BOUNDEDNESS OF THE BERGMAN TYPE OPERATORS
ON MIXED NORM SPACES

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ABSTRACT. Conditions sufficient for boundedness of the Bergman type operators on certain mixed norm spaces $L^{p,q}(\mathbb{D})$ of functions on the unit ball of $\mathbb{C}^n$ are given, and this is used to solve Gleason’s problem for the mixed norm spaces $H^{p,q}(\mathbb{D})$.

1. Introduction

For $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, we shall denote the inner product of $z, w$ by $(z, w) = \sum_{j=1}^{n} z_j \overline{w_j}$ and the norm of $z$ by $\|z\| = \sqrt{(z, z)}$. Let $B = B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the open unit ball in $\mathbb{C}^n$. The space of holomorphic functions on $B$ will be denoted by $H(B)$. A positive continuous function $\varphi$ on $[0,1)$ is normal (see [1]) if there exist $0 < a < b; 0 < r_0 < 1$ such that:

1. $\varphi(r)/(1-r)^a$ is nonincreasing for $r_0 \leq r < 1$ and $\lim_{r \to 1} \varphi(r)/(1-r)^a = 0$;
2. $\varphi(r)/(1-r)^b$ is nondecreasing for $r_0 \leq r < 1$ and $\lim_{r \to 1} \varphi(r)/(1-r)^b = \infty$.

For $0 < p < \infty$, $0 < q < \infty$ and a normal function $\varphi$, let $L_{p,q}(\varphi)$ denote the spaces of measurable functions on $B$ with

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr \right\}^{1/p} < \infty,$$

where

$$M_q^p(r,f) = \left\{ \int_B |f(r\zeta)|^q d\sigma(\zeta) \right\}^{1/q}.$$

Let $1 < q < \infty$. Equipped with the above norm, $L_{p,q}(\varphi)$ is a Banach space for $p \geq 1$. When $0 < p < 1$, $\|\cdot\|_{p,q,\varphi}$ is a quasinorm on $L_{p,q}(\varphi)$. Thus, $L_{p,q}(\varphi)$ is a metric space if supplied with the distance $d(f,g) = \|f-g\|_{p,q,\varphi}$ and the vector space operations are continuous in this metric. That this $d$ is complete is proved in the same way as in the familiar case $p \geq 1$. There is nothing to guarantee that $\|\alpha f\|_{p,q,\varphi} = |\alpha| \|f\|_{p,q,\varphi}$, however, and so $\|\cdot\|_{p,q,\varphi}$ may not be a norm. So $L_{p,q}(\varphi)$ is a Frechet space but not a Banach space. The reader is referred to [2] for the...
basic theory of Frechet and Banach spaces. For \( s \in R, t > 0 \), the operator \( P_{s,t} \) on \( L_{p,q}(\varphi) \) is given by
\[
P_{s,t}f(z) = c_{n,t}(1-\|z\|^2)^{s} \int_B \frac{(1-\|w\|^2)^{t-1}f(w)}{(1-\langle z, w \rangle)^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi),
\]
where the complex power is understood to be principal branches,
\[
c_{n,t} = C_{n+t-1}^n = \frac{\Gamma(n+t)}{\Gamma(t)\Gamma(n+1)}.
\]
G.B. Ren and J.H. Shi in [1] show that: If \( t > b > a > -s \), then \( P_{s,t} \) is a bounded operator of \( L_{p,q}(\varphi) \) into \( L_{p,q}(\varphi) \) (\( 1 \leq p < \infty, 1 \leq q < \infty \)), but the problem which is still unsolved is the case \( 0 < p < 1 \). We know that the main tools for proving the above results are Hölder’s inequality and the results due to Forelli and Rudin (e.g. see [2], Proposition 2.7 or Lemma 3.1, etc.). Because Hölder’s inequality can be used only in the case \( 1 < p < \infty \), the method in [3], [4], [1], [7] and [2] cannot deal with the case \( 0 < p < 1 \). In order to overcome this difficulty, we will exploit the Hardy-Littlewood analytical technique in [9]. Here we discuss the boundedness of the operator \( P_{s,t} \) on \( L_{p,q}(\varphi) \) \( (0 < p < 1, 1 < q < \infty) \) and obtain a sufficient condition. One of the main points of the paper is to extend some results for a Banach space setting to a Frechet space setting. Our main result is:

**Theorem A.** Let \( 0 < p < 1, 1 < q < \infty \). If \( t > b > a > -s \) and \( \varphi \) is a normal function, then \( P_{s,t} : L_{p,q}(\varphi) \to L_{p,q}(\varphi) \) is a bounded operator.

Throughout this paper, the letter \( C \) stands for a positive different constant.

### 2. Background and preliminaries

The following lemmas will be needed in the proof of Theorem A.

**Lemma 1** ([4], Proposition 1.4.10). For \( z \in B, c \) real, \( t > -1 \), define
\[
J_{c,t}(z) = \int_B \frac{(1-\|w\|^2)^{t}}{|1-\langle z, w \rangle|^{n+1+t+c}} dV(w).
\]
Then:

1. when \( c < 0 \), \( J_{c,t} \) is bounded in \( z \);
2. when \( c > 0 \), \( J_{c,t}(z) \sim (1-\|z\|^2)^{-c} \), where \( \sim \) denotes \( (1-\|z\|^2)^c J_{c,t}(z) \) has a positive limit \((\|z\| \to 1)\).

**Lemma 2** ([1], Lemma 2.3). Let \( \varphi \) be a normal function. If \( s + t > b > a > s \), then
\[
\int_0^1 \frac{\varphi^p(\rho)}{(1-\rho)^{ps+1}(1-r\rho)^{pt}} d\rho \leq C \frac{\varphi^p(r)}{(1-r)^{ps+1}} \quad (0 \leq r < 1, p > 0).
\]

**Lemma 3** ([1], Lemma 2.1). If \( s+t > 0, 1 \leq q < \infty \), then
\[
M_q(\rho,P_{s,t}f) \leq C(1-\rho^2)^s \int_0^1 \frac{r^{n-1}(1-r^2)^{t-1}}{(1-r\rho)^{t+s}} M_q(r,f) dr.
\]

This follows from Hölder’s inequality and Lemma 1 (see [1]).

In order to prove our main result, we first prove the following lemma.

**Lemma 4.** If \( t > b > a > -s \), \( 0 < p < 1 \) and \( 1 < q < \infty \), then
\[
M_q^p(\rho,P_{s,t}f) \leq C(1-\rho^2)^{ps} \int_0^1 \frac{r^{ps}(1-r^2)^{pt-1}}{(1-r\rho)^{pt+s}} M_q^p(r,f) dr.
\]
Proof. Denote $s_k = 1 - 2^{-k}$. By the monotonicity of the integral means $M_p^q(r, f)$ with respect to $r$, Lemma 3 and the elementary inequality $(a + b)^p \leq a^p + b^p$ $(a, b \geq 0, 0 < p < 1)$, we obtain

\[
M_p^q(\rho, P_s, f) \leq C(1 - \rho^2)^{ps} \left( \int_0^1 \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r, f) dr \right)^p \leq C(1 - \rho^2)^{ps} \left( \sum_{k=1}^\infty \int_{s_{k-1}}^{s_k} \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r, f) dr \right)^p \leq C(1 - \rho^2)^{ps} \left( \sum_{k=1}^\infty \frac{s_k^{2n-1}(1-s_{k-1})^t}{(1-s_k\rho)^{t+s}} M_q(s_k, f) \right)^p \leq C(1 - \rho^2)^{ps} \int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q(r, f) dr.
\]

This completes the proof of the lemma.

3. PROOF OF THE MAIN THEOREM

We can now prove the main result of this paper.

Proof of Theorem A. For any $f \in L_p,q(\varphi)$, by Lemma 3, Lemma 2, Lemma 4 and Fubini’s Theorem, we have

\[
\|P_s, f\|_{p,q,\varphi}^p = \int_0^1 (1 - \rho)^{-1} \varphi^p(\rho) M_q(\rho, P_s, f) d\rho \leq C \int_0^1 \left( \int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q(r, f) dr \right)^p (1 - \rho)^{p-1} \varphi^p(\rho) d\rho \\
\leq C \int_0^1 \varphi^p(\rho) \left( \int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q(r, f) dr \right) d\rho \\
\leq C \int_0^1 r^{p(2n-1)}(1-r)^{-1} \varphi^p(r) M_q(r, f) dr.
\]

Use the change of variables $r = \rho^{\frac{1}{p}}$ and note that $p < 1$, $\rho^{\frac{1}{p}} < \rho$. Thus

\[
(1 - \rho^{\frac{1}{p}})^{-1} < (1 - \rho)^{-1} \quad \text{and} \quad M_q(\rho^{\frac{1}{p}}, f) < M_q(\rho, f).
\]

Finally, since $\varphi$ is normal,

\[
\varphi^p(\rho) \leq \frac{(1 - \rho^{\frac{1}{p}})^{bp}}{(1 - \rho)^{bp}} \varphi^p(\rho).
\]

That is, $\varphi^p(\rho^{\frac{1}{p}}) \leq C\varphi^p(\rho)$. So

\[
\|P_s, f\|_{p,q,\varphi}^p \leq C \int_0^1 \rho^{2n-1}(1 - \rho^{\frac{1}{p}})^{-1} \varphi^p(\rho^{\frac{1}{p}}) M_q(\rho^{\frac{1}{p}}, f) d\rho \\
\leq C \int_0^1 \rho^{2n-1}(1 - \rho)^{-1} \varphi^p(\rho) M_q(\rho, f) d\rho = C \|f\|_{p,q,\varphi}^p.
\]

Hence the proof of Theorem A follows.
**Remark.** Let

\[ P_{n,t}^s f(z) = c_{n,t}(1 - \|z\|^2)^s \int_B \frac{(1 - \|w\|^2)^{t-1}|f(w)|}{(1 - \langle z, w \rangle)^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi). \]

The proof of Theorem A actually gives us a little more, that is, there exists a constant \( C \) such that \( \|P_{n,t}^s f\|_{p,q,\varphi} \leq C\|f\|_{p,q,\varphi}, \forall f \in L_{p,q}(\varphi). \)

4. **AN APPLICATION**

We denote the holomorphic mixed norm space \( L_{p,q}(\varphi) \cap H(B) \) by \( H_{p,q}(\varphi) \), and the norm in \( H_{p,q}(\varphi) \) is equivalent to

\[ \|f\|_{p,q,\varphi} = \left\{ \int_0^1 (1 - r)^{-1} \varphi^p(r) M_p^q(r,f) dr \right\}^{1/p}. \]

In this section, we investigate Gleason’s problem on \( H_{p,q}(\varphi) \). To the best of our knowledge, this problem has been answered respectively by Zhu in [10], Ortega in [4, 5], Choe in [5], and Ren and Shi in [1, 12, 13] in the case \( 1 \leq p < \infty \). So a natural question comes up: How is the case \( 0 < p < 1 \)? As an application of Theorem A, we obtain the following:

**Theorem B.** Gleason’s problem can be solved on \( H_{p,q}(\varphi) \) \( (0 < p < 1, 1 < q < \infty) \).

More precisely, for any integer \( m > 1 \), there exist bounded linear operators \( A_\alpha \) on \( H_{p,q}(\varphi) \) such that if \( f \in H_{p,q}(\varphi) \), \( D^\alpha f(0) = 0 \) \( (|\alpha| \leq m - 1) \), then \( f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z) \) on \( B \), with \( \alpha = (\alpha_1, \ldots , \alpha_n) \) a multi-index, \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

The fundamental ideas used in arguing this theorem come from the references [10] and [13].

**Proof.** Assume \( m = 1 \). By Leibensson’s technique, we get

\[ f(z) = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k} (rz) dr, \]

where \( f \in H_{p,q}(\varphi) \) and \( f(0) = 0 \). Set \( A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k} (rz) dr, z \in B \). \( A_k \) is obviously linear, so it remains to show that \( A_k \) is bounded on \( H_{p,q}(\varphi) \) \( (0 < p < 1, 1 < q < \infty) \). Given \( f \in H_{p,q}(\varphi) \), let \( f_r(z) = f(rz), r \in (0,1) \). We have \( P_{0,t} f_r = f_r; \) see [1]. Letting \( r \to 1^- \), the boundedness of \( P_{0,t} \) implies that \( P_{0,t} f = f \). Thus

\[ f(z) = c_{n,t} \int_B \frac{(1 - \|w\|^2)^{t-1}f(w)}{(1 - \langle z, w \rangle)^{n+t}} dV(w), \]

where \( t > b, \ t \in \mathcal{N} \).

Differentiating under the integral gives

\[ \frac{\partial f}{\partial z_k}(z) = c_{n,t} \int_B \frac{\partial}{\partial z_k} \left[ \frac{(1 - \|w\|^2)^{t-1}f(w)}{(1 - \langle z, w \rangle)^{n+t+1}} \right] dV(w). \]
This implies that
\[ A_k f(z) = C \int_0^1 \int_B \frac{w^k(1 - \|w\|^2)^{t-1} f(w)}{(1 - r(z, w))^{n+\tau+1}} dV(w) \]
\[ = C \int_B \frac{w^k(1 - \|w\|^2)^{t-1} f(w) dV(w)}{(1 - r(z, w))^{n+\tau+1}} \int_0^1 \frac{1}{(1 - \|w\|^2)^{t-1} f(w)} Q(z, w) dV(w), \]
where
\[ Q(z, w) = \frac{1 - (1 - \langle z, w \rangle)^{n+\tau}}{\langle z, w \rangle} = \sum_{k=1}^{n+\tau-1} (1 - \langle z, w \rangle)^k. \]
Note that \( Q(z, w) \) is a polynomial in \( z \) and \( \bar{w} \). Thus we can find a constant \( C > 0 \) such that \( |A_k f(z)| \leq C |P_n^\tau f(z)| \). By the Remark after Theorem A, we see that \( A_k \) is bounded on \( H_{p,q}(\varphi) \) (\( 0 < p < 1, 1 < q < \infty \)). For \( m \) in the general case, it can be proved by induction. Therefore, the proof is complete.

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REFERENCES


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