$, $w = (w_1, w_2, \cdots, w_n) \in C^n$, we shall denote the inner product of z, w by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm of z by $\|z\| = \sqrt{\langle z, z\rangle}$. Let $B = B_n = \{z \in C^n : \|z\| < 1\}$ be the open unit ball in C^n. The space of holomorphic functions on B will be denoted by $H(B)$. A positive continuous function φ on [0,1) is normal (see [1]) if there exist $0 < a < b, 0 \le r_0 < 1$ such that:

(1) $\frac{\varphi(r)}{(1-r)^a}$ is nonincreasing for $r_0 \le r < 1$ and $\lim_{r\to 1} \frac{\varphi(r)}{(1-r)^a} = 0$;

(2) $\frac{\varphi(r)}{(1-r)^b}$ is nondecreasing for $r_0 \le r < 1$ and $\lim_{r\to 1} \frac{\varphi(r)}{(1-r)^b} = \infty$.

For $0 < p < \infty$, $0 < q < \infty$ and a normal function φ, let $L_{p,q}(\varphi)$ denote the spaces of measurable functions on B with

$$\|f\|_{p,q,\varphi} = \left\{\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr\right\}^{1/p} < \infty,$$

where

$$M_q(r,f) = \left\{\int_{\partial B} |f(r\zeta)|^q d\sigma(\zeta)\right\}^{1/q}.$$

Let $1 < q < \infty$. Equipped with the above norm, $L_{p,q}(\varphi)$ is a Banach space for $p \ge 1$. When $0 < p < 1$, $\|\cdot\|^p_{p,q,\varphi}$ is a quasinorm on $L_{p,q}(\varphi)$. Thus, $L_{p,q}(\varphi)$ is a metric space if supplied with the distance $d(f,g) = \|f-g\|^p_{p,q,\varphi}$ and the vector space operations are continuous in this metric. That this d is complete is proved in the same way as in the familiar case $p \ge 1$. There is nothing to guarantee that $\|\alpha f\|^p_{p,q,\varphi} = |\alpha|\|f\|^p_{p,q,\varphi}$, however, and so $\|\cdot\|^p_{p,q,\varphi}$ may not be a norm. So $L_{p,q}(\varphi)$ is a Frechet space but not a Banach space. The reader is referred to [2] for the

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basic theory of Frechet and Banach spaces. For $s \in R, t > 0$, the operator $P_{s,t}$ on $L_{p,q}(\varphi)$ is given by

$$P_{s,t}f(z) = c_{n,t}(1 - \|z\|^2)^s \int_B \frac{(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w\rangle)^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi),$$

where the complex power is understood to be principal branches,

$$c_{n,t} = C^n_{n+t-1} = \frac{\Gamma(n+t)}{\Gamma(t)\Gamma(n+1)}.$$

G.B. Ren and J.H. Shi in [1] show that: If $t > b > a > -s$, then $P_{s,t}$ is a bounded operator of $L_{p,q}(\varphi)$ into $L_{p,q}(\varphi)$ $(1 \le p < \infty, 1 \le q < \infty)$, but the problem which is still unsolved is the case $0 < p < 1$. We know that the main tools for proving the above results are Hölder's inequality and the results due to Forelli and Rudin (e.g. see [3], Proposition 2.7 or Lemma 3.1, etc.). Because Hölder's inequality can be used only in the case $1 < p < \infty$, the method in [5], [6], [1],[7] and [8] cannot deal with the case $0 < p < 1$. In order to overcome this difficulty, we will exploit the Hardy-Littlewood analytical technique in [9]. Here we discuss the boundedness of the operator $P_{s,t}$ on $L_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$ and obtain a sufficient condition. One of the main points of the paper is to extend some results for a Banach space setting to a Frechet space setting. Our main result is:

Theorem A. *Let $0 < p < 1, 1 < q < \infty$. If $t > b > a > -s$ and φ is a normal function, then $P_{s,t} : L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is a bounded operator.*

Throughout this paper, the letter C stands for a positive different constant.

2. Background and preliminaries

The following lemmas will be needed in the proof of Theorem A.

Lemma 1 ([4], Proposition 1.4.10). *For $z \in B$, c real, $t > -1$, define*

$$J_{c,t}(z) = \int_B \frac{(1 - \|w\|^2)^t}{|1 - \langle z, w\rangle|^{n+1+t+c}} dV(w).$$

Then:

1. *when $c < 0$, $J_{c,t}$ is bounded in z;*
2. *when $c > 0$, $J_{c,t}(z) \sim (1 - \|z\|^2)^{-c}$, where $\sim$ denotes $(1 - \|z\|^2)^c J_{c,t}(z)$ has a positive limit $(\|z\| \to 1)$.*

Lemma 2 ([1], Lemma 2.3). *Let φ be a normal function. If $s + t > b > a > s$, then*

$$\int_0^1 \frac{\varphi^p(\rho)d\rho}{(1-\rho)^{ps+1}(1-r\rho)^{pt}} \le C \frac{\varphi^p(r)}{(1-r)^{p(s+t)}} \ (0 \le r < 1, p > 0).$$

Lemma 3 ([1], Lemma 2.1). *If $s + t > 0, 1 \le q < \infty$, then*

$$M_q(\rho, P_{s,t}f) \le C(1-\rho^2)^s \int_0^1 \frac{r^{2n-1}(1-r^2)^{t-1}}{(1-r\rho)^{t+s}} M_q(r, f)dr.$$

This follows from Hölder's inequality and Lemma 1 (see [1]).

In order to prove our main result, we first prove the following lemma.

Lemma 4. *If $t > b > a > -s$, $0 < p < 1$ and $1 < q < \infty$, then*

$$M_q^p(\rho, P_{s,t}f) \le C(1-\rho^2)^{ps} \int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r, f)dr.$$

Proof. Denote $s_k = 1 - 2^{-k}$. By the monotonicity of the integral means $M_q^p(r, f)$ with respect to r, Lemma 3 and the elementary inequality $(a+b)^p \leq a^p + b^p$ $(a, b \geq 0, 0 < p < 1)$, we obtain

$$\begin{aligned} M_q^p(\rho, P_{s,t}f) &\leq C(1-\rho^2)^{ps}\left(\int_0^1 \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r,f)dr\right)^p \\ &= C(1-\rho^2)^{ps}\left(\sum_{k=1}^{\infty}\int_{s_{k-1}}^{s_k} \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r,f)dr\right)^p \\ &\leq C(1-\rho^2)^{ps}\left(\sum_{k=1}^{\infty} \frac{s_k^{2n-1}(1-s_{k-1})^{t}}{(1-s_k\rho)^{t+s}} M_q(s_k,f)\right)^p \\ &\leq C(1-\rho^2)^{ps}\sum_{k=1}^{\infty} \frac{s_k^{p(2n-1)}(1-s_{k-1})^{pt}}{(1-s_k\rho)^{p(t+s)}} M_q^p(s_k,f) \\ &\leq C(1-\rho^2)^{ps}\int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r,f)dr. \end{aligned}$$

This completes the proof of the lemma.

3. Proof of the main theorem

We can now prove the main result of this paper.

Proof of Theorem A. For any $f \in L_{p,q}(\varphi)$, by Lemma 3, Lemma 2, Lemma 4 and Fubini's Theorem, we have

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\varphi}^p &\leq \int_0^1 (1-\rho)^{-1}\varphi^p(\rho)M_q^p(\rho, P_{s,t}f)d\rho \\ &\leq C\int_0^1\left(\int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r,f)dr\right)(1-\rho)^{ps-1}\varphi^p(\rho)d\rho \\ &= C\int_0^1\left(\int_0^1 \frac{\varphi^p(\rho)}{(1-\rho)^{1-ps}(1-r\rho)^{p(t+s)}}d\rho\right) r^{p(2n-1)}(1-r)^{pt-1}M_q^p(r,f)dr \\ &\leq C\int_0^1 r^{p(2n-1)}(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr. \end{aligned}$$

Use the change of variables $r = \rho^{\frac{1}{p}}$ and note that $p < 1$, $\rho^{\frac{1}{p}} < \rho$. Thus

$$(1-\rho^{\frac{1}{p}})^{-1} < (1-\rho)^{-1} \quad \text{and} \quad M_q^p(\rho^{\frac{1}{p}}, f) < M_q^p(\rho, f).$$

Finally, since φ is normal,

$$\varphi^p(\rho^{\frac{1}{p}}) \leq \frac{(1-\rho^{\frac{1}{p}})^{bp}}{(1-\rho)^{bp}}\varphi^p(\rho).$$

That is, $\varphi^p(\rho^{\frac{1}{p}}) \leq C\varphi^p(\rho)$. So

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\varphi}^p &\leq C\int_0^1 \rho^{2n-1}(1-\rho^{\frac{1}{p}})^{-1}\varphi^p(\rho^{\frac{1}{p}})M_q^p(\rho^{\frac{1}{p}}, f)d\rho \\ &\leq C\int_0^1 \rho^{2n-1}(1-\rho)^{-1}\varphi^p(\rho)M_q^p(\rho, f)d\rho = C\|f\|_{p,q,\varphi}^p. \end{aligned}$$

Hence the proof of Theorem A follows.

Remark. Let

$$P^{\sim}_{s,t}f(z) = c_{n,t}(1-\|z\|^2)^s \int_B \frac{(1-\|w\|^2)^{t-1}|f(w)|}{|1-\langle z,w\rangle|^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi).$$

The proof of Theorem A actually gives us a little more, that is, there exists a constant C such that $\|P^{\sim}_{s,t}f\|_{p,q,\varphi} \le C\|f\|_{p,q,\varphi}$, $\forall f \in L_{p,q}(\varphi)$.

4. An application

We denote the holomorphic mixed norm space $L_{p,q}(\varphi) \cap H(B)$ by $H_{p,q}(\varphi)$, and the norm in $H_{p,q}(\varphi)$ is equivalent to

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 (1-r)^{-1}\varphi^p(r) M_q^p(r,f)dr \right\}^{1/p}.$$

In this section, we investigate Gleason's problem on $H_{p,q}(\varphi)$. To the best of our knowledge, this problem has been answered respectively by Zhu in [10], Ortega in [11], Choe in [5], and Ren and Shi in [1], [12], [13] in the case $1 \le p < \infty$. So a natural question comes up: How is the case $0 < p < 1$? As an application of Theorem A, we obtain the following:

Theorem B. *Gleason's problem can be solved on $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$. More precisely, for any integer $m > 1$, there exist bounded linear operators A_α on $H_{p,q}(\varphi)$ such that if $f \in H_{p,q}(\varphi)$, $D^\alpha f(0) = 0$ $(|\alpha| \le m-1)$, then $f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z)$ on B, with $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.*

The fundamental ideas used in arguing this theorem come from the references [10] and [13].

Proof. Assume $m = 1$. By Leibenson's technique, we get

$$f(z) = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k}(rz)dr,$$

where $f \in H_{p,q}(\varphi)$ and $f(0) = 0$. Set $A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(rz)dr, z \in B$. A_k is obviously linear, so it remains to show that A_k is bounded on $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$. Given $f \in H_{p,q}(\varphi)$, let $f_r(z) = f(rz)$, $r \in (0,1)$. We have $P_{0,t}f_r = f_r$; see [1]. Letting $r \to 1^-$, the boundedness of $P_{0,t}$ implies that $P_{0,t}f = f$. Thus

$$f(z) = c_{n,t}\int_B \frac{(1-\|w\|^2)^{t-1}f(w)}{(1-\langle z,w\rangle)^{n+t}} dV(w),$$

where $t > b$, $t \in \mathcal{N}$.

Differentiating under the integral gives

$$\frac{\partial f}{\partial z_k}(z) = c_{n,t}\int_B \frac{\overline{w_k}(1-\|w\|^2)^{t-1}f(w)}{(1-\langle z,w\rangle)^{n+t+1}} dV(w).$$

This implies that

$$\begin{aligned}A_kf(z) &= C\int_0^1 dr\int_B \frac{\overline{w_k}(1-\|w\|^2)^{t-1}f(w)}{(1-r\langle z,w\rangle)^{n+t+1}}dV(w)\\ &= C\int_B \overline{w_k}(1-\|w\|^2)^{t-1}f(w)dV(w)\int_0^1 \frac{1}{(1-r\langle z,w\rangle)^{n+t+1}}dr\\ &= C\int_B \frac{\overline{w_k}(1-\|w\|^2)^{t-1}f(w)}{(1-\langle z,w\rangle)^{n+t}}Q(z,w)dV(w),\end{aligned}$$

where

$$Q(z,w)=\frac{1-(1-\langle z,w\rangle)^{n+t}}{\langle z,w\rangle}=\sum_{k=1}^{n+t-1}(1-\langle z,w\rangle)^k.$$

Note that $Q(z,w)$ is a polynomial in z and $\overline{w}$. Thus we can find a constant $C>0$ such that $|A_kf(z)|\le C|P_{0,t}^{\sim}f(z)|$. By the Remark after Theorem A, we see that A_k is bounded on $H_{p,q}(\varphi)$ $(0<p<1, 1<q<\infty)$. For m in the general case, it can be proved by induction. Therefore, the proof is complete.

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