WIGNER’S THEOREM IN HILBERT C*-MODULES
OVER C*-ALGEBRAS OF COMPACT OPERATORS

DAMIR BAKIĆ AND BORIS GULJAŠ

(Communicated by David R. Larson)

Abstract. Let W be a Hilbert C*-module over the C*-algebra A ≠ C of all compact operators on a Hilbert space. It is proved that any function T : W → W which preserves the absolute value of the A-valued inner product is of the form
Tv = ϕ(v)Uv, v ∈ W, where ϕ is a phase function and U is an A-linear isometry. The result generalizes Molnár’s extension of Wigner’s classical unitary-antiunitary theorem.

1. Introduction

Wigner’s unitary-antiunitary theorem states that each bijective function T : H → H acting on a complex Hilbert space (H, ⟨·, ·⟩) which satisfies |⟨Tx, Ty⟩| = |⟨x, y⟩|, x, y ∈ H, must be of the form Tx = ϕ(x)Ux, x ∈ H, where U : H → H is either a unitary or antiunitary operator and ϕ : H → C is a phase function (i.e. its values are of modulus 1).

Wigner’s theorem was first published in 1931 ([14]), while in the 1960’s several authors began working on the proof in order to make the argument rigorous (see [11] and references therein).

In [8] and [9] a new, algebraic approach to this theorem is used in proving a natural generalization of Wigner’s theorem for Hilbert C*-modules over matrix algebras M_d. It should also be observed that in the statement of the Theorem in [9] even the surjectivity of the transformation in question is not assumed.

In the present note we give a further generalization of Wigner’s theorem to Hilbert C*-modules over C*-algebras of compact operators.

A (left) Hilbert C*-module over a C*-algebra A is a left A-module W equipped with an A-valued inner product ⟨·, ·⟩ which is linear over A in the first variable and conjugate linear in the second, such that W is a Banach space with the norm ||w|| = ||⟨w, w⟩||^{1/2}. Hilbert C*-modules are introduced and first investigated in [6], [10] and [12]. A Hilbert C*-module is said to be full if the two sided ideal generated by all products ⟨v, w⟩, v, w ∈ W, is dense in A. The basic theory of Hilbert C*-modules can be found in [7] and [13].

We denote by B_a(W) the C*-algebra of all adjointable operators on W (i.e. of all maps A : W → W such that there exists A* : W → W with the property ⟨Av, w⟩ = ⟨v, A*w⟩, ∀v, w ∈ W). It is well known that each adjointable operator is
necessarily bounded and $\mathcal{A}$-linear in the sense $A(\lambda v) = \lambda Av$, $\forall \lambda \in \mathcal{A}, \forall v \in W$. In general, bounded $\mathcal{A}$-linear operators may fail to possess an adjoint. However, if $W$ is a Hilbert $C^*$-module over the $C^*$-algebra $\mathcal{A}$ of all compact operators on a Hilbert space, then it is known that each bounded $\mathcal{A}$-linear operator on $W$ is necessarily adjointable (see for example [3], Remark 5).

Let us also note that an operator $U \in B_a(W)$ on an arbitrary Hilbert $C^*$-module $W$ is an isometry if and only if $U^*U = I$ (with $I$ denoting the identity operator on $W$). Indeed, $U^*U = I$ obviously implies that $U$ is an isometry, while the converse can be proved repeating the nice argument from Theorem 3.5 in [7].

Before stating the main result let us fix the rest of our notation.

Throughout, $|a|$ denotes the unique positive square root of $a^*a$. We denote by $K(H)$ the $C^*$-algebra of all compact operators on a Hilbert space $H$. The field of complex numbers is denoted by $C$.

**Theorem 1.** Let $W$ be a Hilbert $C^*$-module over the $C^*$-algebra $\mathcal{A}$ of all compact operators on a Hilbert space $H$ with $\dim H > 1$. Let $T : W \to W$ be a function satisfying

$$|\langle Tv, Tw \rangle| = |\langle v, w \rangle|, \forall v, w \in W. \tag{1}$$

Then there exist an $\mathcal{A}$-linear isometry $U \in B_a(W)$ and a phase function $\varphi : W \to C$ such that

$$Tv = \varphi(v)Uv, \forall v \in W.$$  

Let us observe that the corresponding result for the case $\dim H = 1$ is in fact the classical Wigner’s theorem. In this situation one cannot exclude the antilinear alternative from the assertion of the theorem. On the other hand, if $\dim H > 1$, then, as noted in [9], the nonappearance of antilinear isometries is the consequence of the noncommutativity of the underlying $C^*$-algebra.

We also note that our theorem includes the Theorem from [9] since the $C^*$-algebra of all compact operators $K(H)$ with $\dim H = d$ is in fact the algebra of all $d \times d$ complex matrices $M_d$. Thus our theorem generalizes the Theorem in [9] from a finite to an arbitrary dimension of the underlying Hilbert space.

The proof of Theorem 1 basically depends on the presence of an orthonormal basis in each Hilbert $C^*$-module $W$ over the $C^*$-algebra of compact operators $K(H)$.

Recall from [2] that a vector $v \in W$ is said to be a basic vector if $e = \langle v, v \rangle$ is a minimal (i.e. one-dimensional) projection in $K(H)$. A system $(v_\lambda)$, $\lambda \in \Lambda$, is an orthonormal system if each $v_\lambda$ is a basic vector and $\langle v_\lambda, v_\mu \rangle = 0$ for all $\lambda \neq \mu$. An orthonormal system $(v_\lambda)$ in $W$ is said to be an orthonormal basis for $W$ if it generates a dense submodule of $W$.

Originally, the concept is introduced in [4] for Hilbert $H^*$-modules. It is proved in [4], Theorem 2, that each Hilbert $K(H)$-module possesses an orthonormal basis. We also note that by Theorem 1 from [3] the following properties of an orthonormal system $(v_\lambda)$, $\lambda \in \Lambda$, are mutually equivalent:

(a) $(v_\lambda)$ is an orthonormal basis for $W$.
(b) $w = \sum_{\lambda \in \Lambda} (w, v_\lambda) v_\lambda$, $\forall w \in W$.
(c) $\langle w, v \rangle = \sum_{\lambda \in \Lambda} \langle w, v_\lambda \rangle \langle v_\lambda, v \rangle$, $\forall w \in W$.
(d) $\langle w_1, w_2 \rangle = \sum_{\lambda \in \Lambda} \langle w_1, v_\lambda \rangle \langle v_\lambda, w_2 \rangle$, $\forall w_1, w_2 \in W$.

Notice that each orthonormal basis is in fact a standard normalized tight frame in the sense of [3] (cf. Definition 2.1 and Proposition 2.2 in [3]). However, we shall
use the name orthonormal basis to emphasize the fact that its members $v_\lambda$ are supported by minimal projections $\langle v_\lambda, v_\lambda \rangle$.

2. Proof

In order to prove Theorem 1 we first state an independent lemma. Although our lemma is in fact proved in [3], p. 363, we include the sketch of a more direct proof.

**Lemma 1.** Let $W$ be a Hilbert $C^*$-module over the $C^*$-algebra $A = K(H)$ of all compact operators on a Hilbert space $H$. Suppose that $x, y \in W$ satisfy $\langle x, v \rangle \langle v, x \rangle = \langle y, v \rangle \langle v, y \rangle$, $\forall v \in W$. Then there exists a complex number $\lambda$ such that $|\lambda| = 1$ and $y = \lambda x$.

**Proof.** First observe that the corresponding statement is obviously true for Hilbert spaces.

Let $(u_j)$ be an orthonormal basis for $W$. By Theorem 1 from [3],

$$\langle x, x \rangle = \sum_j \langle x, u_j \rangle \langle u_j, x \rangle = \sum_j \langle y, u_j \rangle \langle u_j, y \rangle = \langle y, y \rangle.$$ 

Now $\langle x, x \rangle$, being a positive compact operator on $H$, can be written in the form

$$\langle x, x \rangle = \langle y, y \rangle = \sum_i \gamma_i e_i$$

where $\gamma_i > 0$ and $(e_i)$ is a family of pairwise orthogonal minimal projections in $K(H)$. This in turn implies

$$x = \sum_i e_i x, \quad y = \sum_i e_i y.$$ 

Let $e$ be any minimal orthogonal projection in $K(H)$. By assumption we have

$$\langle ex, ev \rangle \langle ev, ex \rangle = \langle ey, ev \rangle \langle ev, ey \rangle, \quad \forall v \in W.$$ 

Recall from [3] (see also [2]) that $W_e := eW$ is a Hilbert space contained in $W$ with a scalar product $(\cdot, \cdot)$ such that $(\cdot, \cdot) = (\cdot, \cdot)e$. The above equality gives

$$\langle ex, ev \rangle (ev, ex) = \langle ey, ev \rangle (ev, ey), \quad \forall v \in W.$$ 

By the correspondent assertion of Lemma 1 for Hilbert spaces one gets $ey = \lambda_e ex$ for some $\lambda_e \in C$ such that $|\lambda_e| = 1$ which, together with (3), implies

$$x = \sum_i e_i x, \quad y = \sum_i \lambda_i e_i x, \quad |\lambda_i| = 1, \quad \forall i.$$ 

It remains to show $\lambda_i = \lambda_j, \forall i, j$. To do this, it suffices to substitute $v = ax$ in the hypothesis of the lemma, where $a = e_i + e_j + h + h^*$ and $h$ is the partial isometry such that $h^* h = e_i, hh^* = e_j$. We omit the details.

Notice that $W$, being nontrivial (as we tacitly assume), must be a full $K(H)$-module since $K(H)$ has no nontrivial closed two sided ideals. This implies that the subspace $W_e$ used in the above proof cannot be trivial. Indeed, $ev = 0$, $\forall v \in W$ would imply $e\langle v, v \rangle = 0$, $\forall v \in W$ and finally $e = 0$ since $W$ is full.

**Proof of Theorem 1.** We assume dim $H = \infty$ since the case dim $H < \infty$ is already proved in [3]. Let $e \in A = K(H)$ be an arbitrary minimal projection. Notice that
Consider again $W_e = eW$ which is a Hilbert space with the scalar product $(x, y) = \text{tr}(\langle x, y \rangle)$, $x, y \in W_e$. Now the equality ([2], Lemma 2)
\begin{equation}
W_e = \{ x \in W : \langle x, x \rangle \in C e \}
\end{equation}
and the assumed property of $T$ imply that $W_e$ is invariant for $T$. Indeed, take any $x \in W_e$ and observe $\langle x, x \rangle = \alpha e$ for some $\alpha \geq 0$. Since $T$ preserves inner squares (this follows immediately from (1)), we have $(Tx, Tx) = \langle x, x \rangle = \alpha e$ and, by (6), $Tx \in W_e$.

Thus we may consider the induced function $T_e = T|W_e : W_e \rightarrow W_e$. Moreover, since $\langle x, y \rangle = (x, ye)$, $\forall x, y \in W_e$, we conclude $|(Tx, Ty)| = |(x, y)|$, $\forall x, y \in W_e$. Now we can apply the classical Wigner's theorem on the Hilbert space $X$.

Let us denote by $h$ since $e$ orthogonal to $f$ and let $b \in K(H)$ be the partial isometry such that $b^*b = e$, $bb^* = f$. Let us denote by $M_2 \subseteq K(H)$ the algebra spanned by matrix units $e, f, b, b^*$. Now define
\begin{equation}
X = \{ x \in W : \langle x, x \rangle \in M_2 \}.
\end{equation}
Obviously, $X$ is closed and, by (6), contains $W_e$. Denoting $p = e + f$ one finds $px = x$, $\forall x \in X$ and $X = pW$. Now it is easy to conclude that $X$ is a Hilbert module over the matrix algebra $M_2$ with the $M_2$-valued inner product inherited from $W$. We omit the details.

Since $T$ by assumption preserves the inner squares, (8) implies that $X$ is invariant for $T$. Now we apply the Theorem from [9] to the induced function $T_2 = T|X : X \rightarrow X$: there exist a phase function $\varphi_2 : X \rightarrow C$ and a $M_2$-linear (!) isometry $U_2 : X \rightarrow X$ such that $T_2 x = \varphi_2(x) U_2 x$, $\forall x \in X$. By $M_2$-linearity $W_e$ remains invariant under the action of $U_2$ since $W_e = eW$. This shows that the pair $(\varphi_2, U_e)$ from the first part of the proof can actually be chosen by restricting the action of $\varphi_2$ and $U_2$ to $W_e$.

After all, we can apply Theorem 5 from [3] (see also Theorem 1 in [2]): there exists a unique isometry $U \in B_a(W)$ such that $U|W_e = U_e$. Define a new function $T'$ on $W$ putting $T' = U^*T$.

We claim
\begin{equation}
\langle T'v, w \rangle e \langle w, T'v \rangle = \langle v, w \rangle e \langle w, v \rangle, \forall v, w \in W.
\end{equation}
Indeed,
\begin{align*}
\langle T'v, w \rangle e \langle w, T'v \rangle &= \langle T'v, w \rangle e \varphi_e(ew)e \varphi_e(ew)e \langle w, T'v \rangle \\
&= \langle U^*Tv, \varphi_e(ew)ew \rangle \langle \varphi_e(ew)ew, U^*Tv \rangle \\
&= \langle Tv, \langle Tv, ew \rangle \rangle \\
&= \langle v, ew \rangle \langle w, v \rangle = \langle v, w \rangle e \langle w, v \rangle .
\end{align*}
Now observe that each minimal projection $f \in K(H)$ can be written in the form $f = a^*ea$ for a suitably chosen partial isometry $a \in K(H)$. Substituting $aw$ for $w$ in (9) we get
\begin{equation}
\langle T'v, w \rangle f \langle w, T'v \rangle = \langle v, w \rangle f \langle w, v \rangle, \forall v, w \in W.
\end{equation}
Since there exists an approximate unit \((u_{\lambda})\) for \(K(H)\) whose elements are finite linear combinations of minimal projections, this gives
\[
\langle T'v, w \rangle u_{\lambda} \langle w, T'v \rangle = \langle v, w \rangle u_{\lambda} \langle w, v \rangle, \forall \lambda, \forall v, w \in W.
\]
Finally, (11) obviously implies
\[
\langle T'v, w \rangle \langle w, T'v \rangle = \langle v, w \rangle \langle w, v \rangle, \forall v, w \in W.
\]
Now we apply Lemma 1. There exists a complex number (depending on \(v\)) \(\lambda_v =: \varphi(v)\) of modulus 1 such that
\[
T'v = \varphi(v)v \quad \text{or} \quad U^*Tv = \varphi(v)v, \forall v \in W.
\]
This gives
\[
UU^*Tv = \varphi(v)Uv, \forall v \in W
\]
and, because \(UU^*\) is the projection to \(\text{Im} \, U\), the proof will be finished by showing \(T(W) \subseteq \text{Im} \, U\).

To do this, let us take an orthonormal basis \((u_j)\) for \(W\) such that \(\langle u_j, u_j \rangle = e, \forall j\).
(There exists an orthonormal basis for \(W\) with this property by [2], Proposition 2 and [3], Theorem 2.) Observe that the system \((Tu_j)\) is also an orthonormal system in \(W\). Indeed,
\[
\langle Tu_i, Tu_j \rangle = \langle u_i, u_j \rangle = \delta_{ij}e, \forall i, j \quad \text{(with} \delta \quad \text{denoting the Kronecker symbol).}
\]
Moreover, for each \(v \in W\) we have (see [3], Theorem 1)
\[
\langle Tv, Tv \rangle = \langle v, v \rangle = \sum_j \langle v, u_j \rangle \langle u_j, v \rangle = \sum_j |\langle u_j, v \rangle|^2 = \sum_j |\langle Tu_j, Tv \rangle|^2
\]
\[
= \sum_j \langle Tv, Tu_j \rangle \langle Tu_j, Tv \rangle.
\]
This is enough to conclude
\[
T_v = \sum_j \langle Tv, Tu_j \rangle Tu_j.
\]
Notice that \(\langle u_j, u_j \rangle = e\) implies \(u_j = eu_j \in eW = W_e\), hence we may use (7) to rewrite (13):
\[
T_v = \sum_j \langle Tv, \varphi_e(u_j)U_e(u_j) \rangle \varphi_e(u_j)U_e(u_j) = \sum_j \langle Tv, Uu_j \rangle Uu_j
\]
\[
= \sum_j U(\langle Tv, Uu_j \rangle u_j).
\]
Since \(U\) is an isometry, its image is closed. This completes the proof.

**Remark 1.** Let \(W\) be a Hilbert \(C^*\)-module over an arbitrary \(C^*\)-algebra \(A\) of compact operators and let \(T : W \to W\) be a function satisfying condition (1) from Theorem 1.

It is well known ([3], Theorem 1.4.5) that \(A\) must be of the form
\[
A = \bigoplus_{j \in J} K(H_j),
\]
i.e. \(A\) is a direct sum of \(C^*\)-algebras \(K(H_j)\) of all compact operators acting on Hilbert spaces \(H_j, j \in J\). We may assume that \(W\) is a full Hilbert \(A\)-module by dropping unnecessary summands (i.e. those \(K(H_j)\) which act on \(W\) as the zero operator) from the above decomposition of \(A\). For each \(j \in J\) consider the
associated ideal submodule \( W_j = \mathcal{K}(H_j)W \). Notice that \( W_j \), regarded as a Hilbert \( K(H_j) \)-module, is full. Obviously, \((W_j)\) is a family of pairwise orthogonal closed submodules of \( W \) and it is well known (cf. [3]) that \( W \) admits a decomposition into the (outer) direct sum

\[
W = \bigoplus_{j \in J} W_j, \quad W_j = \mathcal{K}(H_j)W.
\]

(15)

Now observe that each \( W_j \) satisfies

\[
W_j = \{ w \in W : \langle w, w \rangle \in \mathcal{K}(H_j) \}.
\]

(16)

Since each function satisfying condition (1) preserves \( \mathcal{A} \)-valued inner squares, (16) shows that each \( W_j \) is invariant for \( T \). Applying Theorem 1 to all induced functions \( T_j = T|W_j : W_j \rightarrow W_j \) we obtain the factorization \( T_j(v) = \varphi_j(v)U_jv, \forall v \in W_j \) on each component \( W_j \).

Observe that the family \((U_j)\) defines an isometry \( U \in \mathcal{B}_a(W) \) ([3], Theorem 8). However, the complex numbers \( \varphi_j(w) \) might be different for fixed \( w \) and varying \( j \in J \), so a unique global choice for \( \varphi \) as a scalar-valued function might be impossible. Therefore we cannot obtain a global factorization for \( T \) as in Theorem 1.

**Remark 2.** Let us note that our Theorem 1 (as well as the above Remark 1) also holds true for functions satisfying condition (1) which are defined on \( H^* \)-modules. Namely, if \( W \) is an arbitrary \( H^* \)-module over the \( H^* \)-algebra of all Hilbert Schmidt operators acting on some Hilbert space, then the same proof applies using the corresponding results concerned with operators on \( H^* \)-modules (cf. [2]).

To conclude, we shall mention a possible extension of Wigner’s theorem to more general Hilbert \( C^* \)-modules. A good candidate is the class of Hilbert \( C^* \)-modules over concrete \( C^* \)-algebras which contain the ideal of all compact operators. There is some evidence along this line.

**Acknowledgment**

The authors wish to thank the referee for several valuable suggestions for improving the original manuscript.

**References**

5. M. Frank, D. R. Larson, *Frames in Hilbert \( C^* \)-modules and \( C^* \)-algebras*, preprint, University of Houston, Houston, and Texas A&M University, College Station, Texas, USA, 1998.

Department of Mathematics, University of Zagreb, Bijenička c. 30, 10000 Zagreb, Croatia

E-mail address: bakic@math.hr

Department of Mathematics, University of Zagreb, Bijenička c. 30, 10000 Zagreb, Croatia

E-mail address: guljas@math.hr