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WIGNER'S THEOREM IN HILBERT C^*-MODULES OVER C^*-ALGEBRAS OF COMPACT OPERATORS

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ABSTRACT. Let W be a Hilbert C^*-module over the C^*-algebra $\mathcal{A} \neq \boldsymbol{C}$ of all compact operators on a Hilbert space. It is proved that any function $T : W \to W$ which preserves the absolute value of the $\mathcal{A}$-valued inner product is of the form $Tv = \varphi(v)Uv$, $v \in W$, where φ is a phase function and U is an $\mathcal{A}$-linear isometry. The result generalizes Molnár's extension of Wigner's classical unitary-antiunitary theorem.

1. Introduction

Wigner's unitary-antiunitary theorem states that each bijective function $T : H \to H$ acting on a complex Hilbert space $(H, (\cdot,\cdot))$ which satisfies $|(Tx, Ty)| = |(x, y)|$, $x, y \in H$, must be of the form $Tx = \varphi(x)Ux$, $x \in H$, where $U : H \to H$ is either a unitary or antiunitary operator and $\varphi : H \to \boldsymbol{C}$ is a phase function (i.e. its values are of modulus 1).

Wigner's theorem was first published in 1931 ([14]), while in the 1960's several authors began working on the proof in order to make the argument rigorous (see [11] and references therein).

In [8] and [9] a new, algebraic approach to this theorem is used in proving a natural generalization of Wigner's theorem for Hilbert C^*-modules over matrix algebras M_d. It should also be observed that in the statement of the Theorem in [9] even the surjectivity of the transformation in question is not assumed.

In the present note we give a further generalization of Wigner's theorem to Hilbert C^*-modules over C^*-algebras of compact operators.

A (left) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module W equipped with an $\mathcal{A}$-valued inner product $\langle \cdot,\cdot \rangle$ which is linear over $\mathcal{A}$ in the first variable and conjugate linear in the second, such that W is a Banach space with the norm $\|w\| = \|\langle w, w\rangle\|^{1/2}$. Hilbert C^*-modules are introduced and first investigated in [6], [10] and [12]. A Hilbert C^*-module is said to be full if the two sided ideal generated by all products $\langle v, w\rangle$, $v, w \in W$, is dense in $\mathcal{A}$. The basic theory of Hilbert C^*-modules can be found in [7] and [13].

We denote by $\boldsymbol{B}_a(W)$ the C^*-algebra of all adjointable operators on W (i.e. of all maps $A : W \to W$ such that there exists $A^* : W \to W$ with the property $\langle Av, w\rangle = \langle v, A^*w\rangle$, $\forall v, w \in W$). It is well known that each adjointable operator is

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necessarily bounded and $\mathcal{A}$-linear in the sense $A(av) = aAv, \forall a \in \mathcal{A}, \forall v \in W$. In general, bounded $\mathcal{A}$-linear operators may fail to possess an adjoint. However, if W is a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$ of all compact operators on a Hilbert space, then it is known that each bounded $\mathcal{A}$-linear operator on W is necessarily adjointable (see for example [3], Remark 5).

Let us also note that an operator $U \in \boldsymbol{B}_a(W)$ on an arbitrary Hilbert C^*-module W is an isometry if and only if $U^*U = I$ (with I denoting the identity operator on W). Indeed, $U^*U = I$ obviously implies that U is an isometry, while the converse can be proved repeating the nice argument from Theorem 3.5 in [7].

Before stating the main result let us fix the rest of our notation.

Throughout, $|a|$ denotes the unique positive square root of a^*a. We denote by $\boldsymbol{K}(H)$ the C^*-algebra of all compact operators on a Hilbert space H. The field of complex numbers is denoted by $\boldsymbol{C}$.

Theorem 1. *Let W be a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$ of all compact operators on a Hilbert space H with $\dim H > 1$. Let $T : W \to W$ be a function satisfying*

$$|\langle Tv, Tw\rangle| = |\langle v, w\rangle|, \ \forall v, w \in W. \tag{1}$$

Then there exist an $\mathcal{A}$-linear isometry $U \in \boldsymbol{B}_a(W)$ and a phase function $\varphi : W \to \boldsymbol{C}$ such that

$$Tv = \varphi(v)Uv, \ \forall v \in W.$$

Let us observe that the corresponding result for the case $\dim H = 1$ is in fact the classical Wigner's theorem. In this situation one cannot exclude the antilinear alternative from the assertion of the theorem. On the other hand, if $\dim H > 1$, then, as noted in [9], the nonappearance of antilinear isometries is the consequence of the noncommutativity of the underlying C^*-algebra.

We also note that our theorem includes the Theorem from [9] since the C^*-algebra of all compact operators $\boldsymbol{K}(H)$ with $\dim H = d$ is in fact the algebra of all $d \times d$ complex matrices M_d. Thus our theorem generalizes the Theorem in [9] from a finite to an arbitrary dimension of the underlying Hilbert space.

The proof of Theorem 1 basically depends on the presence of an orthonormal basis in each Hilbert C^*-module W over the C^*-algebra of compact operators $\boldsymbol{K}(H)$.

Recall from [3] that a vector $v \in W$ is said to be a basic vector if $e = \langle v, v\rangle$ is a minimal (i.e. one-dimensional) projection in $\boldsymbol{K}(H)$. A system (v_λ), $\lambda \in \Lambda$, is an orthonormal system if each v_λ is a basic vector and $\langle v_\lambda, v_\mu\rangle = 0$ for all $\lambda \neq \mu$. An orthonormal system (v_λ) in W is said to be an orthonormal basis for W if it generates a dense submodule of W.

Originally, the concept is introduced in [4] for Hilbert H^*-modules. It is proved in [3], Theorem 2, that each Hilbert $\boldsymbol{K}(H)$-module possesses an orthonormal basis. We also note that by Theorem 1 from [3] the following properties of an orthonormal system (v_λ), $\lambda \in \Lambda$, are mutually equivalent:

(a) (v_λ) is an orthonormal basis for W.
(b) $w = \sum_{\lambda\in\Lambda} \langle w, v_\lambda\rangle v_\lambda, \ \forall w \in W$.
(c) $\langle w, w\rangle = \sum_{\lambda\in\Lambda} \langle w, v_\lambda\rangle\langle v_\lambda, w\rangle, \ \forall w \in W$.
(d) $\langle w_1, w_2\rangle = \sum_{\lambda\in\Lambda} \langle w_1, v_\lambda\rangle\langle v_\lambda, w_2\rangle, \ \forall w_1, w_2 \in W$.

Notice that each orthonormal basis is in fact a standard normalized tight frame in the sense of [5] (cf. Definition 2.1 and Proposition 2.2 in [5]). However, we shall use the name orthonormal basis to emphasize the fact that its members v_λ are supported by minimal projections $\langle v_\lambda, v_\lambda\rangle$.

2. Proof

In order to prove Theorem 1 we first state an independent lemma. Although our lemma is in fact proved in [8], p. 363, we include the sketch of a more direct proof.

Lemma 1. *Let W be a Hilbert C^*-module over the C^*-algebra $\mathcal{A} = \boldsymbol{K}(H)$ of all compact operators on a Hilbert space H. Suppose that $x, y \in W$ satisfy $\langle x, v\rangle\langle v, x\rangle = \langle y, v\rangle\langle v, y\rangle$, $\forall v \in W$. Then there exists a complex number λ such that $|\lambda| = 1$ and $y = \lambda x$.*

Proof. First observe that the corresponding statement is obviously true for Hilbert spaces.

Let (u_j) be an orthonormal basis for W. By Theorem 1 from [3],

$$\langle x, x\rangle = \sum_j \langle x, u_j\rangle\langle u_j, x\rangle = \sum_j \langle y, u_j\rangle\langle u_j, y\rangle = \langle y, y\rangle.$$

Now $\langle x, x\rangle$, being a positive compact operator on H, can be written in the form

$$\langle x, x\rangle = \langle y, y\rangle = \sum_i \gamma_i e_i \tag{2}$$

where $\gamma_i > 0$ and (e_i) is a family of pairwise orthogonal minimal projections in $\boldsymbol{K}(H)$. This in turn implies

$$x = \sum_i e_i x, \ y = \sum_i e_i y. \tag{3}$$

Let e be any minimal orthogonal projection in $\boldsymbol{K}(H)$. By assumption we have

$$\langle ex, ev\rangle\langle ev, ex\rangle = \langle ey, ev\rangle\langle ev, ey\rangle, \ \forall v \in W. \tag{4}$$

Recall from [3] (see also [2]) that $W_e := eW$ is a Hilbert space contained in W with a scalar product $(\cdot,\cdot)$ such that $\langle\cdot,\cdot\rangle = (\cdot,\cdot)e$. The above equality gives

$$(ex, ev)(ev, ex) = (ey, ev)(ev, ey), \ \forall v \in W.$$

By the correspondent assertion of Lemma 1 for Hilbert spaces one gets $ey = \lambda_e ex$ for some $\lambda_e \in \boldsymbol{C}$ such that $|\lambda_e| = 1$ which, together with (3), implies

$$x = \sum_i e_i x, \ y = \sum_i \lambda_i e_i x, \ |\lambda_i| = 1, \ \forall i. \tag{5}$$

It remains to show $\lambda_i = \lambda_j$, $\forall i, j$. To do this, it suffices to substitute $v = ax$ in the hypothesis of the lemma, where $a = e_i + e_j + h + h^*$ and h is the partial isometry such that $h^*h = e_i, hh^* = e_j$. We omit the details. □

Notice that W, being nontrivial (as we tacitly assume), must be a full $\boldsymbol{K}(H)$-module since $\boldsymbol{K}(H)$ has no nontrivial closed two sided ideals. This implies that the subspace W_e used in the above proof cannot be trivial. Indeed, $ev = 0$, $\forall v \in W$ would imply $e\langle v, v\rangle = 0$, $\forall v \in W$ and finally $e = 0$ since W is full.

Proof of Theorem 1. We assume $\dim H = \infty$ since the case $\dim H < \infty$ is already proved in [9]. Let $e \in \mathcal{A} = \boldsymbol{K}(H)$ be an arbitrary minimal projection. Notice that

$e\boldsymbol{K}(H)e = \boldsymbol{C}e$. Consider again $W_e = eW$ which is a Hilbert space with the scalar product $(x, y) = \mathrm{tr}(\langle x, y\rangle)$, $x, y \in W_e$. Now the equality ([2], Lemma 2)

$$W_e = \{x \in W : \langle x, x\rangle \in \boldsymbol{C}e\} \tag{6}$$

and the assumed property of T imply that W_e is invariant for T. Indeed, take any $x \in W_e$ and observe $\langle x, x\rangle = \alpha e$ for some $\alpha \geq 0$. Since T preserves inner squares (this follows immediately from (1)), we have $\langle Tx, Tx\rangle = \langle x, x\rangle = \alpha e$ and, by (6), $Tx \in W_e$.

Thus we may consider the induced function $T_e = T|W_e : W_e \to W_e$. Moreover, since $\langle x, y\rangle = (x, y)e$, $\forall x, y \in W_e$, we conclude $|(Tx, Ty)| = |(x, y)|$, $\forall x, y \in W_e$. Now we can apply the classical Wigner's theorem on the Hilbert space W_e: there exist an isometry (either linear or antilinear) U_e on W_e and a phase function $\varphi_e : W_e \to \boldsymbol{C}$ such that

$$Tx = \varphi_e(x)U_e x,\ \forall x \in W_e. \tag{7}$$

We claim that the isometry U_e must be linear. To see this, we shall "add one more dimension" and apply the same trick. Let f be a minimal projection in $\boldsymbol{K}(H)$ orthogonal to e and let $b \in \boldsymbol{K}(H)$ be the partial isometry such that $b^*b = e, bb^* = f$. Let us denote by $M_2 \subseteq \boldsymbol{K}(H)$ the algebra spanned by matrix units e, f, b, b^*. Now define

$$X = \{x \in W : \langle x, x\rangle \in M_2\}. \tag{8}$$

Obviously, X is closed and, by (6), contains W_e. Denoting $p = e + f$ one finds $px = x$, $\forall x \in X$ and $X = pW$. Now it is easy to conclude that X is a Hilbert module over the matrix algebra M_2 with the M_2-valued inner product inherited from W. We omit the details.

Since T by assumption preserves the inner squares, (8) implies that X is invariant for T. Now we apply the Theorem from [9] to the induced function $T_2 = T|X : X \to X$: there exist a phase function $\varphi_2 : X \to \boldsymbol{C}$ and a M_2-linear (!) isometry $U_2 : X \to X$ such that $T_2 x = \varphi_2(x)U_2 x$, $\forall x \in X$. By M_2-linearity W_e remains invariant under the action of U_2 since $W_e = eW$. This shows that the pair (φ_e, U_e) from the first part of the proof can actually be chosen by restricting the action of φ_2 and U_2 to W_e.

After all, we can apply Theorem 5 from [3] (see also Theorem 1 in [2]): there exists a unique isometry $U \in \boldsymbol{B}_a(W)$ such that $U|W_e = U_e$. Define a new function T' on W putting $T' = U^*T$.

We claim

$$\langle T'v, w\rangle e\langle w, T'v\rangle = \langle v, w\rangle e\langle w, v\rangle,\ \forall v, w \in W. \tag{9}$$

Indeed,

$$\langle T'v, w\rangle e\langle w, T'v\rangle = \langle T'v, w\rangle\overline{\varphi_e(ew)}e\varphi_e(ew)e\langle w, T'v\rangle$$

$$= \langle U^*Tv, \varphi_e(ew)ew\rangle\langle\varphi_e(ew)ew, U^*Tv\rangle = \langle Tv, T(ew)\rangle\langle T(ew), Tv\rangle$$

$$= \langle v, ew\rangle\langle ew, v\rangle = \langle v, w\rangle e\langle w, v\rangle.$$

Now observe that each minimal projection $f \in \boldsymbol{K}(H)$ can be written in the form $f = a^*ea$ for a suitably chosen partial isometry $a \in \boldsymbol{K}(H)$. Substituting aw for w in (9) we get

$$\langle T'v, w\rangle f\langle w, T'v\rangle = \langle v, w\rangle f\langle w, v\rangle,\ \forall v, w \in W. \tag{10}$$

Since there exists an approximate unit (u_λ) for $\boldsymbol{K}(H)$ whose elements are finite linear combinations of minimal projections, this gives

$$\langle T'v, w\rangle u_\lambda \langle w, T'v\rangle = \langle v, w\rangle u_\lambda \langle w, v\rangle,\ \forall\lambda,\ \forall v, w \in W. \tag{11}$$

Finally, (11) obviously implies

$$\langle T'v, w\rangle\langle w, T'v\rangle = \langle v, w\rangle\langle w, v\rangle,\ \forall v, w \in W. \tag{12}$$

Now we apply Lemma 1. There exists a complex number (depending on v) $\lambda_v =: \varphi(v)$ of modulus 1 such that

$$T'v = \varphi(v)v \ \text{ or } \ U^*Tv = \varphi(v)v,\ \forall v \in W.$$

This gives

$$UU^*Tv = \varphi(v)Uv,\ \forall v \in W$$

and, because UU^* is the projection to $\operatorname{Im} U$, the proof will be finished by showing $T(W) \subseteq \operatorname{Im} U$.

To do this, let us take an orthonormal basis (u_j) for W such that $\langle u_j, u_j\rangle = e, \forall j$. (There exists an orthonormal basis for W with this property by [2], Proposition 2 and [3], Theorem 2.) Observe that the system (Tu_j) is also an orthonormal system in W. Indeed, $\langle Tu_i, Tu_j\rangle = \langle u_i, u_j\rangle = \delta_{ij}e, \forall i, j$ (with δ denoting the Kronecker symbol). Moreover, for each $v \in W$ we have (see [3], Theorem 1)

$$\langle Tv, Tv\rangle = \langle v, v\rangle = \sum_j \langle v, u_j\rangle\langle u_j, v\rangle = \sum_j |\langle u_j, v\rangle|^2 = \sum_j |\langle Tu_j, Tv\rangle|^2$$

$$= \sum_j \langle Tv, Tu_j\rangle\langle Tu_j, Tv\rangle.$$

This is enough to conclude

$$Tv = \sum_j \langle Tv, Tu_j\rangle Tu_j. \tag{13}$$

Notice that $\langle u_j, u_j\rangle = e$ implies $u_j = eu_j \in eW = W_e$, hence we may use (7) to rewrite (13):

$$Tv = \sum_j \langle Tv, \varphi_e(u_j)U_e(u_j)\rangle\varphi_e(u_j)U_e(u_j) = \sum_j \langle Tv, Uu_j\rangle Uu_j$$

$$= \sum_j U(\langle Tv, Uu_j\rangle u_j).$$

Since U is an isometry, its image is closed. This completes the proof. □

Remark 1. Let W be a Hilbert C^*-module over an arbitrary C^*-algebra $\mathcal{A}$ of compact operators and let $T : W \to W$ be a function satisfying condition (1) from Theorem 1.

It is well known ([1], Theorem 1.4.5) that $\mathcal{A}$ must be of the form

$$\mathcal{A} = \bigoplus_{j \in J} \boldsymbol{K}(H_j), \tag{14}$$

i.e. $\mathcal{A}$ is a direct sum of C^*-algebras $\boldsymbol{K}(H_j)$ of all compact operators acting on Hilbert spaces H_j, $j \in J$. We may assume that W is a full Hilbert $\mathcal{A}$ -module by dropping unnecessary summands (i.e. those $\boldsymbol{K}(H_i)$ which act on W as the zero operator) from the above decomposition of $\mathcal{A}$. For each $j \in J$ consider the associated ideal submodule $W_j = \overline{[\boldsymbol{K}(H_j)W]}$. Notice that W_j, regarded as a Hilbert $\boldsymbol{K}(H_j)$-module, is full. Obviously, (W_j) is a family of pairwise orthogonal closed submodules of W and it is well known (cf. [3]) that W admits a decomposition into the (outer) direct sum

$$W = \bigoplus_{j\in J} W_j, \quad W_j = \overline{[\boldsymbol{K}(H_j)W]}. \tag{15}$$

Now observe that each W_j satisfies

$$W_j = \{w \in W : \langle w, w\rangle \in \boldsymbol{K}(H_j)\}. \tag{16}$$

Since each function satisfying condition (1) preserves $\mathcal{A}$-valued inner squares, (16) shows that each W_j is invariant for T. Applying Theorem 1 to all induced functions $T_j = T|W_j : W_j \to W_j$ we obtain the factorization $T_j(v) = \varphi_j(v)U_j v, \forall v \in W_j$ on each component W_j.

Observe that the family (U_j) defines an isometry $U \in \boldsymbol{B}_a(W)$ ([3], Theorem 8). However, the complex numbers $\varphi_j(w)$ might be different for fixed w and varying $j \in J$, so a unique global choice for φ as a scalar-valued function might be impossible. Therefore we cannot obtain a global factorization for T as in Theorem 1.

Remark 2. Let us note that our Theorem 1 (as well as the above Remark 1) also holds true for functions satisfying condition (1) which are defined on H^*-modules. Namely, if W is an arbitrary H^*-module over the H^*-algebra of all Hilbert Schmidt operators acting on some Hilbert space, then the same proof applies using the corresponding results concerned with operators on H^*-modules (cf. [2]).

To conclude, we shall mention a possible extension of Wigner's theorem to more general Hilbert C^*-modules. A good candidate is the class of Hilbert C^*-modules over concrete C^*-algebras which contain the ideal of all compact operators. There is some evidence along this line.

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