BOUNDEDNESS OF THE BERGMAN TYPE OPERATORS 
ON MIXED NORM SPACES

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(Communicated by Joseph A. Ball)

ABSTRACT. Conditions sufficient for boundedness of the Bergman type operators on certain mixed norm spaces $L^{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$) of functions on the unit ball of $C^n$ are given, and this is used to solve Gleason’s problem for the mixed norm spaces $H^{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$).

1. Introduction

For $z = (z_1, z_2, \ldots, z_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, we shall denote the inner product of $z, w$ by $(z, w) = \sum_{j=1}^{n} z_j \overline{w_j}$ and the norm of $z$ by $\|z\| = \sqrt{(z, z)}$. Let $B = B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the open unit ball in $\mathbb{C}^n$. The space of holomorphic functions on $B$ will be denoted by $H(B)$. A positive continuous function $\varphi$ on $[0,1)$ is normal (see [1]) if there exist $0 < a < b; 0 < r_0 < 1$ such that:

1. $\varphi(r) \frac{(1-r)}{(1-r^a)}$ is nonincreasing for $r_0 \leq r < 1$ and $\lim_{r \to 1^-} \varphi(r) \frac{(1-r)}{(1-r^a)} = 0$;

2. $\frac{(1-r)}{(1-r^b)}$ is nondecreasing for $r_0 \leq r < 1$ and $\lim_{r \to 1^-} \varphi(r) \frac{(1-r)}{(1-r^b)} = \infty$.

For $0 < p < \infty, 0 < q < \infty$ and a normal function $\varphi$, let $L_{p,q}(\varphi)$ denote the spaces of measurable functions on $B$ with

$$
\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{2n-1}(1-r)^{-1} \varphi^p(r) M_q^p(r, f) dr \right\}^{1/p} < \infty,
$$

where

$$
M_q^p(r, f) = \left\{ \int_B |f(r\zeta)|^q d\sigma(\zeta) \right\}^{1/q}.
$$

Let $1 < q < \infty$. Equipped with the above norm, $L_{p,q}(\varphi)$ is a Banach space for $p \geq 1$. When $0 < p < 1$, $\| \cdot \|_{p,q,\varphi}$ is a quasinorm on $L_{p,q}(\varphi)$. Thus, $L_{p,q}(\varphi)$ is a metric space if supplied with the distance $d(f, g) = \| f - g \|_{p,q,\varphi}$ and the vector space operations are continuous in this metric. That this $d$ is complete is proved in the same way as in the familiar case $p \geq 1$. There is nothing to guarantee that $\| \alpha f \|_{p,q,\varphi} = |\alpha| \| f \|_{p,q,\varphi}$ however, and so $\| \cdot \|_{p,q,\varphi}$ may not be a norm. So $L_{p,q}(\varphi)$ is a Frechet space but not a Banach space. The reader is referred to [2] for the...
basic theory of Frechet and Banach spaces. For \( s \in \mathbb{R}, t > 0 \), the operator \( P_{s,t} \) on \( L_{p,q}(\varphi) \) is given by

\[
P_{s,t}f(z) = c_{n,t}(1-\|z\|^2)^s \int_B \frac{(1-\|w\|^2)^t-1} {(1-\langle z,w \rangle)^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi),
\]

where the complex power is understood to be principal branches,

\[
c_{n,t} = C_{n+t-1} = \frac{\Gamma(n+t)} {\Gamma(t)\Gamma(n+1)}.
\]

G.B. Ren and J.H. Shi in [1] show that: If \( t > b > a > -s \), then \( P_{s,t} \) is a bounded operator of \( L_{p,q}(\varphi) \) into \( L_{p,q}(\varphi) \) (1 \( \leq p < \infty, 1 \leq q < \infty \)), but the problem which is still unsolved is the case \( 0 < p < 1 \). We know that the main tools for proving the above results are Hölder’s inequality and the results due to Forelli and Rudin (e.g. see [3], Proposition 2.7 or Lemma 3.1, etc.). Because Hölder’s inequality can be used only in the case \( 1 < p < \infty \), the method in [5], [6], [1], [7] and [8] cannot deal with the case \( 0 < p < 1 \). In order to overcome this difficulty, we will exploit the Hardy-Littlewood analytical technique in [9]. Here we discuss the boundedness of the operator \( P_{s,t} \) on \( L_{p,q}(\varphi) \) \( (0 < p < 1, 1 < q < \infty) \) and obtain a sufficient condition. One of the main points of the paper is to extend some results for a Banach space setting to a Frechet space setting. Our main result is:

**Theorem A.** Let \( 0 < p < 1, 1 < q < \infty \). If \( t > b > a > -s \) and \( \varphi \) is a normal function, then \( P_{s,t} : L_{p,q}(\varphi) \rightarrow L_{p,q}(\varphi) \) is a bounded operator.

Throughout this paper, the letter \( C \) stands for a positive different constant.

2. BACKGROUND AND PRELIMINARIES

The following lemmas will be needed in the proof of Theorem A.

**Lemma 1** ([4], Proposition 1.4.10). For \( z \in B, c \) real, \( t > -1 \), define

\[
J_{c,t}(z) = \int_B \frac{(1-\|w\|^2)^t} {\|1-\langle z,w \rangle\|^{n+1+t+c}} dV(w).
\]

Then:

1. when \( c < 0 \), \( J_{c,t} \) is bounded in \( z \);
2. when \( c > 0 \), \( J_{c,t}(z) \sim (1-\|z\|^2)^{-c} \), where \( \sim \) denotes \( (1-\|z\|^2)^c J_{c,t}(z) \) has a positive limit \( (\|z\| \rightarrow 1) \).

**Lemma 2** ([1], Lemma 2.3). Let \( \varphi \) be a normal function. If \( s+t > b > a > s \), then

\[
\int_0^1 \frac{\varphi^p(\rho)} {(1-\rho)^{ps+1}(1-r\rho)^{pt}} d\rho \leq C \frac{\varphi^p(r)} {(1-r)^{p(s+t)}} \quad (0 \leq r < 1, p > 0).
\]

**Lemma 3** ([1], Lemma 2.1). If \( s+t > 0, 1 \leq q < \infty \), then

\[
M_q(\rho, P_{s,t}f) \leq C(1-\rho^2)^s \int_0^1 r^{2q-1} \frac{(1-r^2)^{t-1}} {(1-r\rho)^{t+s}} M_q(r, f) dr.
\]

This follows from Hölder’s inequality and Lemma 1 (see [1]).

In order to prove our main result, we first prove the following lemma.

**Lemma 4.** If \( t > b > a > -s \), \( 0 < p < 1 \) and \( 1 < q < \infty \), then

\[
M^q_p(\rho, P_{s,t}f) \leq C(1-\rho^2)^{ps} \int_0^1 \frac{r^{p(2q-1)}(1-r)^{pt-1}} {(1-r\rho)^{p(t+s)}} M^q_p(r, f) dr.
\]
Proof. Denote $s_k = 1 - 2^{-k}$. By the monotonicity of the integral means $M^p_q(r, f)$ with respect to $r$, Lemma 3 and the elementary inequality $(a + b)^p \leq a^p + b^p$ ($a, b \geq 0, 0 < p < 1$), we obtain

$$
M^p_q(\rho, P_s, f) \leq C(1 - \rho^2)^{ps} \left( \int_0^1 \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M^p_q(r, f)dr \right)^p \\
= C(1 - \rho^2)^{ps} \left( \sum_{k=1}^{\infty} \int_{s_{k-1}}^{s_k} \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M^p_q(r, f)dr \right)^p \\
\leq C(1 - \rho^2)^{ps} \left( \sum_{k=1}^{\infty} \frac{s_k^{2n-1}(1-s_{k-1})^t}{(1-s_k\rho)^{t+s}} M^p_q(s_k, f) \right)^p \\
\leq C(1 - \rho^2)^{ps} \sum_{k=1}^{\infty} \frac{s_k^{2n-1}(1-s_{k-1})^t}{(1-s_k\rho)^{p(t+s)}} M^p_q(s_k, f) \\
\leq C(1 - \rho^2)^{ps} \int_0^1 \frac{r^{2n-1}(1-r)^{p(t-1)}}{(1-r\rho)^{p(t+s)}} M^p_q(r, f)dr.
$$

This completes the proof of the lemma.

3. PROOF OF THE MAIN THEOREM

We can now prove the main result of this paper.

Proof of Theorem A. For any $f \in L_{p,q}(\varphi)$, by Lemma 3, Lemma 2, Lemma 4 and Fubini’s Theorem, we have

$$
\|P_s, f\|_{p,q, \varphi}^p \leq \int_0^1 (1 - \rho)^{-1} \varphi^p(\rho) M^p_q(\rho, P_s, f) d\rho \\
\leq C \int_0^1 \left( \int_0^1 \frac{\varphi^p(\rho)}{(1-r\rho)^{p(t+s)}} M^p_q(r, f) dr \right) (1 - \rho)^{p-1} \varphi^p(\rho) d\rho \\
= C \int_0^1 \left( \int_0^1 \frac{\varphi^p(\rho)}{(1-r\rho)^{p(t+s)}} d\rho \right) \frac{r^{2n-1}(1-r)^{p(t-1)}}{(1-r\rho)^{p(t+s)}} M^p_q(r, f) dr \\
\leq C \int_0^1 r^{2n-1}(1-r)^{-1} \varphi^p(r) M^p_q(r, f) dr.
$$

Use the change of variables $r = \rho^{1\over p}$ and note that $p < 1, \rho ^ {1\over p} < \rho$. Thus

$$
(1 - \rho ^ {1\over p})^{-1} < (1 - \rho)^{-1} \quad \text{and} \quad M^p_q(\rho ^ {1\over p}, f) < M^p_q(\rho, f).
$$

Finally, since $\varphi$ is normal,

$$
\varphi^p(\rho ^ {1\over p}) \leq \frac{(1 - \rho ^ {1\over p})^{bp}}{(1 - \rho)^{bp}} \varphi^p(\rho).
$$

That is, $\varphi^p(\rho ^ {1\over p}) \leq C \varphi^p(\rho)$. So

$$
\|P_s, f\|_{p,q, \varphi}^p \leq C \int_0^1 \rho^{2n-1}(1 - \rho ^ {1\over p})^{-1} \varphi^p(\rho ^ {1\over p}) M^p_q(\rho ^ {1\over p}, f) d\rho \\
\leq C \int_0^1 \rho^{2n-1}(1 - \rho)^{-1} \varphi^p(\rho) M^p_q(\rho, f) d\rho = C \|f\|_{p,q, \varphi}^p.
$$

Hence the proof of Theorem A follows.
The proof of Theorem A actually gives us a little more, that is, there exists a constant $C$ such that $\|P_{n,t}^* f\|_{p,q,\varphi} \leq C \|f\|_{p,q,\varphi}$, $\forall f \in L_{p,q}(\varphi)$.

4. An Application

We denote the holomorphic mixed norm space $L_{p,q}(\varphi) \cap H(B)$ by $H_{p,q}(\varphi)$, and the norm in $H_{p,q}(\varphi)$ is equivalent to

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 (1-r)^{-1} \varphi^p(r) M_q^p(r,f) dr \right\}^{1/p}.$$ 

In this section, we investigate Gleason’s problem on $H_{p,q}(\varphi)$. To the best of our knowledge, this problem has been answered respectively by Zhu in [10], Ortega in [11], Choe in [5], and Ren and Shi in [1], [12], [13] in the case $1 \leq p < \infty$. So a natural question comes up: How is the case $0 < p < 1$? As an application of Theorem A, we obtain the following:

**Theorem B.** Gleason’s problem can be solved on $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$. More precisely, for any integer $m > 1$, there exist bounded linear operators $A_\alpha$ on $H_{p,q}(\varphi)$ such that if $f \in H_{p,q}(\varphi)$, $D^\alpha f(0) = 0$ $(|\alpha| \leq m - 1)$, then $f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z)$ on $B$, with $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The fundamental ideas used in arguing this theorem come from the references [10] and [13].

**Proof.** Assume $m = 1$. By Leibenson’s technique, we get

$$f(z) = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k}(rz) dr,$$

where $f \in H_{p,q}(\varphi)$ and $f(0) = 0$. Set $A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(rz) dr, z \in B$. $A_k$ is obviously linear, so it remains to show that $A_k$ is bounded on $H_{p,q}(\varphi)$ $(0 < p < 1, 1 < q < \infty)$. Given $f \in H_{p,q}(\varphi)$, let $f_r(z) = f(zr), r \in (0,1)$. We have $P_{0,t} f_r = f_r$; see [1]. Letting $r \to 1^-$, the boundedness of $P_{0,t}$ implies that $P_{0,t} f = f$. Thus

$$f(z) = c_{n,t} \int_B \frac{(1-\|w\|^2)^{t-1} f(w)}{(1-\langle z, w \rangle)^{n+t}} dV(w),$$

where $t > b, t \in \mathcal{N}$.

Differentiating under the integral gives

$$\frac{\partial f}{\partial z_k}(z) = c_{n,t} \int_B \frac{\overline{w}_k (1-\|w\|^2)^{t-1} f(w)}{(1-\langle z, w \rangle)^{n+t+1}} dV(w).$$
This implies that
\[
A_k f(z) = C \int_0^1 dr \int_B \frac{\overline{w}(1 - \|w\|^2)^{t-1} f(w)}{(1 - r(z, w))^{n+t+1}} dV(w)
\]
\[
= C \int_B \frac{\overline{w}(1 - \|w\|^2)^{t-1} f(w) dV(w)}{(1 - r(z, w))^{n+t+1}} dr
\]
\[
= C \int_B \frac{\overline{w}(1 - \|w\|^2)^{t-1} f(w)}{(1 - z, w)^{n+t}} Q(z, w) dV(w),
\]
where
\[
Q(z, w) = \frac{1 - (1 - \langle z, w \rangle)^{n+t}}{\langle z, w \rangle} = \sum_{k=1}^{n+t-1} (1 - \langle z, w \rangle)^k.
\]
Note that \(Q(z, w)\) is a polynomial in \(z\) and \(\overline{w}\). Thus we can find a constant \(C > 0\) such that \(|A_k f(z)| \leq C |P_0^m f(z)|\). By the Remark after Theorem A, we see that \(A_k\) is bounded on \(H_{p,q}^2(\varphi)\) \((0 < p < 1, 1 < q < \infty)\). For \(m\) in the general case, it can be proved by induction. Therefore, the proof is complete.

ACKNOWLEDGMENT

The author thanks the referee whose suggestions greatly improved this paper, especially for pointing out the proof of the main theorem to simplify our original proof.

REFERENCES


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