

## SHARP LOCAL ISOPERIMETRIC INEQUALITIES INVOLVING THE SCALAR CURVATURE

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ABSTRACT. We provide sharp local isoperimetric inequalities on Riemannian manifolds involving the scalar curvature, and thus answer a question asked by Johnson and Morgan.

### 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let  $(M, g)$  be a complete Riemannian manifold of dimension  $n \geq 2$  with sectional curvature  $K_g \leq K_0$ . A long-standing conjecture, a formulation of which can be found in [1], asserts that for any  $x \in M$ , there exists  $r_x > 0$  such that for any  $\Omega$  contained in the geodesic ball of center  $x$  and radius  $r_x$ ,

$$|\partial\Omega|_g \geq |\partial B|_{g_0}$$

where  $|\cdot|_g$  (resp.  $|\cdot|_{g_0}$ ) denotes the volume with respect to  $g$  (resp.  $g_0$ ) and  $B$  is a ball of volume  $|\Omega|_g$  in the model space  $(M_0, g_0)$  of constant sectional curvature  $K_0$ . A compact version of this conjecture was proved, with an additional assumption on the Gauss-Bonnet-Chern integrand in even dimensions, in the very nice Johnson and Morgan [10]. A natural question that Johnson and Morgan [10] asked is the following: is the result still true if we assume that the scalar curvature of  $(M, g)$  is such that  $S_g < n(n-1)K_0$  instead of assuming that  $K_g \leq K_0$ ? We answer this question in the affirmative and prove the following:

**Theorem 1.** *Let  $(M, g)$  be a complete Riemannian manifold of dimension  $n \geq 2$  and let  $x \in M$ . Assume that  $S_g(x) < n(n-1)K_0$  for some  $K_0 \in \mathbf{R}$ . Then there exists  $r_x > 0$  such that for any  $\Omega$  contained in the geodesic ball of center  $x$  and radius  $r_x$ ,*

$$|\partial\Omega|_g > |\partial B|_{g_0}$$

where  $B$  is a ball of volume  $|\Omega|_g$  in the model space  $(M_0, g_0)$  of constant sectional curvature  $K_0$ .

In the compact setting, the situation that was actually considered by Johnson and Morgan [10], we have the following:

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**Theorem 2.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 2$  with scalar curvature  $S_g < n(n - 1)K_0$ . There exists  $V > 0$  such that for any subset  $\Omega$  of  $M$  of volume less than or equal to  $V$ ,*

$$|\partial\Omega|_g > |\partial B|_{g_0}$$

where  $B$  is a ball of volume  $|\Omega|_g$  in the model space  $(M_0, g_0)$  of constant sectional curvature  $K_0$ .

These results are optimal in the following sense: if we only assume that the Ricci curvature of  $M$  verifies  $Ric_g \leq (n - 1)K_0$ , the above isoperimetric comparison fails. Indeed, for any  $n$ -manifold  $M$  which is Ricci-flat but not flat (see [3] for examples of such manifolds), one may find a ball  $B_r$  in  $M$  of radius  $r$  as small as we want which verifies

$$|\partial B_r|_g < |\partial B|_\xi$$

where  $B$  is a ball of volume  $|B_r|_g$  in the Euclidean space  $(\mathbf{R}^n, \xi)$ . The above comparison result is also false on  $S^2 \times S^2$ , as noticed in [10]. The proof of Theorem 1 is based on the study of local optimal Sobolev inequalities. The proof relies on PDE techniques and a fine asymptotic analysis of solutions of quasi-elliptic equations involving the  $p$ -Laplacian. Theorem 2 is a consequence of Theorem 1 thanks to geometric measure theory. The relevance of the scalar curvature when studying the validity of sharp Sobolev inequalities was noticed first by the author in [4] and underlined by Hebey in [9].

## 2. SOBOLEV INEQUALITIES AND PROOF OF THEOREM 1

Let  $B$  be a ball in the model space  $(M_0, g_0)$  of constant sectional curvature  $K_0$ . It is not difficult to check that, for balls of small volume,

$$|\partial B|_{g_0}^2 = K(n, 1)^{-2}|B|_{g_0}^{2\frac{n-1}{n}} - \frac{n}{n+2}(n(n-1)K_0)|B|_{g_0}^2 + o(|B|_{g_0}^2).$$

Here,  $n = \dim M_0$  and

$$K(n, 1)^{-1} = n \left( \frac{\omega_{n-1}}{n} \right)^{\frac{1}{n}}.$$

Now, let  $(M, g)$  be a complete Riemannian manifold of dimension  $n \geq 2$  and let  $x_0 \in M$ . In order to prove Theorem 1, it is clearly sufficient to prove that for any  $\varepsilon > 0$ , there exists  $r_\varepsilon > 0$  such that for any  $\Omega \subset B_g(x_0, r_\varepsilon)$ ,

$$(2.1) \quad |\partial\Omega|_g^2 \geq K(n, 1)^{-2}|\Omega|_g^{2\frac{n-1}{n}} - \left( \frac{n}{n+2}S_g(x_0) + \varepsilon \right) |\Omega|_g^2.$$

It is now well known that (2.1) is a consequence of the following Sobolev inequality: for any  $u \in C_c^\infty(B_g(x_0, r_\varepsilon))$ ,

$$\|u\|_{\frac{n}{n-1}}^2 \leq K(n, 1)^2 (\|\nabla u\|_1^2) + \left( \frac{n}{n+2}S_g(x_0) + \varepsilon \right) \|u\|_1^2$$

where  $\|\cdot\|_p$  denotes the  $L^p$ -norm with respect to the Riemannian volume element  $dv_g$ . Indeed,  $\Omega \subset B_g(x_0, r_\varepsilon)$  being given, one may find a sequence  $(u_i)$  of smooth functions with compact support in  $B_g(x_0, r_\varepsilon)$  such that for any  $q \geq 1$ ,

$$\lim_{i \rightarrow +\infty} \int_{B_g(x_0, r_\varepsilon)} |u_i|^q dv_g = |\Omega|_g$$

and

$$\lim_{i \rightarrow +\infty} \int_{B_g(x_0, r_\varepsilon)} |\nabla u_i|_g \, dv_g = |\partial\Omega|_g.$$

Before starting the proof of the above Sobolev inequality, we must set up some notations. For any  $1 \leq p < n$ , we let

$$K(n, p)^{-p} = \inf_{u \in C_c^\infty(\mathbf{R}^n), u \neq 0} \frac{\int_{\mathbf{R}^n} |\nabla u|_\xi^p \, dv_\xi}{\left(\int_{\mathbf{R}^n} |u|^{p^*} \, dv_\xi\right)^{\frac{p}{p^*}}}$$

where  $p^* = \frac{np}{n-p}$  is the critical exponent for the Sobolev embeddings and  $\xi$  is the Euclidean metric. The value of  $K(n, p)$  is explicitly known (see [1] or [15]) but the only point of interest to us is that

$$\lim_{p \rightarrow 1} K(n, p) = K(n, 1) = \frac{1}{n} \left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}.$$

We also let, for  $1 \leq p < n$ ,  $H_1^p(\mathbf{R}^n)$  be the standard Sobolev space of order  $p$ , that is the completion of  $C_c^\infty(\mathbf{R}^n)$  for the norm

$$\|u\|_{H_1^p} = \left(\int_{\mathbf{R}^n} |\nabla u|_\xi^p \, dv_\xi\right)^{\frac{1}{p}}.$$

At last, we let  $BV(\mathbf{R}^n)$  be the space of functions with bounded variations in  $\mathbf{R}^n$ , defined as the completion of  $C_c^\infty(\mathbf{R}^n)$  with respect to the norm

$$\|u\|_{BV} = \sup \left\{ - \int_{\mathbf{R}^n} u \operatorname{div}(X) \, dv_\xi, \|X\|_{L^\infty(\mathbf{R}^n)} \leq 1, \operatorname{div}(X) \in L^n(\mathbf{R}^n) \right\}$$

where  $\operatorname{div}(X) = \partial^i X_i$ . Basic facts about  $BV(\mathbf{R}^n)$  can be found in [7] or [16].

As already mentioned, Theorem 1 reduces to the following proposition:

**Proposition.** *Let  $(M, g)$  be a complete Riemannian manifold of dimension  $n \geq 2$  and let  $x_0 \in M$ . For any  $\varepsilon > 0$ , there exists  $r_\varepsilon > 0$  such that for any  $u$  in  $C_c^\infty(B_g(x_0, r_\varepsilon))$ ,*

$$\|u\|_{\frac{n}{n-1}}^2 \leq K(n, 1)^2 (\|\nabla u\|_1^2 + \alpha_\varepsilon \|u\|_1^2)$$

where  $\alpha_\varepsilon = \frac{n}{n+2} S_g(x_0) + \varepsilon$ .

We prove the Proposition in what follows.

*Proof of the Proposition.* Clearly, we may assume, without loss of generality, that  $M = \mathbf{R}^n$  and that  $x_0 = 0$ . We let, for any  $r > 0$ , any  $p > 1$  and any  $\varepsilon > 0$ ,

$$\lambda_{p,r} = \inf_{u \in C_c^\infty(B_g(0,r)), u \neq 0} \frac{\left(\int_{B_g(0,r)} |\nabla u|_g^p \, dv_g\right)^{\frac{2}{p}} + \alpha_\varepsilon \left(\int_{B_g(0,r)} |u|^p \, dv_g\right)^{\frac{2}{p}}}{\left(\int_{B_g(0,r)} |u|^{p^*} \, dv_g\right)^{\frac{2}{p^*}}}.$$

We proceed by contradiction. We assume that there exists  $\varepsilon_0 > 0$  such that for any  $r > 0$ ,

$$\lambda_{1,r} < K(n, 1)^{-2}.$$

Then, since  $\limsup_{p \rightarrow 1} \lambda_{p,r} \leq \lambda_{1,r}$ , one easily gets that for any  $r > 0$ , there exists  $p_r > 1$  such that

$$(2.2) \quad \lambda_{p_r,r} < K(n, 1)^{-2} \left( \frac{n - p_r}{p_r(n - 1)} \right)^2, \lambda_{p_r,r} < K(n, p_r)^{-2}.$$

We may assume that  $r \rightarrow 0$  and we may choose  $p_r$  decreasing when  $r$  is decreasing. Then we get a sequence  $p > 1$  going to 1 and a sequence  $r_p > 0$  going to 0 as  $p$  goes to 1 which verify (2.2). It is by now classical that the second inequality in (2.2) ensures the existence of a minimizer  $u_p$  which satisfies the following:

$$(2.3) \quad \begin{aligned} C_p \Delta_p u_p + \alpha \|u_p\|_p^{2-p} u_p^{p-1} &= \lambda_p u_p^{p^*-1} \quad \text{in } B_g(0, r_p), \\ u_p &\in C^{1,\eta}(B_g(0, r_p)) \quad \text{for some } \eta > 0, \\ u_p &> 0 \quad \text{in } B_g(0, r_p), \quad u_p = 0 \quad \text{on } \partial B_g(0, r_p), \end{aligned}$$

$$(2.4) \quad \int_{B_g(0, r_p)} u_p^{p^*} dv_g = 1,$$

$$(2.5) \quad \lambda_p < K(n, p)^{-2}, \quad \lambda_p < K(n, 1)^{-2} \left( \frac{n - p}{p(n - 1)} \right)^2,$$

$$(2.6) \quad C_p = \left( \int_{B_g(0, r_p)} |\nabla u_p|_g^p dv_g \right)^{\frac{2-p}{p}}.$$

In the above equations,  $\Delta_p$  is the  $p$ -laplacian with respect to  $g$ , that is  $\Delta_p u = -div_g(|\nabla u|_g^{p-2} \nabla u)$ , and we have set

$$\alpha = \frac{n}{n + 2} S_g(0) + \varepsilon_0.$$

Now the aim is to study this sequence  $(u_p)$  as  $p \rightarrow 1$ . We let  $x_p$  be a point in  $B_g(0, r_p)$  where  $u_p$  achieves its maximum and we also let

$$u_p(x_p) = \mu_p^{1-\frac{n}{p}}.$$

We have

$$1 = \int_{B_g(0, r_p)} u_p^{p^*} dv_g \leq Vol_g(B_g(0, r_p)) \mu_p^{-n}$$

and since  $r_p$  goes to 0,  $\mu_p$  goes to 0 as  $p$  goes to 1. In the same way, thanks to Hölder's inequalities, we get

$$(2.7) \quad \lim_{p \rightarrow 1} \int_{B_g(0, r_p)} u_p^p dv_g = 0.$$

*Step 1.* We first claim that

$$(2.8) \quad \lim_{p \rightarrow 1} \lambda_p = K(n, 1)^{-2}$$

and that

$$(2.9) \quad \lim_{p \rightarrow 1} \int_{B_g(0, r_p)} |\nabla u_p|_g^p dv_g = K(n, 1)^{-1}.$$

Indeed (see for instance [8] for an exposition in book form) for any  $\varepsilon > 0$  there exists  $B_\varepsilon > 0$  such that for any  $p > 1$ ,

$$\left( \int_{B_g(0,r_p)} u_p^{p^*} dv_g \right)^{2\frac{n-1}{n}} \leq (K(n,1) + \varepsilon)^2 \left( \int_{B_g(0,r_p)} \left| \nabla \left( u_p^{\frac{p(n-1)}{n-p}} \right) \right|_g dv_g \right)^2 + B_\varepsilon \left( \int_{B_g(0,r_p)} u_p^{\frac{p(n-1)}{n-p}} dv_g \right)^2$$

which gives with (2.3), (2.4) and Hölder’s inequalities

$$1 \leq (K(n,1) + \varepsilon)^2 \left( \frac{p(n-1)}{n-p} \right)^2 (\lambda_p - \alpha \|u_p\|_p^2) + B_\varepsilon \|u_p\|_p^2.$$

This leads with (2.7) to

$$1 \leq (1 + \varepsilon K(n,1)^{-1})^2 \liminf_{p \rightarrow 1} (\lambda_p K(n,1)^2).$$

Since it is valid for any  $\varepsilon > 0$ , we obtain  $\liminf_{p \rightarrow 1} \lambda_p \geq K(n,1)^{-2}$ . By (2.5), we get that (2.8) is proved. Then (2.9) is an obvious consequence of (2.3), (2.4), (2.7) and (2.8).

Step 2. We let  $\Omega_p = \mu_p^{-1} \exp_{x_p}^{-1}(B_g(0, r_p)) \subset \mathbf{R}^n$  and we set

$$g_p(x) = \exp_{x_p}^* g(\mu_p x) \quad \text{for } x \in \Omega_p$$

and

$$v_p(x) = \mu_p^{\frac{n}{p}-1} u_p(\exp_{x_p}(\mu_p x)) \quad \text{for } x \in \Omega_p, \quad v_p(x) = 0 \quad \text{for } x \in \mathbf{R}^n \setminus \Omega_p.$$

Clearly we have

$$(2.10) \quad C_p \Delta_{p,g_p} v_p + \alpha \mu_p^2 \|v_p\|_p^{2-p} v_p^{p-1} = \lambda_p v_p^{p^*-1} \quad \text{in } \Omega_p$$

with  $v_p = 0$  on  $\partial\Omega_p$  and

$$(2.11) \quad \int_{\Omega_p} v_p^{p^*} dv_{g_p} = 1.$$

We also let

$$(2.12) \quad \tilde{v}_p(x) = v_p(x)^{\frac{p(n-1)}{n-p}}.$$

By the Cartan expansion of a metric in the exponential chart, there exists  $C > 1$  such that

$$\begin{aligned} dv_{g_p} &\geq \left(1 - \frac{1}{C} \mu_p^2\right) dv_\xi, \\ |\nabla \tilde{v}_p|_{g_p} dv_{g_p} &\leq (1 + C \mu_p^2) |\nabla \tilde{v}_p|_\xi dv_\xi \end{aligned}$$

where  $\xi$  is the Euclidean metric. This easily leads with (2.9), (2.11) and Hölder’s inequalities to

$$(2.13) \quad \lim_{p \rightarrow 1} \frac{\int_{\mathbf{R}^n} |\nabla \tilde{v}_p|_\xi^p dv_\xi}{\left(\int_{\mathbf{R}^n} v_p^{p^*} dv_\xi\right)^{\frac{n-1}{n}}} = K(n,1)^{-1}.$$

Remember here that  $r_p \rightarrow 0$  as  $p \rightarrow 1$ . Since  $(\tilde{v}_p)$  is bounded in  $H_1^1(\mathbf{R}^n)$ , there exists  $v_0 \in BV(\mathbf{R}^n)$  such that

$$\lim_{p \rightarrow 1} \tilde{v}_p = v_0 \quad \text{weakly in } BV(\mathbf{R}^n).$$

If we apply the concentration-compactness principle of P.L. Lions ([11], [12], see also [14] for an exposition in book form) to  $|v_p|^{p^*} dv_\xi$ , four situations may occur: compactness, concentration, dichotomy or vanishing. Dichotomy is classically forbidden by (2.13). Concentration cannot happen since  $\sup_{\Omega_p} v_p = v_p(0) = 1$ . As for vanishing, since  $v_p$  is bounded in  $L^\infty$ , by applying Moser's iterative scheme to (2.10), one gets the existence of some  $C > 0$  such that for any  $p > 1$ ,

$$1 = \sup_{\Omega_p \cap B_{g_p}(0,1/2)} v_p \leq C \left( \int_{\Omega_p \cap B_{g_p}(0,1)} v_p^{p^*} dv_\xi \right)^{\frac{1}{p^*}}.$$

Thus vanishing cannot happen. Compactness together with (2.13) just gives

$$(2.14) \quad \lim_{p \rightarrow 1} \tilde{v}_p = v_0 \quad \text{strongly in } BV(\mathbf{R}^n).$$

Then  $v_0$  is a minimizer for the  $H_1^1$  Euclidean Sobolev inequality which verifies  $\int_{\mathbf{R}^n} v_0^{\frac{n}{n-1}} dv_\xi = 1$ . Thus there exists  $y_0 \in \mathbf{R}^n$ ,  $\lambda_0 > 0$  and  $R_0 > 0$  such that

$$(2.15) \quad v_0 = \lambda_0 \mathbf{1}_{B(y_0, R_0)}$$

where  $\mathbf{1}_{B(y_0, R_0)}$  denotes the characteristic function of the Euclidean ball  $B(y_0, R_0)$ . Moreover, since  $v_p \leq 1$  in  $\Omega_p$ , we obtain with (2.14) that  $v_p \rightarrow v_0$  in any  $L^q(\mathbf{R}^n)$ ,  $q \geq \frac{n}{n-1}$ . One can deduce from this that  $\lambda_0 = 1$ . At last, we have:

$$(2.16) \quad Vol_\xi(B(y_0, R_0)) = \frac{\omega_{n-1}}{n} R_0^n = 1.$$

Up to changing  $x_p$  into  $exp_{x_p}(\mu_p y_0)$  in the definition of  $v_p$ ,  $\Omega_p$  and  $g_p$ , we may assume that  $y_0 = 0$ . We have thus obtained that

$$\lim_{p \rightarrow 1} \tilde{v}_p = \mathbf{1}_{B(0, R_0)} \quad \text{strongly in } BV(\mathbf{R}^n).$$

This means in particular that

$$(2.17) \quad \lim_{p \rightarrow 1} \tilde{v}_p = \mathbf{1}_{B(0, R_0)} \quad \text{strongly in } L^{\frac{n}{n-1}}(\mathbf{R}^n)$$

and that for any  $\varphi \in C_c^\infty(\mathbf{R}^n)$ ,

$$(2.18) \quad \lim_{p \rightarrow 1} \int_{\mathbf{R}^n} |\nabla \tilde{v}_p|_\xi \varphi dv_\xi = \int_{\partial B(0, R_0)} \varphi d\sigma_\xi.$$

If we set

$$(2.19) \quad V_p(x) = \left( 1 + \left( \frac{|x|}{R_0} \right)^{\frac{p}{p-1}} \right)^{1-n}, \quad x \in \mathbf{R}^n,$$

a simple application of the concentration-compactness principle, using what we just proved, gives

$$(2.20) \quad \lim_{p \rightarrow 1} \int_{\mathbf{R}^n} |\nabla(\tilde{v}_p - V_p)|_\xi dv_\xi = 0.$$

Applying Moser's iterative scheme to (2.10) with the help of (2.17), we also get that for any  $R > R_0$ ,

$$(2.21) \quad \lim_{p \rightarrow 1} \sup_{\Omega_p \setminus B(0,R)} v_p = 0.$$

*Step 3.* The aim is to transform the  $L^{\frac{n}{n-1}}$ -estimate (2.17) into a pointwise estimate. We follow here [6] (see also [5]). We let

$$w_p(z) = |z|^{\frac{n}{p}-1} v_p(z)$$

and we let  $z_p \in \Omega_p$  be a point where  $w_p$  achieves its maximum. Let us assume by contradiction that

$$\lim_{p \rightarrow 1} w_p(z_p) = +\infty.$$

We set

$$\nu_p^{1-\frac{n}{p}} = v_p(z_p)$$

so that

$$(2.22) \quad \lim_{p \rightarrow 1} \frac{|z_p|}{\nu_p} = +\infty.$$

Independently, since  $v_p \leq 1$  in  $\Omega_p$ ,

$$(2.23) \quad \lim_{p \rightarrow 1} |z_p| = +\infty.$$

Thanks to (2.22) and (2.23), one proves then that  $(\nu_p^{\frac{n}{p}-1} v_p(\exp_{z_p}(\nu_p x)))$  is bounded in  $L^\infty(B(0,1))$ . This allows us to apply Moser's iterative scheme to the equation verified by  $(\nu_p^{\frac{n}{p}-1} v_p(\exp_{z_p}(\nu_p x)))$  and to get the existence of some  $C > 0$  such that

$$\liminf_{p \rightarrow 1} \int_{B_{g_p}(z_p, \nu_p) \cap \Omega_p} v_p^{p^*} dv_g > 0.$$

The contradiction then easily follows from (2.17), (2.22) and (2.23). Thus we have the existence of some  $C > 0$  such that for any  $p > 1$ , any  $z \in \Omega_p$ ,

$$(2.24) \quad |z|^{\frac{n}{p}-1} v_p(z) \leq C.$$

In the same way, using (2.24), one proves thanks to (2.21) that for any  $R > R_0$ ,

$$(2.25) \quad \lim_{p \rightarrow 1} \sup_{\Omega_p \setminus B_{g_p}(0,R)} |z|^{\frac{n}{p}-1} v_p(z) = 0.$$

We refer the reader to [6] for details on such claims.

*Step 4.* We let  $L_p$  be the following operator:

$$L_p u = C_p \Delta_{p, g_p} u + \alpha \mu_p^2 \|v_p\|_p^{2-p} u^{p-1} - \lambda_p v_p^{p^*-p} u^{p-1}.$$

We fix  $0 < \nu < n - 1$  and we set

$$G_p(x) = \theta_p |x|^{-\frac{n-p-\nu}{p-1}}$$

where  $\theta_p$  is some positive constant to be fixed later. Easy computations lead to

$$|x|^{n-\nu} \frac{L_p G_p(x)}{G_p(x)^{p-1}} \geq C_p \nu \left( \frac{n-p-\nu}{p-1} \right)^{p-1} - C \mu_p^2 |x|^2 + \alpha \mu_p^2 \|v_p\|_p^{2-p} |x|^p - \lambda_p |x|^p v_p^{p^*-p}$$

in  $\Omega_p \setminus \{0\}$ . Here  $C$  denotes some constant independent of  $p$ . Thanks to (2.7), (2.8), (2.9), (2.25) and the fact that  $r_p \rightarrow 0$  as  $p \rightarrow 1$ , one gets that for any  $R > R_0$ ,

$$L_p G_p(x) \geq 0 \quad \text{in } \Omega_p \setminus B_{g_p}(0, R)$$

for  $p$  small enough. On the other hand,

$$L_p v_p = 0 \quad \text{in } \Omega_p.$$

At last, it is not difficult to check with (2.21) that

$$v_p \leq \theta_p G_p \quad \text{on } \partial B_{g_p}(0, R)$$

if we take  $\theta_p = R^{\frac{n-p-\nu}{p-1}}$ . Now we may apply the maximum principle as stated for instance in [2] (lemma 3.4) to get, for  $p$  small enough,

$$v_p(y) \leq \left( \frac{R}{|y|} \right)^{\frac{n-p-\nu}{p-1}} \quad \text{in } \Omega_p \setminus B_{g_p}(0, R).$$

Since this inequality obviously holds on  $B_{g_p}(0, R)$ , we have finally obtained the following: for any  $\nu > 0$  and any  $R > R_0$ , there exists  $C(R, \nu) > 0$  such that for any  $p > 1$  and any  $y \in \Omega_p$ ,

$$(2.26) \quad \left( \frac{|y|}{R} \right)^{\frac{n-p-\nu}{p-1}} v_p(y) \leq C(R, \nu).$$

*Step 5.* We conclude the proof of the Proposition. We apply the  $H_1^1$  Euclidean Sobolev inequality to  $\tilde{v}_p$ :

$$(2.27) \quad \left( \int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi \right)^{\frac{n-1}{n}} \leq K(n, 1) \int_{\Omega_p} |\nabla \tilde{v}_p|_\xi dv_\xi.$$

By the Cartan expansion of  $g_p$  around 0, we have

$$(2.28) \quad dv_\xi = \left( 1 + \frac{\mu_p^2}{6} Ric_g(y_p)_{ij} x^i x^j + o(\mu_p^2 |x|^2) \right) dv_{g_p}$$

where  $Ric_g$  denotes the Ricci curvature of  $g$  in the  $exp_{y_p}$ -map. Thus, by (2.11),

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1 + \frac{\mu_p^2}{6} Ric_g(y_p)_{ij} \int_{\Omega_p} x^i x^j v_p^{p^*} dv_{g_p} + o\left( \mu_p^2 \int_{\Omega_p} |x|^2 v_p^{p^*} dv_{g_p} \right).$$

Using (2.17) and (2.26), one gets

$$(2.29) \quad \int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1 + \frac{S_g(0)}{6n(n+2)} \omega_{n-1} R_0^{n+2} \mu_p^2 + o(\mu_p^2).$$

By the Cartan expansion of  $g_p$  around 0, since  $r_p \rightarrow 0$  as  $p \rightarrow 1$ , we also have

$$|\nabla \tilde{v}_p|_\xi^p = |\nabla \tilde{v}_p|_{g_p}^p \left[ 1 - \frac{\mu_p^2}{6} |\nabla \tilde{v}_p|_{g_p}^{-2} Rm_g(y_p)(\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) + o(\mu_p^2 |x|^2) \right]$$

where  $Rm_g$  denotes the Riemann curvature of  $g$  in the  $exp_{y_p}$ -map. Then, using (2.28), we get

$$\begin{aligned} \int_{\Omega_p} |\nabla \tilde{v}_p|_\xi dv_\xi &= \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} dv_{g_p} + \frac{\mu_p^2}{6} Ric_g(y_p)_{ij} \int_{\Omega_p} x^i x^j |\nabla \tilde{v}_p|_\xi dv_\xi \\ &\quad - \frac{\mu_p^2}{6} \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p}^{-1} Rm_g(y_p)(\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) dv_{g_p} \quad (2.30) \\ &\quad + o\left(\mu_p^2 \int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p|_{g_p} dv_{g_p}\right). \end{aligned}$$

Let us now look at the different terms of (2.30). First, by equation (2.10) and relation (2.5), we have

$$\begin{aligned} \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} dv_{g_p} &= \frac{p(n-1)}{n-p} \int_{\Omega_p} v_p^{\frac{n(p-1)}{n-p}} |\nabla v_p|_{g_p} dv_{g_p} \\ &\leq \frac{p(n-1)}{n-p} \left( \int_{\Omega_p} |\nabla v_p|_{g_p}^p dv_{g_p} \right)^{\frac{1}{p}} \\ &\leq K(n, 1)^{-1} (1 - \alpha \mu_p^2 \lambda_p^{-1} \|v_p\|_p^2)^{\frac{1}{2}}. \end{aligned}$$

Since, by (2.17) and (2.26),  $\|v_p\|_p = 1 + o(1)$ , we get

$$(2.31) \quad \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p} dv_{g_p} \leq K(n, 1)^{-1} - \frac{\alpha}{2} K(n, 1) \mu_p^2 + o(\mu_p^2).$$

Independently, by Hölder’s inequalities, we have

$$\int_{\Omega_p} |x|^2 |\nabla \tilde{v}_p|_{g_p} dv_{g_p} \leq \frac{p(n-1)}{n-p} \left( \int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p dv_{g_p} \right)^{\frac{1}{p}}.$$

By equation (2.10), one gets

$$\begin{aligned} \int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p dv_{g_p} &\leq \int_{\Omega_p} |\nabla v_p|_{g_p}^{p-2} (\nabla(|x|^{2p} v_p), \nabla v_p)_{g_p} dv_{g_p} \\ &\quad + C \int_{\Omega_p} |x|^{2p-1} |\nabla v_p|_{g_p}^{p-1} v_p dv_{g_p} \\ &\leq C + C \left( \int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p dv_{g_p} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_p} |x|^p v_p^p dv_{g_p} \right)^{\frac{1}{p}} \end{aligned}$$

where  $C$  denotes some constant independent of  $p$ . Using (2.26) and Young’s inequalities, one deduces that

$$(2.32) \quad \int_{\Omega_p} |x|^{2p} |\nabla v_p|_{g_p}^p dv_{g_p} = O(1).$$

Now, for some  $R > R_0$ , we get by (2.18) that

$$\int_{\Omega_p} |\nabla \tilde{v}_p|_\xi x^i x^j dv_\xi = O\left(\int_{\Omega_p \setminus B(0, R)} |x|^2 |\nabla \tilde{v}_p|_\xi dv_\xi\right) + \int_{\partial B(0, R_0)} x^i x^j d\sigma_\xi + o(1).$$

Using equation (2.10) and relation (2.26), it is easy to check that

$$\lim_{p \rightarrow 1} \int_{\Omega_p \setminus B(0,R)} |x|^2 |\nabla \tilde{v}_p|_\xi dv_\xi = 0$$

so that

$$(2.33) \quad \lim_{p \rightarrow 1} Ric_g(y_p)_{ij} \int_{\Omega_p} |\nabla \tilde{v}_p|_\xi x^i x^j dv_\xi = \frac{\omega_{n-1}}{n} R_0^{n+1} S_g(0).$$

At last, since  $\nabla V_p, V_p$  as in (2.19), and  $x$  are pointwise colinear vector fields, we have

$$Rm_g(y_p)(\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) \leq C|x|^2 |\nabla \tilde{v}_p|_\xi |\nabla(\tilde{v}_p - V_p)|_\xi$$

so that, by (2.10), (2.20) and (2.26),

$$(2.34) \quad \lim_{p \rightarrow 1} \int_{\Omega_p} |\nabla \tilde{v}_p|_{g_p}^{-1} Rm_g(y_p)(\nabla \tilde{v}_p, x, x, \nabla \tilde{v}_p) dv_{g_p} = 0.$$

Coming back to (2.27) with (2.29)-(2.34), we obtain, after easy computations using in particular (2.16),

$$\left( \alpha - \frac{n}{n+2} S_g(0) \right) \mu_p^2 + o(\mu_p^2) \leq 0.$$

This gives the desired contradiction by letting  $p$  go to 0. Remember here that  $\alpha - \frac{n}{n+2} S_g(0) = \varepsilon_0 > 0$ . This ends the proof of the Proposition, hence the proof of Theorem 1. □

### 3. THE COMPACT CASE - PROOF OF THEOREM 2

In order to prove Theorem 2, we let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 2$ . We assume that  $S_g < n(n-1)K_0$ . If we apply Theorem 1 with some  $x$  in  $M$  and  $K_0$ , we get some  $r_x > 0$  such that the isoperimetric comparison (with the model space form of curvature  $K_0$ ) holds for sets contained in the geodesic ball of center  $x$  and radius  $r_x$ . It is clear that  $r_x$  is continuous with respect to  $x$ . Thus, there exists  $d > 0$  such that for any subset  $\Omega$  of  $M$  of diameter less than or equal to  $d$ ,

$$(3.1) \quad |\partial\Omega|_g > |\partial B|_{g_0}$$

where  $B$  is a ball of volume  $|\Omega|_g$  in the model space of constant curvature  $K_0$ . For  $0 < V < |M|_g$ , we let

$$h(V) = \inf \{ |\partial\Omega|_g, \Omega \subset M, |\Omega|_g = V \}.$$

There exists some  $\Omega_V \subset M$  such that

$$|\partial\Omega_V|_g = h(V).$$

The boundary  $\partial\Omega_V$  of  $\Omega_V$  is a smooth hypersurface of constant mean curvature up to a compact set of Hausdorff dimension at most  $n-8$  (see for instance [13]). Now, as a consequence of the work of Johnson and Morgan [10], we know that

$$diam(\Omega_V) \rightarrow 0$$

as  $V \rightarrow 0$ . In fact, Johnson and Morgan proved that  $\Omega_V$  is asymptotically, as  $V \rightarrow 0$ , a ball. In particular, for some  $V_0$  small enough, any  $\Omega_V$  for  $V \leq V_0$  has a diameter less than or equal to  $d$ . We may then apply (3.1) to end the proof of Theorem 2. □

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