SHARP LOCAL ISOPERIMETRIC INEQUALITIES INVOLVING THE SCALAR CURVATURE

OLIVIER DRUET

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Abstract. We provide sharp local isoperimetric inequalities on Riemannian manifolds involving the scalar curvature, and thus answer a question asked by Johnson and Morgan.

1. Introduction and statement of the results

Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) with sectional curvature \(K_g \leq K_0\). A long-standing conjecture, a formulation of which can be found in [1], asserts that for any \(x \in M\), there exists \(r_x > 0\) such that for any \(\Omega\) contained in the geodesic ball of center \(x\) and radius \(r_x\),

\[
|\partial \Omega|_g \geq |\partial B|_{g_0}
\]

where \(|\cdot|_g\) (resp. \(|\cdot|_{g_0}\)) denotes the volume with respect to \(g\) (resp. \(g_0\)) and \(B\) is a ball of volume \(|\Omega|_g\) in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\).

A compact version of this conjecture was proved, with an additional assumption on the Gauss-Bonnet-Chern integrand in even dimensions, in the very nice Johnson and Morgan [10]. A natural question that Johnson and Morgan [10] asked is the following: is the result still true if we assume that the scalar curvature of \((M, g)\) is such that \(S_g < n(n - 1)K_0\) instead of assuming that \(K_g \leq K_0\)? We answer this question in the affirmative and prove the following:

**Theorem 1.** Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(x \in M\). Assume that \(S_g(x) < n(n - 1)K_0\) for some \(K_0 \in \mathbb{R}\). Then there exists \(r_x > 0\) such that for any \(\Omega\) contained in the geodesic ball of center \(x\) and radius \(r_x\),

\[
|\partial \Omega|_g > |\partial B|_{g_0}
\]

where \(B\) is a ball of volume \(|\Omega|_g\) in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\).

In the compact setting, the situation that was actually considered by Johnson and Morgan [10], we have the following:
Theorem 2. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\) with scalar curvature \(S_g < n(n-1)K_0\). There exists \(V > 0\) such that for any subset \(\Omega\) of \(M\) of volume less than or equal to \(V\),

\[
|\partial \Omega|_g > |\partial B|_{g_0}
\]

where \(B\) is a ball of volume \(|\Omega|_g\) in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\).

These results are optimal in the following sense: if we only assume that the Ricci curvature of \(M\) verifies \(\text{Ric}_g \leq (n-1)K_0\), the above isoperimetric comparison fails. Indeed, for any \(n\)-manifold \(M\) which is Ricci-flat but not flat (see [3] for examples of such manifolds), one may find a ball \(B_r\) in \(M\) of radius \(r\) as small as we want which verifies

\[
|\partial B_r|_g < |\partial B|_{\xi}
\]

where \(B\) is a ball of volume \(|B_r|_g\) in the Euclidean space \((\mathbb{R}^n, \xi)\). The above comparison result is also false on \(S^2 \times S^2\), as noticed in [10]. The proof of Theorem 1 is based on the study of local optimal Sobolev inequalities. The proof relies on PDE techniques and a fine asymptotic analysis of solutions of quasi-elliptic equations involving the \(p\)-Laplacian. Theorem 2 is a consequence of Theorem 1 thanks to geometric measure theory. The relevance of the scalar curvature when studying the validity of sharp Sobolev inequalities was noticed first by the author in [4] and underlined by Hebey in [9].

2. Sobolev inequalities and proof of Theorem 1

Let \(B\) be a ball in the model space \((M_0, g_0)\) of constant sectional curvature \(K_0\). It is not difficult to check that, for balls of small volume,

\[
|\partial B|_{g_0}^2 = K(n, 1)^{-2}|B_{g_0}|_{g_0}^{\frac{n+1}{n-1}} - \frac{n}{n+2}(n(n-1)K_0)|B|_{g_0}^2 + o(|B|_{g_0}^2).
\]

Here, \(n = \dim M_0\) and

\[
K(n, 1)^{-1} = n \left( \frac{\omega_{n-1}}{n} \right)^{\frac{1}{n}}.
\]

Now, let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(x_0 \in M\). In order to prove Theorem 1, it is clearly sufficient to prove that for any \(\varepsilon > 0\), there exists \(r_\varepsilon > 0\) such that for any \(\Omega \subset B_g(x_0, r_\varepsilon)\),

\[
|\partial \Omega|_g^2 \geq K(n, 1)^{-2}|\Omega|_g^{\frac{n+1}{n-1}} - \left( \frac{n}{n+2}S_g(x_0) + \varepsilon \right)|\Omega|_g^2.
\]

It is now well known that (2.1) is a consequence of the following Sobolev inequality: for any \(u \in C^\infty_c(B_g(x_0, r_\varepsilon))\),

\[
\|u\|_{p+1}^{p+1} \leq K(n, 1)^2 (\|\nabla u\|_1^2) + \left( \frac{n}{n+2}S_g(x_0) + \varepsilon \right) \|u\|_1^2
\]

where \(\| \cdot \|_p\) denotes the \(L^p\)-norm with respect to the Riemannian volume element \(dv_g\). Indeed, \(\Omega \subset B_g(x_0, r_\varepsilon)\) being given, one may find a sequence \((u_i)\) of smooth functions with compact support in \(B_g(x_0, r_\varepsilon)\) such that for any \(q \geq 1\),

\[
\lim_{i \to +\infty} \int_{B_g(x_0, r_\varepsilon)} |u_i|^q dv_g = |\Omega|_g
\]
and
\[
\lim_{i \to +\infty} \int_{B_g(x_0, r_i)} |\nabla u_i|_g \, dv_g = |\partial \Omega|_g.
\]

Before starting the proof of the above Sobolev inequality, we must set up some notations. For any \(1 < p < n\), we let
\[
K(n, p)^{-p} = \inf_{u \in C_c^\infty(\mathbb{R}^n), u \not\equiv 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^p_\xi \, dv \xi}{\int_{\mathbb{R}^n} |u|^p \, dv}.
\]
where \(p^* = \frac{np}{n-p}\) is the critical exponent for the Sobolev embeddings and \(\xi\) is the Euclidean metric. The value of \(K(n, p)\) is explicitly known (see [1] or [15]) but the only point of interest to us is that
\[
\lim_{p \to 1} K(n, p) = K(n, 1) = \frac{1}{n} \left( \frac{n}{\omega_{n-1}} \right)^{1\over n}.
\]

We also let, for \(1 < p < n\), \(H^p_1(\mathbb{R}^n)\) be the standard Sobolev space of order \(p\), that is the completion of \(C_c^\infty(\mathbb{R}^n)\) for the norm
\[
\|u\|_{H^p_1} = \left( \int_{\mathbb{R}^n} |\nabla u|^p_\xi \, dv \xi \right)^{1\over p}.
\]

At last, we let \(BV(\mathbb{R}^n)\) be the space of functions with bounded variations in \(\mathbb{R}^n\), defined as the completion of \(C_c^\infty(\mathbb{R}^n)\) with respect to the norm
\[
\|u\|_{BV} = \sup \left\{ -\int_{\mathbb{R}^n} u \, div(X) \, dv \xi, \|X\|_{L^\infty(\mathbb{R}^n)} \leq 1, div(X) \in L^n(\mathbb{R}^n) \right\}
\]
where \(div(X) = \partial X_i\). Basic facts about \(BV(\mathbb{R}^n)\) can be found in [7] or [16].

As already mentioned, Theorem 1 reduces to the following proposition:

**Proposition.** Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(x_0 \in M\). For any \(\varepsilon > 0\), there exists \(r_\varepsilon > 0\) such that for any \(u \in C_c^\infty(B_g(x_0, r_\varepsilon))\),
\[
\|u\|_{L^2}^2 \leq K(n, 1)^2 \left( \|\nabla u\|_1^2 + \alpha_\varepsilon \|u\|_1^2 \right)
\]
where \(\alpha_\varepsilon = \frac{n}{n-2} S_g(x_0) + \varepsilon\).

We prove the Proposition in what follows.

**Proof of the Proposition.** Clearly, we may assume, without loss of generality, that \(M = \mathbb{R}^n\) and that \(x_0 = 0\). We let, for any \(r > 0\), any \(p > 1\) and every \(\varepsilon > 0\),
\[
\lambda_{p, r} = \inf_{u \in C_c^\infty(B_g(0, r)), u \not\equiv 0} \frac{\left( \int_{B_g(0, r)} |\nabla u|^p_\xi \, dv \xi \right)^{\frac{1}{p}} + \alpha_\varepsilon \left( \int_{B_g(0, r)} |u|^p \, dv \right)^{\frac{1}{p}}}{\left( \int_{B_g(0, r)} |u|^{p^*} \, dv \right)^{\frac{1}{p^*}}}.
\]
We proceed by contradiction. We assume that there exists \(\varepsilon_0 > 0\) such that for any \(r > 0\),
\[
\lambda_{1, r} < K(n, 1)^{-2}.
\]
Then, since \( \limsup_{p \to 1} \lambda_{p,r} \leq \lambda_{1,r} \), one easily gets that for any \( r > 0 \), there exists \( p_r > 1 \) such that

\[
\lambda_{p_r,r} < K(n,1)^{-2} \left( \frac{n-p_r}{p_r(n-1)} \right)^2, \quad \lambda_{p,r} < K(n,p_r)^{-2}.
\]

We may assume that \( r \to 0 \) and we may choose \( p_r \) decreasing when \( r \) is decreasing. Then we get a sequence \( p > 1 \) going to 1 and a sequence \( r > 0 \) going to 0 as \( p \) goes to 1 which verify (2.2). It is by now classical that the second inequality in (2.2) ensures the existence of a minimizer \( u_p \) which satisfies the following:

\[
C_p \Delta_p u_p + \alpha \| u_p \|^{2-p} u_p^{p-1} = \lambda_p u_p^{p-1} \quad \text{in } B_g (0,r_p),
\]

\[ u_p \in C^{1,\eta} (B_g (0,r_p)) \text{ for some } \eta > 0, \]

\[ u_p > 0 \quad \text{in } B_g (0,r_p), \quad u_p = 0 \quad \text{on } \partial B_g (0,r_p), \]

\[
\int_{B_g (0,r_p)} u_p^{p^*} \, dv_g = 1,
\]

\[
\lambda_p < K(n,p)^{-2}, \quad \lambda_p < K(n,1)^{-2} \left( \frac{n-p}{p(n-1)} \right)^2,
\]

\[
C_p = \left( \int_{B_g (0,r_p)} | \nabla u_p |^p \, dv_g \right)^{\frac{p}{p-1}}.
\]

In the above equations, \( \Delta_p \) is the \( p \)-laplacian with respect to \( g \), that is \( \Delta_p u = -\text{div}_g (| \nabla u |^{p-2} \nabla u) \), and we have set

\[
\alpha = \frac{n}{n+2} S_g (0) + \varepsilon_0.
\]

Now the aim is to study this sequence \( (u_p) \) as \( p \to 1 \). We let \( x_p \) be a point in \( B_g (0,r_p) \) where \( u_p \) achieves its maximum and we also let

\[
\mu_p = u_p(x_p) = \frac{1}{p^{1-p}}.
\]

We have

\[
1 = \int_{B_g (0,r_p)} u_p^{p^*} \, dv_g \leq Vol g (B_g (0,r_p)) \mu_p^{-n}
\]

and since \( r_p \) goes to 0, \( \mu_p \) goes to 0 as \( p \) goes to 1. In the same way, thanks to Hölder’s inequalities, we get

\[
\lim_{p \to 1} \int_{B_g (0,r_p)} u_p^p \, dv_g = 0.
\]

**Step 1.** We first claim that

\[
\lim_{p \to 1} \lambda_p = K(n,1)^{-2}
\]

and that

\[
\lim_{p \to 1} \int_{B_g (0,r_p)} | \nabla u_p |^p \, dv_g = K(n,1)^{-1}.
\]
Indeed (see for instance [8] for an exposition in book form) for any \( \varepsilon > 0 \) there exists \( B_{\varepsilon} > 0 \) such that for any \( p > 1 \),
\[
\left( \int_{B_{\varepsilon}(0,r_p)} u_p^p \, dv_g \right)^{\frac{2-n}{n}} \leq (K(n,1) + \varepsilon)^2 \left( \int_{B_{\varepsilon}(0,r_p)} |\nabla \left( u_p^{(n-1)} \right) \right) dv_g \right)^2 \\
+ B_{\varepsilon} \left( \int_{B_{\varepsilon}(0,r_p)} u_p^{(n-1)} \, dv_g \right)^2
\]
which gives with (2.3), (2.4) and Hölder’s inequalities
\[
1 \leq (K(n,1) + \varepsilon)^2 \left( \frac{p(n-1)}{n-p} \right)^2 (\lambda_p - \alpha \|u_p\|_p^2) + B_{\varepsilon}\|u_p\|_p^2.
\]
This leads with (2.7) to
\[
1 \leq (1 + \varepsilon K(n,1)^{-1})^2 \liminf_{p \to 1} (\lambda_p K(n,1)^2).
\]
Since it is valid for any \( \varepsilon > 0 \), we obtain \( \liminf_{p \to 1} \lambda_p \geq K(n,1)^{-2} \). By (2.5), we get that (2.8) is proved. Then (2.9) is an obvious consequence of (2.3), (2.4), (2.7) and (2.8).

Step 2. We let \( \Omega_p = \mu_p^{-1} \exp_{x_p}^{-1} (B_g(0,r_p)) \subset \mathbb{R}^n \) and we set
\[
g_p(x) = \exp_{x_p}^* g(\mu_p x) \quad \text{for} \quad x \in \Omega_p
\]
and
\[
v_p(x) = \mu_p^{-1} u_p \left( \exp_{x_p}(\mu_p x) \right) \quad \text{for} \quad x \in \Omega_p, \quad v_p(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \Omega_p.
\]
Clearly we have
\[
C_p \Delta_{p,g_p} v_p + \alpha \mu_p^2 \|v_p\|_p^{2-p} v_p^{p-1} = \lambda_p v_p^{p-1} \quad \text{in} \quad \Omega_p
\]
with \( v_p = 0 \) on \( \partial \Omega_p \) and
\[
\int_{\Omega_p} v_p^{p-1} \, dv_g = 1.
\]
We also let
\[
\tilde{v}_p(x) = v_p(x)^{\frac{p(n-1)}{n-p}}.
\]
By the Cartan expansion of a metric in the exponential chart, there exists \( C > 1 \) such that
\[
dv_{g_p} \geq \left( 1 - \frac{1}{C^2} \right) dv_\xi,
\]
\[
|\nabla \tilde{v}_p|_{g_p} dv_{g_p} \leq \left( 1 + C \mu_p^2 \right) |\nabla \tilde{v}_p|_\xi dv_\xi
\]
where \( \xi \) is the Euclidean metric. This easily leads with (2.9), (2.11) and Hölder’s inequalities to
\[
\lim_{p \to 1} \frac{\int_{\mathbb{R}^n} |\nabla \tilde{v}_p|_\xi^p \, dv_\xi}{\left( \int_{\mathbb{R}^n} v_p^{p-1} \, dv_\xi \right)^\frac{n}{n-1}} = K(n,1)^{-1}.
\]
Remember here that $r_p \to 0$ as $p \to 1$. Since $(\tilde{v}_p)$ is bounded in $H^1_1(\mathbb{R}^n)$, there exists $v_0 \in BV(\mathbb{R}^n)$ such that
\[ \lim_{p \to 1} \tilde{v}_p = v_0 \text{ weakly in } BV(\mathbb{R}^n). \]

If we apply the concentration-compactness principle of P.L. Lions ([11], [12], see also [14] for an exposition in book form) to $|v_p|^p \, dv_\xi$, four situations may occur: compactness, concentration, dichotomy or vanishing. Dichotomy is classically forbidden by (2.13). Concentration cannot happen since $\sup_{\Omega_p} v_p = v_p(0) = 1$. As for vanishing, since $v_p$ is bounded in $L^\infty$, by applying Moser’s iterative scheme to (2.10), one gets the existence of some $C > 0$ such that for any $p > 1$,
\[ 1 = \sup_{\Omega_p \cap B_{g_p}(0,1/2)} v_p \leq C \left( \int_{\Omega_p \cap B_{g_p}(0,1)} v_p^{q^*} \, dv_\xi \right)^{\frac{1}{q^*}}. \]
Thus vanishing cannot happen. Compactness together with (2.13) just gives
\[ \lim_{p \to 1} \tilde{v}_p = v_0 \text{ strongly in } BV(\mathbb{R}^n). \]

Then $v_0$ is a minimizer for the $H^1_1$ Euclidean Sobolev inequality which verifies \( \int_{\mathbb{R}^n} v_0^\omega \, dv_\xi = 1 \). Thus there exists $y_0 \in \mathbb{R}^n$, $\lambda_0 > 0$ and $R_0 > 0$ such that
\[ v_0 = \lambda_0 1_{B(y_0,R_0)} \]
where $1_{B(y_0,R_0)}$ denotes the characteristic function of the Euclidean ball $B(y_0,R_0)$. Moreover, since $v_p \leq 1$ in $\Omega_p$, we obtain with (2.14) that $v_p \to v_0$ in any $L^q(\mathbb{R}^n)$, $q \geq \frac{n}{n-1}$. One can deduce from this that $\lambda_0 = 1$. At last, we have:
\[ Vol_\xi(B(y_0,R_0)) = \frac{\omega_{n-1}}{n} R_0^n = 1. \]
Up to changing $x_p$ into $exp_{x_p}(\mu_p y_0)$ in the definition of $v_p$, $\Omega_p$ and $g_p$, we may assume that $y_0 = 0$. We have thus obtained that
\[ \lim_{p \to 1} \tilde{v}_p = 1_{B(0,R_0)} \text{ strongly in } BV(\mathbb{R}^n). \]

This means in particular that
\[ \lim_{p \to 1} \tilde{v}_p = 1_{B(0,R_0)} \text{ strongly in } L^{\frac{n}{n-1}}(\mathbb{R}^n) \]
and that for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,
\[ \lim_{p \to 1} \int_{\mathbb{R}^n} |\nabla \tilde{v}_p| \xi \varphi \, dv_\xi = \int_{\partial B(0,R_0)} \varphi \, d\sigma_\xi. \]
If we set
\[ V_p(x) = \left( 1 + \left( \frac{|x|}{R_0} \right) \right)^{1-n}, \quad x \in \mathbb{R}^n, \]
a simple application of the concentration-compactness principle, using what we just proved, gives
\[ \lim_{p \to 1} \int_{\mathbb{R}^n} |\nabla (\tilde{v}_p - V_p) \xi \, dv_\xi = 0. \]
Applying Moser’s iterative scheme to (2.10) with the help of (2.17), we also get that for any \( R > R_0 \),

\[
\lim_{p \to 1} \sup_{B(0,R) \setminus \Omega_p} v_p = 0.
\]

Step 3. The aim is to transform the \( L^\infty \)-estimate (2.17) into a pointwise estimate. We follow here [6] (see also [5]). We let

\[ w_p(z) = |z|^{\frac{n}{p} - 1} v_p(z) \]

and we let \( z_p \in \Omega_p \) be a point where \( w_p \) achieves its maximum. Let us assume by contradiction that

\[ \lim_{p \to 1} w_p(z_p) = +\infty. \]

We set

\[ v_p^{1 - \frac{n}{p}} = v_p(z_p) \]

so that

\[
\lim_{p \to 1} |z_p| = +\infty.
\]

Independently, since \( v_p \leq 1 \) in \( \Omega_p \),

\[ \lim_{p \to 1} |z_p| = +\infty. \]

Thanks to (2.22) and (2.23), one proves then that \( (v_p^{\frac{n}{p} - 1} v_p(exp_{z_p}(v_p x))) \) is bounded in \( L^\infty (B(0,1)) \). This allows us to apply Moser’s iterative scheme to the equation verified by \( (v_p^{\frac{n}{p} - 1} v_p (exp_{z_p}(v_p x))) \) and to get the existence of some \( C > 0 \) such that

\[
\liminf_{p \to 1} \int_{B_{g_p}(z_p,v_p) \cap \Omega_p} v_p^{p^*} \, dv_g > 0.
\]

The contradiction then easily follows from (2.17), (2.22) and (2.23). Thus we have the existence of some \( C > 0 \) such that for any \( p > 1 \), any \( z \in \Omega_p \),

\[
|z|^{\frac{n}{p} - 1} v_p(z) \leq C.
\]

In the same way, using (2.24), one proves thanks to (2.21) that for any \( R > R_0 \),

\[
\lim_{p \to 1} \sup_{B(0,R) \setminus \Omega_p} |z|^{\frac{n}{p} - 1} v_p(z) = 0.
\]

We refer the reader to [6] for details on such claims.

Step 4. We let \( L_p \) be the following operator:

\[
L_p u = C_p \Delta_{p,g_p} u + \alpha \mu^2_p \|v_p\|_p^{2 - p} u^{p - 1} - \lambda_p v_p^{p^* - p} u^{p - 1}.
\]

We fix \( 0 < \nu < n - 1 \) and we set

\[
G_p(x) = \theta_p |x|^{-\frac{n - \nu}{p - 1}}.
\]
where $\theta_p$ is some positive constant to be fixed later. Easy computations lead to

$$|x|^{n-\nu} \frac{L_p G_p(x)}{G_p(x)^{p-1}} \geq C_p \mu^\nu \left( \frac{n - p - \nu}{p - 1} \right)^{p-1} - C \mu^2 |x|^2 + \alpha \mu^2 v_p \|v_p\|^{2-p} |x|^p - \lambda_p |x|^p v_p^{p-p}$$

in $\Omega_p \setminus \{0\}$. Here $C$ denotes some constant independent of $p$. Thanks to (2.7), (2.8), (2.9), (2.25) and the fact that $p_p \to 0$ as $p \to 1$, one gets that for any $R > R_0$,

$$L_p G_p(x) \geq 0 \quad \text{in } \Omega_p \setminus B_{g_p}(0, R)$$

for $p$ small enough. On the other hand,

$$L_p v_p = 0 \quad \text{in } \Omega_p.$$

At last, it is not difficult to check with (2.21) that

$$v_p \leq \theta_p G_p \quad \text{on } \partial B_{g_p}(0, R)$$

if we take $\theta_p = R^{-\frac{p-n}{p-1}}$. Now we may apply the maximum principle as stated for instance in [2] (lemma 3.4) to get, for $p$ small enough,

$$v_p(y) \leq \left( \frac{R}{|y|} \right)^{\frac{n-p-\nu}{p-1}} v_p(0) \quad \text{in } \Omega_p \setminus B_{g_p}(0, R).$$

Since this inequality obviously holds on $B_{g_p}(0, R)$, we have finally obtained the following: for any $\nu > 0$ and any $R > R_0$, there exists $C(R, \nu) > 0$ such that for any $p > 1$ and any $y \in \Omega_p$,

$$v_p(y) \leq C(R, \nu). \quad (2.26)$$

Step 5. We conclude the proof of the Proposition. We apply the $H^1$ Euclidean Sobolev inequality to $\tilde{v}_p$:

$$\left( \int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-p}} \, dv_{\xi} \right)^{\frac{n-p}{n}} \leq K(n, 1) \int_{\Omega_p} |\nabla \tilde{v}_p|_{\xi} \, dv_{\xi}. \quad (2.27)$$

By the Cartan expansion of $g_p$ around $0$, we have

$$dv_{\xi} = \left( 1 + \frac{\mu^2}{6} Ric_g(y_p)_{ij} x^i x^j + o \left( \mu^2 |x|^2 \right) \right) dv_{g_p} \quad (2.28)$$

where $Ric_g$ denotes the Ricci curvature of $g$ in the $exp_{y_p}$-map. Thus, by (2.11),

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-p}} \, dv_{\xi} = 1 + \frac{\mu^2}{6} Ric_g(y_p)_{ij} \int_{\Omega_p} x^i x^j v_p^{p *} dv_{g_p} + o \left( \mu^2 \int_{\Omega_p} |x|^2 v_p^{p *} dv_{g_p} \right).$$

Using (2.17) and (2.26), one gets

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-p}} \, dv_{\xi} = 1 + \frac{S_g(0)}{6n(n+2)} \omega_{n-1} R_0^{n-1} \mu^2 + o \left( \mu^2 \right). \quad (2.29)$$

By the Cartan expansion of $g_p$ around $0$, since $r_p \to 0$ as $p \to 1$, we also have

$$|\nabla \tilde{v}_p|_{y_p}^{g_p} = |\nabla \tilde{v}_p|_{y_p}^{g_p} \left[ 1 - \frac{\mu^2}{6} \nabla \tilde{v}_p |x|^{-2} Rm_g(y_p) (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) + o \left( \mu^2 |x|^2 \right) \right]$$
where $Rm_g$ denotes the Riemann curvature of $g$ in the $exp_{y_p}$-map. Then, using (2.28), we get

$$
\int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} = \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} + \frac{\mu_p^2}{6} Ric_g(y_p)_{ij} \int_{\Omega_p} x^i x^j |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} \\
- \frac{\mu_p^2}{6} \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p}^{-1} Rm_g(y_p)(\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) \, dv_{g_p} + \mu_p \int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p}.
$$

(2.30)

Let us now look at the different terms of (2.30). First, by equation (2.10) and relation (2.5), we have

$$
\int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} = \frac{p(n-1)}{n-p} \int_{\Omega_p} \frac{(p-1)}{p} \nabla \nabla v_{g_p} \, dv_{g_p} \\
\leq \frac{p(n-1)}{n-p} \left( \int_{\Omega_p} |\nabla v_{g_p}|^{p} \, dv_{g_p} \right)^{\frac{1}{p}} \\
\leq K(n,1)^{-1} \left( 1 - \alpha \mu_p^2 \lambda_p^{-1} \|v_p\|_{p}^{2} \right)^{\frac{1}{p}}.
$$

(2.31)

Since, by (2.17) and (2.26), $\|v_p\|_{p} = 1 + o(1)$, we get

Independently, by Hölder’s inequalities, we have

$$
\int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} \leq \frac{p(n-1)}{n-p} \left( \int_{\Omega_p} |x|^{2p} |\nabla v_{g_p}|^{p} \, dv_{g_p} \right)^{\frac{1}{p}}.
$$

By equation (2.10), one gets

$$
\int_{\Omega_p} |x|^{2p} |\nabla v_{g_p}|^{p} \, dv_{g_p} \leq \int_{\Omega_p} |\nabla v_{g_p}|^{p-2} \left( |x|^{2p} v_p \right) \, dv_{g_p} \\
+ C \int_{\Omega_p} |x|^{2p-1} |\nabla v_{g_p}|^{p-1} v_p \, dv_{g_p} \\
\leq C + C \left( \int_{\Omega_p} |x|^{2p} |\nabla v_{g_p}|^{p} \, dv_{g_p} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_p} |x|^{p} v_p^p \, dv_{g_p} \right)^{\frac{1}{p}}
$$

where $C$ denotes some constant independent of $p$. Using (2.26) and Young’s inequalities, one deduces that

$$
\int_{\Omega_p} |x|^{2p} |\nabla v_{g_p}|^{p} \, dv_{g_p} = O(1).
$$

(2.32)

Now, for some $R > R_0$, we get by (2.18) that

$$
\int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} x^i x^j \, dv_{g_p} = O \left( \int_{\Omega_p \setminus B(0,R)} |x|^2 |\nabla \tilde{v}_p|_{g_p} \, dv_{g_p} \right) + \int_{\partial B(0,R_0)} x^i x^j d\sigma_{g_p} + o(1).
$$
Using equation (2.10) and relation (2.26), it is easy to check that
\[ \lim_{p \to 1} \int_{\Omega_p \setminus B(0,R)} |x|^2 |\nabla \tilde{v}_p| d\xi = 0 \]
so that
\[ \lim_{p \to 1} \int_{\Omega_p} |\nabla \tilde{v}_p|_{\xi} x^i x^j d\xi = \frac{\omega_{n-1}}{n} R_0^{n+1} S_g(0). \]
At last, since \( \nabla V_p \), \( V_p \) as in (2.19), and \( x \) are pointwise colinear vector fields, we have
\[ \text{Ric}_g (y_p) (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) \leq C|x|^2 |\nabla \tilde{v}_p|_{\xi} |\nabla (\tilde{v}_p - V_p)|_{\xi} \]
so that, by (2.10), (2.20) and (2.26),
\[ \lim_{p \to 1} \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p}^{-1} \text{Ric}_g (y_p) (\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) d\nu_{g_p} = 0. \]
Coming back to (2.27) with (2.29)-(2.34), we obtain, after easy computations using in particular (2.16),
\[ \left( \alpha - \frac{n}{n+2} S_g(0) \right) \mu_p^2 + o(\mu_p^2) \leq 0. \]
This gives the desired contradiction by letting \( p \) go to 0. Remember here that \( \alpha - \frac{n}{n+2} S_g(0) = \varepsilon_0 > 0 \). This ends the proof of the Proposition, hence the proof of Theorem 1.

3. The compact case - Proof of Theorem 2

In order to prove Theorem 2, we let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 2 \). We assume that \( S_g < n(n-1)K_0 \). If we apply Theorem 1 with some \( x \) in \( M \) and \( K_0 \), we get some \( r_x > 0 \) such that the isoperimetric comparison (with the model space form of curvature \( K_0 \)) holds for sets contained in the geodesic ball of center \( x \) and radius \( r_x \). It is clear that \( r_x \) is continuous with respect to \( x \). Thus, there exists \( d > 0 \) such that for any subset \( \Omega \) of \( M \) of diameter less than or equal to \( d \),
\[ |\partial \Omega|_g > |\partial B|_{g_0} \]
where \( B \) is a ball of volume \( |\Omega|_g \) in the model space of constant curvature \( K_0 \). For \( 0 < V < |M|_g \), we let
\[ h(V) = \inf \{|\partial \Omega|_g, \ \Omega \subset M, \ |\Omega|_g = V\}. \]
There exists some \( \Omega_V \subset M \) such that
\[ |\partial \Omega_V|_g = h(V). \]
The boundary \( \partial \Omega_V \) of \( \Omega_V \) is a smooth hypersurface of constant mean curvature up to a compact set of Hausdorff dimension at most \( n-8 \) (see for instance [13]). Now, as a consequence of the work of Johnson and Morgan [10], we know that
\[ \text{diam} (\Omega_V) \to 0 \]
as \( V \to 0 \). In fact, Johnson and Morgan proved that \( \Omega_V \) is asymptotically, as \( V \to 0 \), a ball. In particular, for some \( V_0 \) small enough, any \( \Omega_V \) for \( V \leq V_0 \) has a diameter less than or equal to \( d \). We may then apply (3.1) to end the proof of Theorem 2.
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References


DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ DE CERGY-PONTOISE, SITE DE SAINT-MARTIN,
2 AVENUE ADOLPHE CHAUVIN, 95302 CERGY-PONTOISE CEDEX, FRANCE
E-mail address: Olivier.Druet@math.u-cregy.fr