ON THE CELLULAR DECOMPOSITION
OF THE EXCEPTIONAL LIE GROUP $G_2$

MAMORU MIMURA AND TETSU NISHIMOTO

(Communicated by Paul Goerss)

Abstract. The present note is to give a cellular decomposition of the compact
connected exceptional Lie group $G_2$.

1. Introduction

Let us denote by $G_2$ the compact connected exceptional Lie group of rank 2. By
definition $G_2$ is the automorphism group $\text{Aut}(\mathfrak{C})$, where $\mathfrak{C}$ is the Cayley algebra. It
has been long known that $G_2$ has the homotopy type of
$$S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$ The purpose of the present note is to give a cellular decomposition of $G_2$:
$$G_2 = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

2. Preliminary

Let us recall known results on $G_2$ which will be needed later.
The Cayley algebra $\mathfrak{C}$ is isomorphic to $\mathbb{R}^8$ as an $\mathbb{R}$-module and we denote its basis
by $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Notice that the multiplication is not associative.
The element $e_0$ is the unit of the algebra which we denote by 1. The multiplication
of the remaining basis is given in the following diagram:

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Then the exceptional Lie group $G_2$ is defined to be

$$G_2 = \{ g \in SO(8) \mid g(x)g(y) = g(xy), x, y \in \mathfrak{C} \} = \text{Aut}(\mathfrak{C}).$$

Since $g(1) = 1$ for any $g \in G_2$, we may regard $G_2$ as a subgroup of $SO(7)$. So from now on, we express an element of $G_2$ as that of $SO(7)$. The subgroup of elements in $G_2$ fixing $e_1$ is known to be isomorphic to $SU(3)$. The subgroup of elements in $G_2$ fixing $e_1$ and $e_2$ is known to be isomorphic to $SU(2)$. Thus we regard $SU(3)$ and $SU(2)$ as subgroups of $G_2$. Let $S^6$ be the unit sphere of $\mathbb{R}^7$ whose basis is $\{ e_i \mid 1 \leq i \leq 7 \}$ and $S^5$ be the unit sphere of $\mathbb{R}^6$ whose basis is $\{ e_i \mid 2 \leq i \leq 7 \}$. Then there are two principal fiber bundles over them:

$$SU(3) \longrightarrow G_2 \underset{p_1}{\longrightarrow} S^6,$$

$$SU(2) \longrightarrow SU(3) \underset{p_2}{\longrightarrow} S^5,$$

where $p_i(g) = g(e_i)$ for $i = 1, 2$. Let $H$ be the subgroup of $G_2$ defined by

$$H = G_2 \cap (SO(3) \oplus SO(4)).$$

**Lemma 2.1.** $hgh^{-1} \in SU(2)$ for any $h \in H$ and $g \in SU(2)$.

**Proof.** It is obvious, since $SU(2) = G_2 \cap \{1\} \oplus SO(4)$, where $\{1\}$ denotes the subgroup of $SO(3)$ consisting of the identity element.

In the remainder of the section, we will construct cells of $G_2$. Let $D^i$ $(1 \leq i \leq 3)$ be the $i$-dimensional discs defined respectively by

\[
D^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}, \\
D^2 = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\}, \\
D^1 = \{z_1 \in \mathbb{R} \mid z_1^2 \leq 1\}.
\]

We define $V^3$, $V^5$ and $V^6$ as follows:

$$V^3 = D^3, \quad V^5 = D^3 \times D^2, \quad V^6 = D^3 \times D^2 \times D^1,$$

and put $X, Y$ and $Z$ as

$$X = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}, \quad Y = \sqrt{1 - y_1^2 - y_2^2}, \quad Z = \sqrt{1 - z_1^2}.$$
We define three maps $A, B, C$ respectively of $D^i$ to $G_2$ for $1 \leq i \leq 3$ as follows:

$$A(x_1, x_2, x_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 - 2X^2 & -2x_1X & -2x_2X & -2x_3X \\ 2x_1X & 1 - 2X^2 & 2x_3X & -2x_2X \\ 2x_2X & -2x_3X & 1 - 2X^2 & 2x_1X \\ 2x_3X & 2x_2X & -2x_1X & 1 - 2X^2 \end{pmatrix},$$

$$B(y_1, y_2) = \begin{pmatrix} 1 & y_1 & y_2 & -Y & 0 \\ y_2 & y_1 & 0 & -Y & Y \\ Y & 0 & y_1 & y_2 & 0 \\ 0 & Y & -y_2 & y_1 & 1 \end{pmatrix},$$

$$C(z_1) = \begin{pmatrix} z_1 & 0 & -Z \\ 0 & 1 & 0 \\ Z & 0 & z_1 \\ 1 & z_1 & 0 & -Z \\ 0 & 1 & 0 \\ Z & 0 & z_1 \end{pmatrix},$$

where blanks consist of the zero element. After having prepared these definitions, we will construct some cells of $G_2$. Let $\varphi_i$ be a map of $V^i$ to $G_2$ for $i = 3, 5, 6$ defined respectively by

$$\varphi_3(x_1, x_2, x_3) = A(x_1, x_2, x_3),$$

$$\varphi_5(x_1, x_2, x_3, y_1, y_2) = B(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1},$$

$$\varphi_6(x_1, x_2, x_3, y_1, y_2, z_1) = C(z_1)B(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1}C(z_1)^{-1}.$$

We define eight cells $e^j$ for $j = 0, 3, 5, 6, 8, 9, 11, 14$ as follows:

$$e^0 = \{1\}, \quad e^3 = \text{Im } \varphi_3, \quad e^5 = \text{Im } \varphi_5, \quad e^6 = \text{Im } \varphi_6, \quad e^8 = e^5 e^3, \quad e^9 = e^6 e^3, \quad e^{11} = e^6 e^5, \quad e^{14} = e^6 e^5 e^3.$$

We denote the boundary and the interior of a cell $e^i$ simply by $\partial e^i$ and by $\overline{e^i}$ respectively.

3. A cellular decomposition of $SU(3)$

Yokota [Y1, Y3] constructed a cellular decomposition of $SU(n)$. In this section, we reconstruct a cellular decomposition of $SU(3)$ for our purpose, which is essentially the same as Yokota’s decomposition.

As is known, $SU(2)$ is homeomorphic to $S^3$, and hence $e^0 \cup e^3$ is a cellular decomposition of $SU(2)$.

**Lemma 3.1.** The composite map $p_2 \varphi_5 : (V^5, \partial V^5) \to (S^5, \{e_2\})$ is a relative homeomorphism.
Proof. We express the map \((p_2\varphi_5)_{|V^5 \setminus \partial V^5}\) as follows:

\[
\begin{pmatrix}
0 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7
\end{pmatrix}
= p_2\varphi_5(x_1, x_2, x_3, y_1, y_2) = \begin{pmatrix}
0 \\
1 - 2X^2Y^2 \\
2x_1XY^2 \\
2Y(y_1X^2 - x_1y_2X) \\
-2Y(x_1y_1X + y_2X^2) \\
-2x_2XY \\
-2x_3XY
\end{pmatrix},
\]

and hence

\[
\begin{pmatrix}
0 \\
1 - a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7
\end{pmatrix}
= 2XY \begin{pmatrix}
XY \\
x_1Y \\
y_1X - x_1y_2 \\
y_1X + y_2X \\
x_2 \\
x_3
\end{pmatrix}.
\]

Since \(X > 0, Y > 0\) and \(1 - a_2 > 0\), an easy calculation from the second component in the above equation gives the following equation:

\[
XY = \frac{\sqrt{1 - a_2}}{\sqrt{2}},
\]

from which we easily obtain

\[
x_2 = \frac{-a_6}{\sqrt{2(1 - a_2)}},
\]

\[
x_3 = \frac{-a_7}{\sqrt{2(1 - a_2)}}.
\]

Further we obtain two more equalities from the above equation:

\[
(1 - a_2)^2 + a_3^2 = 4X^2Y^4(x_1^2 + X^2),
\]

\[
a_4^2 + a_5^2 = 4X^2Y^2(y_1^2 + y_2^2)(x_1^2 + X^2) = 4X^2Y^2(1 - Y^2)(x_1^2 + X^2).
\]

Using these two equalities, we obtain

\[
Y^2 = \frac{(1 - a_2)^2 + a_3^2}{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}.
\]

It follows from (3.1) and (3.4) that

\[
X^2 = \frac{(1 - a_2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2((1 - a_2)^2 + a_3^2)}.
\]

It follows from (3.2), (3.3) and (3.5) that

\[
x_1^2 = \frac{a_2((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2(1 - a_2)((1 - a_2)^2 + a_3^2)}.
\]

Since \(Y > 0\), (3.4) implies that

\[
Y = \frac{\sqrt{(1 - a_2)^2 + a_3^2}}{\sqrt{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}}.
\]
Since \( X > 0 \), (3.5) implies that
\[
X = \frac{\sqrt{(1 - a_2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}{2((1 - a_2)^2 + a_3^2)}.
\]
Since the signs of \( x_1 \) and \( a_3 \) are the same, (3.6) implies that
\[
x_1 = \frac{a_3\sqrt{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}}{2(1 - a_2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}.
\]
Now we determine \( y_2 \); we have
\[
a_4x_1 + a_5X = -2XY(x_1^2 + X^2)y_2.
\]
Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain
\[
y_2 = \frac{-a_3a_4 - (1 - a_2)a_5}{((1 - a_2)^2 + a_3^2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}.
\]
Finally we determine \( y_1 \); we have
\[
a_4X - a_5x_1 = 2XY(x_1^2 + X^2)y_1.
\]
Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain
\[
y_1 = \frac{(1 - a_2)a_4 - a_3a_5}{((1 - a_2)^2 + a_3^2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}.
\]
Thus we have expressed \( x_1, x_2, x_3, y_1, y_2 \) in terms of \( a_2, \ldots, a_7 \), that is, the inverse map has been constructed, which completes the proof. \( \square \)

**Proposition 3.2.** \( e^0 \cup e^3 \cup e^5 \cup e^8 \) is a cellular decomposition of \( SU(3) \).

**Proof.** First we will show that \( \mathcal{e}^i \cap \mathcal{e}^j = \emptyset \) if \( i \neq j \). We consider the following three cases:

1. For the case where \( i = 0 \) and \( j = 3 \), it is obvious that \( \mathcal{e}^0 \cap \mathcal{e}^3 = \emptyset \) since \( e^0 \cup e^3 \) is a cellular decomposition of \( SU(2) \).
2. For the case where \( i \in \{0,3\} \) and \( j \in \{5,8\} \), we have \( p_2(\mathcal{e}^i) = e_2 \) and \( p_2(\mathcal{e}^j) = S^5 \setminus \{e_2\} \). Then we have \( \mathcal{e}^i \cap \mathcal{e}^j = \emptyset \).
3. For the case where \( i = 5 \) and \( j = 8 \), suppose that \( A \in \mathcal{e}^5 \cap \mathcal{e}^8 \). Since \( \mathcal{e}^8 = \mathcal{e}^5 \mathcal{e}^3 \), we can put \( A = A_1A_2 \) where \( A_1 \in \mathcal{e}^5 \) and \( A_2 \in \mathcal{e}^3 \). We have \( A = A_1 \) since \( p_2(A) = p_2(A_1A_2) = p_2(A_1) \) and \( p_2|_{\mathcal{e}^5} \) is monic. Then we have \( A_2 = 1 \in \mathcal{e}^3 \), which is a contradiction. Thus \( \mathcal{e}^5 \cap \mathcal{e}^8 = \emptyset \).

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. It is obvious that the boundary \( \mathcal{e}^3 \) is included in \( e^0 \). Observe that the boundary \( \mathcal{e}^5 \) is a union of the following two sets:
\[
\{BAB^{-1} \mid A \in A(\mathcal{D}^1), B \in B(D^2)\}, \quad \{BAB^{-1} \mid A \in A(D^3), B \in B(\mathcal{D}^2)\}.
\]
The first set contains only the identity element since \( A \) is the identity element. Lemma 2.1 implies that the second set is contained in \( SU(2) \) since \( B \) is contained in \( H \). Thus we have \( \mathcal{e}^5 \subset \mathcal{e}^3 \). Further we have \( \mathcal{e}^8 = \mathcal{e}^5 \mathcal{e}^3 \mathcal{e}^5 \mathcal{e}^3 \subset \mathcal{e}^2 \mathcal{e}^3 \mathcal{e}^5 \mathcal{e}^0 = e^3 \mathcal{e}^5 \).

Finally, we will show that the inclusion map \( e^0 \cup e^3 \cup e^5 \cup e^8 \to SU(3) \) is epic. Let \( g \in SU(3) \). If \( p_2(g) = e_2 \), then \( g \) is contained in \( SU(2) = e^0 \cup e^3 \). Suppose that \( p_2(g) \neq e_2 \). There is an element \( h \in e^5 \) such that \( p_2(h) = p_2(g) \). Then we have
$h^{-1}g \in SU(2) = e^0 \cup e^3$, since $p_2(h^{-1}g) = e_2$. Therefore we have $g \in h(e^0 \cup e^3) \subset e^0 \cup e^3 \cup e^5 \cup e^8$.

4. A cellular decomposition of $G_2$

First we need to show

**Lemma 4.1.** The composite map $p_1\varphi_6 : (V^6, \partial V^6) \to (S^6, \{e_1\})$ is a relative homeomorphism.

**Proof.** We express the map $(p_1\varphi_6)|_{V^6 \setminus \partial V^6}$ as follows:

\[
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ \end{pmatrix} = p_1\varphi_6(x_1, x_2, x_3, y_1, y_2, z_1) = \begin{pmatrix} 1 - 2X^2Y^2Z^2 \\ 2x_1XYZ \\ 2z_1X^2Y^2Z \\ -2XYZ(x_1y_1 + y_2X) \\ -2XYZ(y_1z_1X - x_1y_2z_1 + x_2Z) \\ -2XYZx_3 \\ -2XYZ(y_1XZ - x_1y_2Z - x_2z_1) \\ \end{pmatrix},
\]

and hence

\[
\begin{pmatrix} 1 - a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ \end{pmatrix} = 2XYZ \begin{pmatrix} XYZ \\ x_1Y \\ z_1XY \\ -x_1y_1 - y_2X \\ -y_1z_1X + x_1y_2z_1 - x_2Z \\ -x_3 \\ -y_1XZ + x_1y_2Z + x_2z_1 \\ \end{pmatrix}.
\]

We set for simplicity

\[
\begin{align*}
\alpha_1 &= (1 - a_1)^2 + a_3^2, \\
\alpha_2 &= (1 - a_1)^2 + a_2^2 + a_3^2, \\
\alpha_3 &= (1 - a_1)^2 + a_2^2 + a_3^2 + a_4^2, \\
\beta_1 &= a_3a_5 + (1 - a_1)a_7, \\
\beta_2 &= a_2a_4 + a_3a_5 + (1 - a_1)a_7.
\end{align*}
\]

Since $1 - a_1 > 0$, we have $\alpha_i > 0$ for $i = 1, 2, 3$. By an easy calculation one can obtain the following three equations:

\[
\begin{align*}
Z^2 &= \frac{(1 - a_1)^2}{\alpha_1}, \\
X^2Y^2 &= \frac{1 - a_1}{2Z^2} = \frac{\alpha_1}{2(1 - a_1)}, \\
z_1^2 - 1 &= Z^2 = \frac{a_2^2}{\alpha_1}.
\end{align*}
\]

Since $Z \geq 0$ and $1 - a_1 \geq 0$, (4.1) implies that

\[
Z = \frac{1 - a_1}{\sqrt{\alpha_1}}.
\]

Since the signs of $z_1$ and $a_3$ are the same, (4.3) implies that

\[
z_1 = \frac{a_3}{\sqrt{\alpha_1}}.
\]
We easily have
\[(4.6) \quad x_3 = \frac{-a_6}{2YZ} = \frac{-a_6}{\sqrt{2(1 - a_1)}}.\]

Next we determine \(X\) and \(Y\); we have
\[(a_1^2 + (a_5 z_1 + a_7 Z)^2)Y^2 = 4X^4Y^2 Z^2(y_1^2 + y_2^2)(x_1^2 + X^2),\]
\[a_2^2(1 - Y^2) = 4X^2Y^4Z^2(1 - Y^2)x_1^2.\]

It follows from these two equalities that
\[(a_1^2 + (a_5 z_1 + a_7 Z)^2)Y^2 - a_2^2(1 - Y^2) = 4X^4Y^2 Z^2(1 - Y^2).\]

Substituting the equations \((4.2)\), \((4.4)\) and \((4.5)\) in the above equation, we obtain
\[a_4^2Y^2 + \left(\frac{a_3 a_5 + (1 - a_1)a_7}{a_1}\right)^2 Y^2 + a_2^2Y^2 - a_2^2 = \alpha_1(1 - Y^2),\]
whence we have
\[Y^2 = \frac{\alpha_1 \alpha_2}{\alpha_1 \alpha_3 + \beta_1^2}.\]

Since \(Y \geq 0\), we have
\[(4.7) \quad Y = \frac{\sqrt{\alpha_1 \alpha_2}}{\sqrt{\alpha_1 \alpha_3 + \beta_1^2}}.\]

Since \(X \geq 0\), \((4.2)\) and \((4.7)\) imply that
\[(4.8) \quad X = \frac{\sqrt{\alpha_1 \alpha_3 + \beta_1^2}}{\sqrt{2(1 - a_1)\alpha_2}}.\]

We easily have
\[(4.9) \quad x_1 = \frac{a_2}{2XY^2 Z} = \frac{a_2}{\sqrt{2(1 - a_1)\alpha_1 \alpha_2}}.\]

Now we determine \(y_1\) and \(y_2\); we have
\[a_4 x_1 + a_5 z_1 X + a_7 X Z = -2XY Z(x_1^2 + X^2)y_1,\]
into which substituting \((4.3)\), \((4.5)\), \((4.7)\), \((4.8)\) and \((4.9)\) we obtain
\[(4.10) \quad y_1 = \frac{-\beta_2 \sqrt{\alpha_1}}{\sqrt{2(1 - a_1)\alpha_1 \alpha_2}}.\]

Quite similarly the equation
\[a_4 X - a_5 x_1 z_1 - a_7 x_1 Z = -2XY Z(x_1^2 + X^2)y_2\]
gives rise to
\[(4.11) \quad y_2 = \frac{a_2 \beta_1 - a_4 \alpha_1}{\sqrt{2(1 - a_1)\alpha_1 \alpha_2}}.\]

Finally we determine \(x_2\); we have
\[a_5 Z - a_7 z_1 = -2XY Z x_2,\]
which gives

\[ x_2 = \frac{a_3a_7 - (1 - a_1)a_5}{\sqrt{2(1 - a_1)a_1}}. \]

Thus we have expressed \( x_1, x_2, x_3, y_1, y_2, z_1 \) in terms of \( a_1, \ldots, a_7 \), that is, the inverse map has been constructed and this completes the proof. \( \square \)

The following is our main result.

**Theorem 4.2.** The cell complex \( e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14} \) thus constructed gives a cellular decomposition of \( G_2 \).

**Proof.** First we show that \( \dot{e}^i \cap \dot{e}^j = \emptyset \) if \( i \neq j \). We consider the following three cases:

1. For the case where \( i, j \in \{0, 3, 5, 8\} \), both cells \( \dot{e}^i \) and \( \dot{e}^j \) are in \( SU(3) \) and \( e^0 \cup e^3 \cup e^5 \cup e^8 \) is a cellular decomposition of \( SU(3) \). Then we have \( \dot{e}^i \cap \dot{e}^j = \emptyset \) if \( i \neq j \).
2. For the case where \( i \in \{0, 3, 5, 8\} \) and \( j \in \{6, 9, 11, 14\} \), we have \( p_1(\dot{e}^i) = \{e_1\} \) and \( p_1(\dot{e}^j) = S^0 \setminus \{e_1\} \). Then we have \( \dot{e}^i \cap \dot{e}^j = \emptyset \).
3. For the case where \( i, j \in \{6, 9, 11, 14\} \), suppose that \( A \in \dot{e}^i \cap \dot{e}^j \). Since \( \dot{e}^i = \dot{e}^6 \dot{e}^0 \dot{e}^3 \) and \( \dot{e}^j = \dot{e}^6 \dot{e}^0 \dot{e}^3 \), we can put \( A = A_1A_2 = A'_1A'_2 \) where \( A_1, A'_1 \in \dot{e}^6 \), \( A_2, A'_2 \in \dot{e}^0 \dot{e}^3 \). We have \( A_1 = A'_1 \), since \( p_1(A_1) = p_1(A'_1) = p_1(A'_1A'_2) = p_1(A_1) \) and \( p_1_{|\dot{e}^6} \) is monic. Then we have \( A_2 = A'_2 \) and the first case shows that \( i - 6 = j - 6 \), that is, \( i = j \). Thus \( \dot{e}^i \cap \dot{e}^j = \emptyset \) if \( i \neq j \).

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In the proof of Proposition 3.2 it is proved that the boundaries \( \dot{e}^3, \dot{e}^5 \) and \( \dot{e}^8 \) are included in the lower dimensional cells. Observe that the boundary \( \dot{e}^6 \) is a union of the following three sets:

\[ \{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(D^1)\}, \]
\[ \{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(D^1)\}, \]
\[ \{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(D^1)\}. \]

The first set contains only the identity element, since \( A \) is the identity element. Lemma 2.1 implies that the second set is contained in \( SU(2) \), since \( B \) and \( C \) are contained in the subgroup \( H \). We consider the third set. If \( C = C(1) = 1 \), it is obvious that \( CBAB^{-1}C^{-1} = BAB^{-1} \in \dot{e}^5 \). Suppose that \( C = C(-1) \). It is easy to check that

\[ CB(y_1, y_2)C^{-1} = B(y_1, -y_2), \]
\[ CA(x_1, x_2, x_3)C^{-1} = A(-x_1, x_2, -x_3). \]

Thus the third set is contained in \( e^5 \), since we have

\[ CB(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1}C^{-1} \]
\[ =(CB(y_1, y_2)C^{-1})(CA(x_1, x_2, x_3)C^{-1})(CB(y_1, y_2)^{-1}C^{-1}) \]
\[ =B(y_1, -y_2)A(-x_1, x_2, -x_3)B(y_1, -y_2)^{-1}. \]

We have \( \dot{e}^3 = e^6e^3 \cup e^6e^3 \subset e^6 \cup e^6e^3 = e^6 \cup e^8 \). We also have \( \dot{e}^{11} = e^6e^5 \cup e^6e^5 \subset e^5e^5 \cup e^5e^3 = e^5 \cup e^9 \), and \( \dot{e}^{14} = e^6e^5e^3 \cup e^6e^5e^3 \cup e^6e^5e^3 \subset e^5e^5e^3 \cup e^6e^5e^3 \cup e^6e^5 = e^8 \cup e^9 \cup e^{11} \).
Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14} \to G_2$ is epic. Let $g \in G_2$. If $p_1(g) = e_1$, then $g$ is contained in $SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$. Suppose that $p_1(g) \neq e_1$. There is an element $h \in e^6$ such that $p_1(h) = p_1(g)$. Thus we have $h^{-1} g \in SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$ since $p_1(h^{-1} g) = e_1$. Therefore we have $g \in h(e^0 \cup e^3 \cup e^5 \cup e^8) \subset e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$. 

References


