ANTI-WICK QUANTIZATION WITH SYMBOLS IN $L^p$ SPACES

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Abstract. We give a classification of pseudo-differential operators with anti-Wick symbols belonging to $L^p$ spaces: if $p = 1$ the corresponding operator belongs to trace classes; if $1 \leq p \leq 2$ we get Hilbert-Schmidt operators; finally, if $p < \infty$, the operator is compact. This classification cannot be improved, as shown by some examples.

Introduction: Weyl and anti-Wick quantization

A common feature of all types of quantization proposed in the literature is that they establish a correspondence between self-adjoint operators and classical observables, i.e. real functions on the phase space. In general in the frame of pseudo-differential calculus the correspondence between symbols and operators does not fulfill this requirement, that is, the operator fails to be self-adjoint when the symbol is real valued. However this is true for the the Weyl and the anti-Wick symbols, as is shown for instance in [10]. If on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi = \mathbb{R}^{2n}_z, z = (x, \xi)$, we consider the Shubin classes

$$\Gamma^m_\rho = \{ a(z) \in C^\infty(\mathbb{R}^{2n}) : |\partial_z^\alpha a(z)| \leq C_\gamma |z|^{m-\rho|\gamma|} \}$$

with $m \in \mathbb{R}, \rho \in (0,1], \langle z \rangle = \sqrt{1 + |z|^2}$, then a satisfactory pseudo-differential calculus, both in the case of Weyl and anti-Wick operators, has been developed, see for instance [10].

M.W. Wong analysed in [13] the case of Weyl quantization with symbols in $L^p(\mathbb{R}^{2n})$ and gave conditions for the operators to be continuous, compact and Hilbert-Schmidt. In this work we also consider symbols belonging to $L^p(\mathbb{R}^{2n})$ and study the behaviour of the corresponding anti-Wick operators. We obtain a classification of the operators according to the $L^p(\mathbb{R}^{2n})$ space of the symbols.

To be more precise, given a function $a(z), z = (x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi = \mathbb{R}^{2n}_z$, consider the Weyl operator:

$$Op_w[a]u(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi,$$

as well as the anti-Wick operator:

$$Op_{aww}[a]u(x) = \int_{\mathbb{R}^{2n}} a(z) P_z u dz,$$
where \( P_z u(x) = (u, \Phi_z) \Phi_z(x) \) are orthogonal projections in \( L^2(\mathbb{R}^n) \) on the functions \( \Phi_z(x) = \pi^{-\frac{n}{2}} e^{ix\eta} e^{-|x|^2} \) depending on the parameter \( z = (y, \eta) \in \mathbb{R}^{2n} \). If \( a(z) \) belongs to a Shubin class \( \Gamma^\rho_0 \), then both operators are continuous maps from \( S(\mathbb{R}^n) \) to itself, extending to continuous maps on the duals \( S'(\mathbb{R}^n) \). The adjoint operator is in both cases the operator associated with \( \overline{a}(z) \), the complex conjugate of the symbol \( a(z) \), so that, as we mentioned before, real symbols correspond to self-adjoint operators. Moreover the anti-Wick operator satisfies the additional property: \( a(z) \geq 0 \) (\( a(z) > 0 \)) for all \( z \in \mathbb{R}^{2n} \) implies \( Op_{aw}[a] \geq 0 \) (\( Op_{aw}[a] > 0 \)) and, if \( a(z) > 0 \) and elliptic, the operator is a bijection of \( S(\mathbb{R}^n) \) extending to a bijection of \( S'(\mathbb{R}^n) \); see for reference [2], [8]. The relation between Weyl and anti-Wick operators is given by the equality \( Op_{aw}[a] = Op_{aw}[(2\pi)^{-n} a \ast \sigma] \); that means every anti-Wick operator is also a Weyl operator with Weyl symbol given by the convolution between its anti-Wick symbol and the Gaussian function \( \sigma(z) = 2^n e^{-|z|^2} \). The converse is true only modulo regularizing operators; that is, every Weyl operator can be written as the sum of an anti-Wick operator plus a regularizing operator, for details see [10], [2]. If \( a(z) \) is in the Shubin class \( \Gamma^\rho_\rho \), then the convolution \( (a \ast \sigma)(z) \) is also in the same class, so that in this case the properties of Weyl and anti-Wick operators are very similar. But, if we consider \( a \in L^p(\mathbb{R}^n) \), then in general we only have \( a \ast \sigma \in \Gamma^\rho_0 \), that is, we are in the “limit” case \( \rho = 0 \) of Shubin classes (see in the next section the second proof of Theorem 1.2 and Lemma 1.1); however all the continuity properties mentioned above remain valid in view of the Calderon-Vaillancourt theorem; see [8], [4], [5]. We obtain in this case an interesting class of compact operators on \( L^2(\mathbb{R}^n) \). Operators of this type, called also localization operators, were introduced by Daubechies as filters in signal analysis (see [6]); they were studied by Wong in [13], where the continuity of the map \( a(z) \in L^p(\mathbb{R}^{2n}) \rightarrow Op_{aw}[a] \in B(L^2) \) is obtained by means of interpolation methods.

We follow here a direct approach, and in section 2 we present two different proofs of the continuity of this map, which we consider of interest for their particular simplicity; as a corollary we obtain the compactness property as in [13]. For Weyl symbols, on the contrary, we remark that the boundedness of the map \( b(z) \in L^p(\mathbb{R}^{2n}) \rightarrow Op_{aw}[b] \in B(L^2) \) fails if \( p > 2 \); see [11]. In section 3 we study Hilbert-Schmidt and trace class properties of these operators, pointing out the following differences between Weyl and anti-Wick symbols. An operator is Hilbert-Schmidt if and only if its Weyl symbol is in \( L^2 \), while we show that for anti-Wick symbols in \( L^p \) with \( p \in [1, 2] \) one has always Hilbert-Schmidt operators; for \( 2 \leq p \leq \infty \) one can find operators with anti-Wick symbol in \( L^p \) which are not Hilbert-Schmidt; a sufficient condition for an operator to be trace class is that its anti-Wick symbol belongs to \( L^1 \), while the Weyl symbol needs to be in \( L^1 \) together with all its derivatives; see for instance Robert [9]. For \( p > 1 \), there exist operators with anti-Wick symbol in \( L^p \) which are not trace class.

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1. Theorems on boundedness and compactness

In this section we first give a direct proof of the continuity of the map \( a \rightarrow Op_{aw}[a] \), for \( a \in L^p(\mathbb{R}^{2n}) \). Second, we prove the same result as a consequence of the Calderon-Vaillancourt theorem. Finally, we show the compactness of \( Op_{aw}[a] \).
The following lemma, used in Theorem 1.2, clarifies the relation between $L^p$ anti-Wick and Weyl symbols in the $\Gamma^p$ classes.

**Lemma 1.1.** Let $k \geq 0$ and $|z|^k a(z) \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$. Then, for any multi-index $\alpha$ there exist positive functions $\Psi_\alpha(z)$ converging to zero as $|z| \to \infty$ such that $|z|^k \partial^\alpha z (a \ast \sigma)(z) | \leq |z|^k a(z) \|_{L^p} \Psi_\alpha(z)$. In particular, $a \ast \sigma \in \Gamma^p$. 

**Proof.** Using Peetre’s inequality $|z|^k \leq 2^k (w)^k (z - w)^k$, we have, for suitable $M > 0$,

\[
|z|^k \partial^\alpha z (a \ast \sigma)(z) | \leq 2^k \int (z - w)^{-M} |a(w)| (w)^k (z - w)^{M+k} \partial^\alpha z (z - w) dw \\
\leq 2^k \| (z - w)^{-M} \|_{L^p} \left( \int |a(w)|^p (w)^p (z - w)^{p(M+k)} (\partial^\alpha z (z - w))^p dw \right)^{\frac{1}{p}} \\
\leq 2^k \| (z - w)^{-M} \|_{L^p} |a(w) (w)^k| \|\partial^\alpha z (z - w)\|_{L^{2p}} \\
\times \left( \int |a(w)|^p (w)^p (z - w)^{2p(M+k)} (\partial^\alpha z (z - w))^{2p} dw \right)^{\frac{1}{2p}},
\]

where $\frac{1}{q} + \frac{1}{p} = 1$ and we have used Hölder’s inequality. An application of the dominated convergence theorem shows that the last integral tends to zero as $|z| \to +\infty$. \hfill $\square$

**Theorem 1.2.** Let $a \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then the linear map $T : a \in L^p(\mathbb{R}^n) \to Op_{\text{aw}}[a] \in B(L^2(\mathbb{R}^n))$ is continuous.

**First proof.** Letting $u \in S(\mathbb{R}^n)$, we consider two different estimates of $|\langle u, \Phi_2 \rangle|$:

\[
|\langle u, \Phi_2 \rangle| = \pi^{-n/4} \left| \int_{\mathbb{R}^n} e^{-inx} e^{-\frac{|x-y|^2}{2}} u(x) dx \right| \\
\leq \pi^{-n/4} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2}} |u(x)| dx = (\Phi_0 \ast |u|)(y),
\]

with $\Phi_0(x) = \pi^{-n/4} e^{-\frac{|x|^2}{2}}$.

\[
|\langle u, \Phi_2 \rangle| = \left| \mathcal{F}_{x \to \eta}(\tau_y \Phi_0)(\eta) \right| \\
= \left| \mathcal{F}_{x \to \eta}(\tau_y \Phi_0)(\eta) \ast \hat{u}(\eta) \right| \\
\leq (2\pi)^n/2 \left| \Phi_0(\eta) \ast \hat{u}(\eta) \right| \leq (2\pi)^n/2 (\Phi_0 \ast |\hat{u}|)(\eta),
\]

where $\mathcal{F}$ denotes the Fourier transform and $\tau_y$ the translation by $y$.

Using (1.1) and (1.2), we can estimate the $L^q$ norm of the orthogonal projection $P_2 u$ onto the vector $\Phi_2$ uniformly with respect to $z$:

\[
\| \langle u, \Phi_2 \rangle \Phi_2(x) \|_{L^q} = \pi^{-n/4} \left( \int \|\langle u, \Phi_2 \rangle \|_{q} e^{-q|z|^2/2} dz \right)^{\frac{1}{q}} \\
\leq (2\pi)^n/4 \left( \int |\Phi_0 \ast |u||_q^q/2 |\hat{u}|(\eta)| \eta/2 dy \right)^{\frac{1}{q}} \\
= (2\pi)^n/4 \| \Phi_0 \ast |u| \|_{L^q} \| \Phi_0 \ast |\hat{u}| \|_{L^q} := K(u).
\]

Now, let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of functions converging to zero in $L^p(\mathbb{R}^2)$ as $n \to \infty$, and such that $A_n := T a_n$ tends to an operator $B$ in $B(L^2(\mathbb{R}^n))$; we will prove that $B = 0$, so that the closed graph theorem gives the continuity of the map $T$. 

Using (1.3) and Hölder’s inequality, we get, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$|A_n u(x)| = \left| \int_{\mathbb{R}^n} a_n(z) P_z u \, dz \right| \leq \int |a_n(z)| \left| (u, \Phi_z(x)) \right| \, dz \leq \|a_n\|_p \| (u, \Phi_z(x)) \|_q \leq K(u) \|a_n\|_p.$$ 

Therefore the sequence of functions $A_n u(x)$ converges to the null function uniformly with respect to $x$.

If $u, v \in S(\mathbb{R}^n)$, we get

$$\left| (v, A_n u) \right| \leq \int_{\mathbb{R}^n} \left| v(x) \right| |A_n u(x)| \, dx \leq \|v\|_{L^1} \|A_n u\|_{L^\infty};$$

hence $(v, A_n u) \to 0$ as $n \to \infty$. For $u, v \in S(\mathbb{R}^n)$ we have

$$\left| (v, (A_n - B) u) \right| \leq \|v\|_{L^2} \|u\|_{L^2} \|A_n - B\|_{B(L^2)};$$

since $\|A_n - B\|_{B(L^2)} \to 0$ as $n \to \infty$, it follows $B = 0$.

**Second proof.** In $\Gamma_0^0$ we consider the topology induced by the seminorms:

$$\sup_{\alpha \in \mathbb{Z}^n_+} \left| D^n b(z) \right|, \quad k \in \mathbb{N}, \; \alpha \in \mathbb{Z}^n_+.$$

From the Calderon-Vaillancourt theorem (see [8], [4], [5]) we get the continuity of the map $b \in \Gamma_0^0 \to Op_{aw}[b] \in L^2(\mathbb{R}^n)$ with the norm estimate

$$\|Op_{aw}[b]\| \leq C \sup_{\alpha \in \mathbb{Z}^n_+} \left| D^n b(z) \right|$$

for suitable $C > 0$.

Using Lemma 1.1 with $k = 0$, we have the estimate $\left| D^n b(z) \right| \leq C \|a\|_{L^p}$ (which holds also for $p = +\infty$, with a slight modification of the argument of Lemma 1.1); therefore from (1.4) it follows that $\|Op_{aw}[a]\| \leq C \|a\|_{L^p}$, for suitable constants $C > 0$.

If $b \in L^p(\mathbb{R}^{2n})$ has compact support and for any $\phi \in S(\mathbb{R}^{2n})$ we have $\phi \ast b \in S(\mathbb{R}^{2n})$, then the following lemma holds:

**Lemma 1.3.** Let $b \in L^p(\mathbb{R}^{2n}), 1 \leq p \leq +\infty$. If $\text{supp } b$ is compact, then $B = Op_{aw}[b]$ is a regularizing operator, i.e. an operator with Weyl symbol in $S(\mathbb{R}^{2n})$.

**Corollary 1.4.** Let $a \in L^p(\mathbb{R}^{2n}), 1 \leq p < +\infty$. Then the operator $Op_{aw}[a] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.

**Proof.** Let $\chi_M(z)$ the characteristic function of the set $\{z \in \mathbb{R}^{2n} : |z| \leq M\}$. Then from Lemma 1.3 the operator $A_M = Op_{aw}[a \chi_M]$ is compact. By Theorem 1.2 $\|A - A_M\|_{B(L^2)} \leq C \|a - a \chi_M\|_{L^p} \to 0$ as $M \to +\infty$; hence $A$ is compact.

More generally we can consider anti-Wick symbols in the sum of $L^p$ spaces, setting by definition $Op_{aw}[a + b] = Op_{aw}[a] + Op_{aw}[b]$ if $a \in L^{p_1}$ and $b \in L^{p_2}$, $1 \leq p_1, \; p_2 \leq +\infty$. Then the following result holds:

**Corollary 1.5.** Let $p \in [1, +\infty)$ and $a(z) \in L^p(\mathbb{R}^{2n})$, with $a(z) \geq 0$. Then, for every $\epsilon > 0$, $Op_{aw}[a + \epsilon]$ is a bicontinuous bijection on $L^2(\mathbb{R}^{2n})$. 


Proof. Since $a(z) + \epsilon > 0$, we have $Op_{aw}[a + \epsilon] > 0$; so it is injective on $L^2(\mathbb{R}^{2n})$. As $Op_{aw}[a]$ is compact, $Op_{aw}[a + \epsilon] = Op_{aw}[a] + \mathcal{O}$ is a Fredholm operator of index zero, so it is also surjective. The continuity of $Op_{aw}[a + \epsilon]^{-1}$ follows from the inverse mapping theorem.

We notice that, extending to $\Gamma^0_0$ the argument in [10], Chapter 4, the compactness of Weyl operators also follows from Lemma [11].

2. HILBERT-SCHMIDT AND TRACE CLASS OPERATORS

Let $S_2(L^2(\mathbb{R}^n))$ be the space of Hilbert-Schmidt operators and $S_1(L^2(\mathbb{R}^n))$ the space of trace class operators on $L^2(\mathbb{R}^n)$. We give sufficient conditions for operators with anti-Wick symbols in $L^p(\mathbb{R}^{2n})$ to belong to these classes. Our results are the following.

**Theorem 2.1.** Let $a \in L^p(\mathbb{R}^{2n})$, $p \in [1, 2]$. Then $A = Op_{aw}[a] \in S_2(L^2(\mathbb{R}^n))$ and $||A||_2 = ||b||_{L^2(\mathbb{R}^{2n})}$, where $b = (2\pi)^{-\frac{n}{2}} a * \sigma \in L^2(\mathbb{R}^{2n})$ is the Weyl symbol of $A$.

**Proof.** Using the same arguments as in [10], we have also for symbols $b$ in the limit class $\Gamma^0_0$ that $b \in L^2(\mathbb{R}^{2n})$ if and only if $Op_{aw}[b] \in S_2(L^2(\mathbb{R}^n))$, with $||A||_2 = ||b||_{L^2(\mathbb{R}^{2n})}$. Using Young’s inequality $||b||_2 \leq (2\pi)^{-\frac{n}{2}} \|a\|_p \|\sigma\|_q$, $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we then have $b \in L^2(\mathbb{R}^{2n})$ under the condition $p \in [1, 2]$.

The previous result cannot be improved, as shown by the following:

**Theorem 2.2.** For all $p > 2$, there exist $a \in L^p(\mathbb{R}^{2n})$ such that $Op_{aw}[a] \notin S_2(L^2(\mathbb{R}^n))$.

**Proof.** Assume, to the contrary, that there exists $p > 2$ such that $Op_{aw}[a] \in S_2(L^2(\mathbb{R}^n))$ for all $a \in L^p$. Then, for all $a \in L^p$, the convolution $b = (2\pi)^{-\frac{n}{2}} a * \sigma$ should belong to $L^2(\mathbb{R}^{2n})$, since every symbol $b \in \Gamma^0_0$ satisfies $b \in L^2 \Leftrightarrow Op_{aw}[b] \in S_2(L^2(\mathbb{R}^n))$. Then the map $a \in L^p \rightarrow a * \sigma \in L^2, (p > 2)$ is linear and continuous; moreover it commutes with translations, so by a property of the convolution transform it would be the null operator (see [12]), a contradiction.

Now we give an explicit example:

**Example 2.3.** For all $p > 2$ we determine anti-Wick symbols $a \in L^p(\mathbb{R}^{2n})$ such that $Op_{aw}(a * \sigma) \notin S_2(L^2(\mathbb{R}^n))$. Using again the property $Op_{aw}(a * \sigma) \in S_2(L^2(\mathbb{R}^n)) \Leftrightarrow a * \sigma \in L^2(\mathbb{R}^{2n})$, we search for an $a$ such that $|a * \sigma|^2 \notin L^1(\mathbb{R}^{2n})$.

Let $\langle z \rangle^{-s} = (1 + |z|^2)^{-s/2}$; since $\langle z \rangle^{-s} > 0$, it follows that

$$|\langle z \rangle^{-s} * \sigma(z)|^2 = 2^n \left| \int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} dt \right|^2 = 2^n \left( \int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} dt \right)^2.$$

We want to determine $s$ such that

$$\left( \int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} dt \right)^2 \geq \frac{C}{(1 + |z|^2)^a} \notin L^1(\mathbb{R}^{2n}).$$
i.e. \( \int_{\mathbb{R}^{2n}} e^{-|t|^2} (z-t)^{-s}|z|^n dt \geq C \). Using Peetre’s inequality, we get

\[
\int (1 + |z|^2)^{(n-s)/2} (1 + |t|^2)^{-s/2} e^{-|t|^2} (1 + |z-t|^2)^{-s/2} dt \\
\geq 2^{-s} \int (1 + |z|^2)^{(n-s)/2} (1 + |t|^2)^{-s/2} e^{-|t|^2} dt \\
= C_s(1 + |z|^2)^{(n-s)/2} > C
\]

if \( s \leq n \). On the other hand, since \( |z|^{-s} \in L^p(\mathbb{R}^{2n}) \) we get \( s > \frac{2n}{p} \). Therefore, 
\[ a(z) = |z|^{-s} \] is the example we have searched for, under the condition \( \frac{2n}{p} < s \leq n \), which we can satisfy for some \( s > 2 \).

Now we state our result about trace class operators.

**Theorem 2.4.** Let \( a \in L^1(\mathbb{R}^{2n}) \). Then \( Op_{aw}[a] \in S_1(L^2(\mathbb{R}^n)) \) and

\[
Tr(Op_{aw}[a]) = (2\pi)^{-2n} \int a(y, \eta) \sigma(x - y, \xi - \eta) \, dx dy d\xi d\eta.
\]

**Proof.** Let \( b = (2\pi)^{-n} a * \sigma \). By Young’s inequality \( \|b\|_1 \leq (2\pi)^{-n} \|a\|_1 \|\sigma\|_1 \), and, for all \( \alpha \in \mathbb{Z}_+^n \), \( \|\partial^{\alpha} b\|_1 \leq (2\pi)^{-n} \|a\|_1 \|\partial^{\alpha} \sigma\|_1 \). Then \( b \in L^1 \) and \( \partial^{\alpha} b \in L^1 \), \( \forall \alpha \in \mathbb{Z}_+^n \), and we have that \( Op_{aw}[b] \in S_1(L^2(\mathbb{R}^n)) \) and trace’s formula (2.1) holds (see Theorem II-53 of [9]).

We notice that a similar result under different hypothesis is in Berezin [1]. We also remark that, with a minor modification of the same argument, Theorem 2.4 also holds for complex Borel measures as well as for distributions with compact support.

**Remark 2.5.** Let \( a \in L^p(\mathbb{R}^{2n}) \), \( p > 1 \). Then \( Op_{aw}[a] \) is not, in general, a trace class operator. An easy example can be constructed from the harmonic oscillator (for simplicity in dimension 1) \( H = -\frac{\partial^2}{\partial z^2} + |z|^2 \). Denote by \( G^m_\rho \) the space of pseudo-differential operators with Weyl symbols in \( \Gamma^m_\rho \), and consider the operator \( A = H + I \in G^1_1 \); its Weyl symbol is \( h_0(x, \xi) = \xi^2 + x^2 + 1 = |z|^2 + 1 \), with \( z = (x, \xi) \). The operator \( A^{-1} = (H + I)^{-1} \in G^{-1}_1 \) has Weyl symbol \( \sigma_w(A^{-1}) = (|z|^2 + 1)^{-1} + r(z) \), with \( r(z) \in \Gamma^{-1}_1 \).

Then \( \sigma_w(A^{-1}) \in L^p(\mathbb{R}^2) \) for \( p > 1 \), but \( \sigma_w(A^{-1}) \notin L^1(\mathbb{R}^2) \), and, therefore, \( A^{-1} \notin S_1(L^2(\mathbb{R})) \), but \( A^{-1} \notin S_1(L^2(\mathbb{R})) \) because \( \|A^{-1}\|_{tr} = \sum_{j \in \mathbb{N}} \frac{1}{2^j} = +\infty \).

Every \( K \in G^m_\rho \) has an anti-Wick symbol, if we consider it modulo regularizing operators (see, for instance, [2], [10]). Hence there exists \( \tilde{A}^{-1} \in G^{-1}_1 \), such that \( \tilde{A}^{-1} - A^{-1} \in G^{-\infty} \), and its anti-Wick symbol is \( \tilde{a}(z) = (1 + |z|^2)^{-1} + r(z) + \tilde{r}(z) \), with \( \tilde{r}(z) \in \Gamma^{-3}_1 \).

Then, \( \tilde{a}(z) = (1 + |z|^2)^{-1} + r(z) \), with \( r(z) \in \Gamma^{-3}_1 \). Therefore \( \tilde{A}^{-1} \notin S_1(L^2(\mathbb{R})) \), and its anti-Wick symbol satisfies \( \tilde{a}(z) \notin L^1(\mathbb{R}^2) \) but \( \tilde{a}(z) \in L^p(\mathbb{R}^2), \forall p > 1 \).

**References**


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