A COMBINATORIAL PROOF OF ANDREWS’ PARTITION FUNCTIONS RELATED TO SCHUR’S PARTITION THEOREM

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Abstract. We construct an involution to show equality between partition functions related to Schur’s second partition theorem.

1. Introduction

In 1926, I. Schur [2] proved the following theorem on partitions.

Theorem 1. Let $A(n)$ be the number of partitions of $n$ into parts congruent to 1 or 5 (mod 6), $B(n)$ the number of partitions of $n$ into distinct nonmultiples of 3, and $D(n)$ the number of partitions of $n$ of the form $b_1 + b_2 + \cdots + b_s$ such that $b_i - b_{i+1} \geq 3$ with strict inequality if $3 \mid b_i$. Then

$$A(n) = B(n) = D(n).$$

G. E. Andrews [1] found two partition functions equal to the partition functions in Schur’s Theorem. One is $C(n)$, the number of partitions of $n$ into odd parts, none appearing more than twice, and the other is $E(n)$, the number of partitions of $n$ in which no part appears more than twice, odd parts appear at most once, the difference between two parts can never be 1, and can be 2 only if both are odd, with weight $(−1)^e$ when partitions have exactly $e$ different parts that appear twice.

In the sequel, we call partitions enumerated by $X(n)$ partitions of type $X$ for $X = A, B, C, D, E$.

The equality of $C(n)$ with one of the partition functions in Schur’s Theorem can be easily obtained from their generating functions. However, the case of $E(n)$ is quite obscure and mysterious. Andrews first showed that $c_n(q)$, the generating function of partitions of type $E$ whose parts are less than or equal to $n$, satisfies the equality

$$\lim_{n \to \infty} \frac{c_{2n-1}(q)}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1 + q^{2n+1} + q^{4n+2}}{1 - q^{2n+2}},$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. From this it can be shown that the generating function of $E(n)$ is the same as that of $C(n)$.
At the conclusion of his paper, Andrews [1] asked, “First, is there a purely combinatorial or bijective way of proving that $E(n)$ equals any of the other partition functions in Theorem 1?” In Section 2, we show that $D(n) = E(n)$ by constructing an involution in the set of partitions of type $E$ whose invariant set is the set of partitions of type $D$.

Furthermore, Andrews [1, Theorem 2] gave a relation between polynomial generating functions $d_n(q)$ and $e_n(q)$, where $d_n(q)$ is the generating function for all partitions of type $D$ whose parts are less than or equal to $n$.

In the conclusion of his paper, Andrews confessed, “Theorem 2 is still rather a mystery. The proof is purely a verification. Is there an underlying partition-theoretic explanation of Theorem 2?” In Section 3, using the involution we construct in Section 2, we provide a combinatorial proof of his Theorem 2.

2. Relation between $D(n)$ and $E(n)$

In this section, we will prove the following theorem:

**Theorem 2.** For all $n$

$$D(n) = E(n).$$

We can easily check that the partitions of type $D$ satisfy the condition for partitions of type $E$. In other words, there is a sign reversing involution $v$ in the set of partitions of type $E$ such that $v$ is the identity map under the set of partitions of type $D$. Now, let us describe our involution $v$. In this paper, we assume that the parts of a partition are ordered weakly decreasing.

**Proof of Theorem 2.** Let a partition $\lambda = \lambda_1\lambda_2 \cdots \lambda_l$ of type $E$ be given. Assume that $\lambda_0 = \infty$ and $\lambda_{l+1} = -\infty$ for convention. First, consider the largest pair $\lambda_i, \lambda_{i+1}$ among the parts that appear twice and are less than $\lambda_{i-1}$ by at least four, and consider the largest pair $\lambda_j, \lambda_{j+1}$ among the consecutive odd parts. If $\lambda_i > \lambda_j$, then add one to $\lambda_i$ from $\lambda_{i+1}$. Otherwise, add one to $\lambda_j$ from $\lambda_{j+1}$. Then the signs of our new and old partitions are different, and so they are cancelled out in $E(n)$.

Since we choose the largest pairs in both cases, our map is reversible. The only remaining partitions are those whose parts differ by at least 3 or which have three consecutive parts $2k + 3, 2k, 2k$. We need to map a partition with parts appearing twice to a partition having consecutive multiples of 3 as parts.

Now, we consider pairs of parts which are consecutive multiples of 3, and let $\lambda_l = 3m, \lambda_{l+1} = 3m - 3$ be the largest pair. Also, let $\lambda_s = \lambda_{s+1} = 2k$ be the largest even part appearing twice. Since $\lambda_s$ is the largest, all parts greater than $\lambda_s$ differ by at least 3, so we can write parts greater than $\lambda_s$ as

$$(1) \quad \lambda_{s-1} = \lambda_s + 3i + r_i, \quad r_i \geq 0, \quad \text{for } i \geq 0.$$ 

Then $r_0 = r_1 = 0$ since $\lambda_{s-1} = \lambda_s + 3$, and $r_s = \infty$. Let $a$ be the smallest nonnegative integer $i$ satisfying

$$\lambda_s + 3i + k = 3k + 3i < \lambda_{s-i-1} - 3,$$

i.e., $k < r_{i+1}$. Compare $3(k + a)$ and $3m$. We consider two cases: (i) $3(k + a) \geq 3m$ and (ii) $3(k + a) < 3m$. 

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Case (i): $3(k + a) \geq 3m$. In this case, we define our new partition $\mu$ as follows:

$$
\mu_{s-i} = \begin{cases} 
\lambda_{s-i+1} & \text{if } i \geq -1, \\
\lambda_s + 3i + r_{i+2} & \text{if } 0 \leq i \leq a - 2, \\
\lambda_s + 3i + k & \text{if } i = a - 1, a, \\
\lambda_s - i, & \text{otherwise.}
\end{cases}
$$

Since we removed the part $\lambda_{s+1} = 2k$, our new partition $\mu$ has an opposite sign to $\lambda$. Also, we can easily check that $\mu_{s-a}$ not only is greater than $\mu_{s-a+1}$ by exactly 3 but also is a multiple of 3: $\mu_{s-a} = 3(k + a)$ and $\mu_{s-a+1} = 3(k + a) - 3$.

Case (ii): $3(k + a) < 3m$. In this case, as we did above, we first write parts less than $\lambda$ as

$$
\lambda_{t+i} = \lambda_t - 3i - \tilde{r}_i, \quad \tilde{r}_i \geq 0, \quad \text{for } i \geq 0.
$$

Then $\tilde{r}_0 = \tilde{r}_1 = 0$ since $\lambda_{t+1} = \lambda_t - 3$, and $\tilde{r}_{t+1-i} = \infty$. Let us compare $2(m - i)$ and $\lambda_{t+i+1}$ for the nonnegative integer $i$, and let $b$ be the smallest $i$ satisfying $2(m - i) \geq \lambda_{t+i+1} + 3$, i.e., $\tilde{r}_{t+1} \geq m - i$. We define $\mu$ as follows:

$$
\mu_{t+i} = \begin{cases} 
\lambda_{t+i-1}, & \text{if } i \geq b + 2, \\
2(m - b), & \text{if } i = b + 1, \\
\lambda_t - 3i - (m - i), & \text{if } i = b - 1, b, \\
\lambda_t - 3i - \tilde{r}_{i+2} & \text{if } 0 \leq i \leq b - 2, \\
\lambda_{t+i}, & \text{otherwise.}
\end{cases}
$$

Since $\mu_{t+b-1} = 2(m - b) + 3$ and $\mu_{t+b} = \mu_{t+b+1} = 2(m - b)$, our new partition $\mu$ has exactly one more part appearing twice than $\lambda$ does.

Now, we show that $v$ is reversible. Assume that we get a new partition $\mu$ from Case (i). As we noted, $\mu_{s-a} = 3(k + a)$ and $\mu_{s-a+1} = 3(k + a) - 3$ are a pair of consecutive multiples of 3. Let us show that $\mu_{s-a}$ and $\mu_{s-a+1}$ are the largest pair of consecutive multiples of 3. Assume that there is a pair $\mu_i$ and $\mu_{i+1}$ of consecutive multiples of 3 for an $i < s - a$. This implies that $\mu_i > \mu_{s-a}$. However, this is a contradiction, since $\mu_i = \lambda_t$ and $\mu_{s-a} = 3(k + a)$, i.e., the larger one of the largest consecutive multiples of 3 in $\lambda$. So, $\mu_{s-a}$ and $\mu_{s-a+1}$ are the largest pair of consecutive multiples of 3.

We need to investigate the largest triple with even parts appearing twice in $\mu$; let $\mu_{s'-1} = 2k' + 3$ and $\mu_{s'} = \mu_{s'+1} = 2k'$ be the largest triple. Let us write the parts larger than $\mu_{s'}$ as

$$
\mu_{s'-i} = \mu_{s'} + 3i + r'_{i}
$$

as we did in (1) and let $a'$ be the smallest integer $i$ satisfying $k' < r'_{i+1}$. What we want to show is that $3(k + a) > 3(k' + a')$. Then we can apply Case (ii).

From the maximality of $\lambda_s$ and the definition of $v$, $\lambda_s > \lambda_{s'+1} = \mu_{s'}$, i.e., $2k > 2k'$. In fact, we can see that $2k - 2k' > 3(s' - s)$ since $\mu_{s'} = \lambda_{s'+1}$ and all parts between $\lambda_{s+1}$ and $\lambda_{s'+1}$ differ by at least 3. From (5),

$$
\mu_{s-a} = \mu_{s'} + 3(s' - s + a) + r'_{s'-s+a}.
$$

Since $\mu_{s-a} = 3(k + a)$, $\mu_{s'} = 2k'$ and $2k - 2k' \geq 3(s' - s)$, we get

$$
3(k + a) = \mu_{s'} + 3(s' - s) + 3a + r'_{s'-s+a} \leq \mu_{s'} + 2k - 2k' + 3a + r'_{s'-s+a} = 2k + 3a + r'_{s'-s+a}.
$$
This implies that \( k \leq r'_3-s+a \), and since \( k' < k \), we get \( k' < r'_3-s+a \). From the definition of \( a' \), \( a' \leq s'-s+a-1 \). So,

\[
\mu_{s-a} = \mu_{s'} + 3(s'-s+a) + r'_{s'-s+a} > 2k' + 3a' + k',
\]
i.e., \( 3(k+a) > 3(k'+a') \). Thus the partition \( \mu \) satisfies the condition for applying Case (ii).

Now, we write parts of \( \mu \) less than \( \mu_{s-a} \) as in (3):

\[
\mu_{s-a+j} = \mu_{s-a} - 3j - \tilde{r}_j.
\]

To determine \( b \), we need to compare \( \mu_{s-a+j+1} \) and \( 2(k+a-j) \). Let us consider the case when \( j = 0 \). Since \( \mu_{s-a+1} = 3(k+a) > 2(k+a) \), \( b \) must be greater than 0. Let us suppose that there is an \( h \) for \( 1 \leq h < a \) satisfying

\[
\mu_{s-a+h+1} + 3 \leq 2(k+a-h).
\]

From (2) and (4), we get that \( \tilde{r}_j = k - r_{a-j+2} \) for \( 2 \leq j \leq a \), so (2) becomes \(-r_{a-h+1} = a-h \); it is impossible since \( r_{a-h+1} \geq 0 \) and \( a > h \). On the other hand, \( \mu_{s+1} = \lambda_{s+2} \). From this, \( \mu_{s+1} + 3 \leq \lambda_a = 2k \). Hence, \( b \) is replaced by \( a \). Using (4) we have produced the required partition \( \lambda \).

Similarly, we can show that \( \lambda \) is also restored when \( \mu \) is obtained from \( \lambda \) under Case (ii).

**Example.** Let us consider some partitions of 38:

\[
\begin{align*}
21 + 8 + 4 + 4 + 1 & \quad \overset{v}{\rightarrow} \quad 21 + 8 + 5 + 3 + 1, \\
15 + 9 + 6 + 6 + 2 & \quad \overset{v}{\rightarrow} \quad 15 + 12 + 9 + 2.
\end{align*}
\]

In the first example, the partition \( 21 + 8 + 4 + 4 + 1 \) has an even part appearing twice and the part 8 is greater than 4+3, so we get \( 21 + 8 + 5 + 3 + 1 \) by adding 1 to the third part 4 from the following part 4. Also, the partition \( 21 + 8 + 5 + 3 + 1 \) has the consecutive odd parts 5,3, so by adding 1 to the part 3 from the part 5, we get the partition \( 21 + 8 + 4 + 4 + 1 \). However, these two partitions \( 21 + 8 + 4 + 4 + 1 \) and \( 21 + 8 + 5 + 3 + 1 \) have opposite signs. In other words, they are cancelled in \( E(38) \).

The partition \( 15 + 9 + 6 + 6 + 2 \) in the second row has even part 6 appearing twice, but the part 9 is greater than 6 by exactly 3, i.e., \( s = 3 \). Also the parts 9 and 6 are consecutive multiples of 3, i.e., \( t = 2 \), but \( 9 + 3 \geq 9 \). So, we can apply Case (i); \( s = 3 \) and \( k = 3 \). Let us write the part 15 as

\[
15 = 6 + 3 \times 2 + 3.
\]

Since \( r_2(= 3) \) is not greater than \( k(= 3) \), \( a = 2 \). We add \( r_2(= 3) \) to the part 6 from 15, and then \( k(= 3) \) to \( 12 (= 15 - r_2) \) and 9. Thus we get the partition \( 15 + 12 + 9 + 2 \). On the other hand, the partition \( 15 + 12 + 9 + 2 \) has two parts 15,12 which are consecutive multiples of 3; \( t = 1 \) and \( m = 5 \). Let us write parts 9 and 2 as

\[
\begin{align*}
9 &= 15 - 2 \times 3 - 0, \\
2 &= 15 - 3 \times 3 - 4,
\end{align*}
\]
i.e., \( \tilde{r}_2 = 0 \) and \( \tilde{r}_3 = 4 \). From this, \( \tilde{r}_2(= 0) < m - 1 = 4 \), but \( \tilde{r}_3(= 4) \geq m - 2 = 3 \). Thus 2 becomes \( b \). We get the partition \( 15 + 9 + 6 + 6 + 2 \) from (4). They are also cancelled in \( E(38) \) due to their signs.
Example. There are 7 partitions of 9 of type \(E\), and they are mapped by the involution \(v\) as follows:

\[
\begin{align*}
9 & \iff 9, \\
8 + 1 & \iff 8 + 1, \\
7 + 2 & \iff 7 + 2, \\
6 + 3 & \iff 5 + 2 + 2, \\
5 + 3 + 1 & \iff 4 + 4 + 1.
\end{align*}
\]

The first three partitions 9, 8 + 1, and 7 + 2 have no parts which are even appearing twice, consecutive odds, or consecutive multiples of 3. The set of partitions of 9 of type \(D\) consists of the partitions 9, 8 + 1, and 7 + 2. Hence, \(D(9) = E(9)\).

3. Relation between \(d_n(q)\) and \(e_n(q)\)

Now, let us restrict the size of parts. Then some partitions of type \(E\) do not have their images under the involution \(v\) we constructed in Section 2. We investigate what partitions that are not of type \(D\) still remain under \(v\).

Recall the definitions of \(d_n(q)\) and \(e_n(q)\): the generating functions for all partitions of type \(D\) and \(E\) whose parts are less than or equal to \(n\), respectively. For convention, we let \(d_n = e_n = 1\) for \(n \leq 0\).

**Theorem 3** (Andrews [1]). If \(d_m(q)\) is defined for all \(m\), then

\[
\begin{align*}
(8) \quad d_{2n-1}(q) &= \sum_{j \geq 0} q^{6nj-6j^2+3j-6} \frac{\binom{1}{j}}{j!} e_{2n-6j-1}(q), \\
(9) \quad d_2(q) &= \sum_{j \geq 0} q^{6nj-6j^2+3j-6} \frac{\binom{2}{j}}{j!} \left( e_{2n-6j-1}(q) + q^{2n-6j} e_{2n-6j-3}(q) \right),
\end{align*}
\]

where

\[
\binom{x}{y}_q = \begin{cases} 
\frac{(q;q)_x}{(q;q)_y (q;q)_{x-y}}, & \text{if } 0 \leq x \leq y, \\
0, & \text{otherwise}.
\end{cases}
\]

In preparation for the proof of Theorem 3, let \(\mathcal{P}(E;2n-1)\) be a set of partitions of type \(E\) with parts less than or equal to \(2n-1\). From the definition of \(v\), partitions with consecutive odd parts or equal even parts less than previous parts by at least 4 are cancelled out in \(e_{2n-1}(q)\). Hence, we only consider the subset of \(\mathcal{P}(E;2n-1)\) which includes only partitions whose parts differ by at least 3 or which have three consecutive parts \(2k + 3, 2k, 2k\). In the sequel, we regard \(\mathcal{P}(E;2n-1)\) as the very subset. Among partitions in \(\mathcal{P}(E;2n-1)\), if, under the mapping \(v\), the number of even parts appearing twice increases, then the partitions have their images in \(\mathcal{P}(E;2n-1)\) since their parts would not exceed \(2n-1\) under \(v\). Hence, they are cancelled in \(e_{2n-1}(q)\); we only consider partitions with even parts appearing twice whose number decreases under \(v\).

**Proof of Theorem 3.** We only show the case \(2n - 1\). Let \(\mathcal{P}(D;2n-1)\) be a set of partitions of type \(D\) with parts less than or equal to \(2n-1\). Let us consider three cases: \(n = 3m, 3m + 1\) and \(3m + 2\).
Case (i): $n = 3m$. In this case, equation (8) becomes

$$d_{6m-1} = e_{6m-1} + \sum_{j \geq 1} q^{12mj-6j^2+3j} \binom{m}{j} q^j e_{6m-6j-1}(q).$$

First, let us consider what partitions do not have their images in $\mathcal{P}(E; 6m - 1)$. Suppose that a partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$ is in $\mathcal{P}(E; 6m - 1)$ with three consecutive parts $2k + 3, 2k, 2k$, and $v(\lambda) = \mu$ such that $\mu_1 = 3k_1 + 3$ and $\mu_2 = 3k_1$. Then from the definition of $v$, either $\mu_3 = \lambda_3$ or $\mu_3 = \lambda_1 - 6$. Hence the partition $\mu_3 \mu_4 \cdots \mu_{l-1}$ is in $\mathcal{P}(E; 6m - 7)$. When $k_1 \geq 2m - 1$, we can see that $3k_1 + 3 \geq 6m$. In other words, partitions with these three consecutive parts have no images in $\mathcal{P}(E; 6m - 1)$. Also, since $2k_1 + 3 \leq 6m - 1$, $2m - 1 \leq k_1 \leq 3m - 2$. Let $\mathcal{P}_1$ be the set of all partitions $\mu$ satisfying $\mu_1 = 3k_1 + 3, \mu_2 = 3k_1$ for $2m - 1 \leq k_1 \leq 3m - 2$ and $\mu_3 \cdots \mu_{l-1} \in \mathcal{P}(E; 6m - 7)$. Then the generating function for partitions in $\mathcal{P}_1$ is

$$q^{12m-3} \binom{m}{1} q^1 e_{6m-7}(q).$$

Unfortunately, the set of images of that kind of partition in $\mathcal{P}(E; 6m - 1)$ is a subset of $\mathcal{P}_1$. To determine those partitions in $\mathcal{P}_1$ which do not have their image under $v$ in $\mathcal{P}(E; 6m - 1)$, let us apply $v$ to a partition $\nu \in \mathcal{P}_1$, where $\nu_1 = 3k_1 + 3, \nu_2 = 3k_1$. Let $k'$ be a positive integer such that $\nu$ has three consecutive parts $2k' + 3, 2k', 2k'$. From the definition of $v$, if $3(k' + a') \geq 3k_1$, where $a'$ is the same as $a$ in the definition of $v$, then the image of $\nu$ is not in $\mathcal{P}(E; 6m - 1)$. In this case, we have to apply Case (i) of $v$. In other words, if we let $\sigma$ be the image of $\nu$, then the $\sigma_i$ are multiples of 3 for $i = 1, 2, 3, 4$, and $\sigma_5 \cdots \sigma_{\ell(\nu) - 1}$ is in $\mathcal{P}(E; 6m - 13)$, where $\ell(\nu)$ is the number of parts of $\nu$. Since $\nu_1 = 3k_1 + 3, \nu_2 = 3k_1$ and $\nu_2 \neq \nu_3$, the part $2k' + 3 \leq \nu_3$, so $a'$ must be greater than 1. Let us replace $k' + a' - 2$ by $k_2$. This implies that if $k_1 - 2 \leq k_2 \leq 3m - 5$ and $2m - 1 \leq k_1 \leq 3m - 2$, then these partitions do not have their images in $\mathcal{P}(E; 6m - 1)$. Let us denote the set of all those partitions by $\mathcal{P}_2$, and then calculate the generating function. By substituting $k_1$ for $k_1 - 2$, we find that $2m - 3 \leq k_1 \leq k_2 \leq 3m - 5$. Using Gaussian coefficients, we deduce that

$$\sum_{k_1, k_2} q^{6(k_1+1)+3+6k_2+3} = q^{12m-3+12m-15} \binom{m}{2} q^{-3}.$$

Therefore, the generating function for partitions in $\mathcal{P}_2$ is

$$q^{12m-3+12m-15} \binom{m}{2} q^{-3} e_{6m-13}(q),$$

which is the same as the third term on the right side of (10). However, again, some partitions in $\mathcal{P}_2$ do not have their images in $\mathcal{P}_1$. Let us consider the images of that kind of partition $\sigma \in \mathcal{P}_2$. In similar way, we find that if $\sigma$ has three consecutive parts $2k'' + 3, 2k'', 2k''$ for $k'' \leq 3m - 8$ such that $3(k'' + a'') \geq 3k_2$, where $a''$ is the same as $a$ in the definition of $v$, then we have to apply Case (i) of $v$. In other words, the image of $\sigma$ is not in $\mathcal{P}_2$. Also, since $a'' > 1$, let us replace $k'' + a'' - 2$ by $k_3$. From this, we deduce that

$$k_2 - 2 \leq k_3 \leq 3m - 8 \quad \text{and} \quad 2m - 5 \leq k_1 - 4 \leq k_2 - 2 \leq 3m - 7.$$

Let us substitute $k_1, k_2$ for $k_1 - 4, k_2 - 2$, respectively. Then we obtain

$$\sum_{k_1, k_2, k_3} q^{6(k_1+4)+3+6(k_2+2)+3+6k_3+3} = q^{12m-3+12m-15+12m-27} \binom{m}{3} q^{-3}.$$
Let $\mathcal{P}_3$ be the set of all partitions $\rho$ such that $\rho_{2i-1} = 3k_i + 3$ and $\rho_{2i} = 3k_i$, where $k_i$'s satisfy (11) for $i = 1, 2, 3$, and $\rho_7 \cdots \rho_{l-3} \in \mathcal{P}(E; 6m - 19)$. We obtain the correct generating function for partitions in $\mathcal{P}_3$

$$q^{12m-3+12m-15+12m-27}\binom{m}{3} q^{6m-19}(q).$$

Iterating this process, we consider $\mathcal{P}_j$, the set of all partitions such that its $(2i-1)$st part is $3k_i + 3$ and $2i$th part is $3k_i$ for $2m - 2j + 1 \leq k_1 \leq k_2 \cdots \leq k_j \leq 3m - 3j + 1$, and the remainder of the parts satisfy the condition for $\mathcal{P}(E; 6(m - j) - 1)$. The generating function for partitions in $\mathcal{P}_j$ is

$$q^{12mj - 6j^2 + 3j}\binom{m}{j} q^{6m - 6j - 1}(q).$$

This coincides with the $j$th term in the sum on the right side of (11). At last, only the partitions of type $D$ with parts less than $6m - 1$ remain after cancellation. We have thus completed the proof of (11). Similarly, we can prove the other cases: $n = 3m + 1$ and $3m + 2$. 

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