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ANTI-WICK QUANTIZATION WITH SYMBOLS IN L^p SPACES

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ABSTRACT. We give a classification of pseudo-differential operators with anti-Wick symbols belonging to L^p spaces: if $p = 1$ the corresponding operator belongs to trace classes; if $1 \leq p \leq 2$ we get Hilbert-Schmidt operators; finally, if $p < \infty$, the operator is compact. This classification cannot be improved, as shown by some examples.

INTRODUCTION: WEYL AND ANTI-WICK QUANTIZATION

A common feature of all types of quantization proposed in the literature is that they establish a correspondence between self-adjoint operators and classical observables, i.e. real functions on the phase space. In general in the frame of pseudo-differential calculus the correspondence between symbols and operators does not fulfill this requirement, that is, the operator fails to be self-adjoint when the symbol is real valued. However this is true for the the Weyl and the anti-Wick symbols, as is shown for instance in [10]. If on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi = \mathbb{R}^{2n}_z$, $z = (x, \xi)$, we consider the Shubin classes

$$\Gamma^m_\rho = \{a(z) \in C^\infty(\mathbb{R}^{2n}) : |\partial^\gamma_z a(z)| \leq C_\gamma \langle z \rangle^{m-\rho|\gamma|}\} \tag{0.1}$$

with $m \in \mathbb{R}$, $\rho \in (0, 1]$, $\langle z \rangle = \sqrt{1 + |z|^2}$, then a satisfactory pseudo-differential calculus, both in the case of Weyl and anti-Wick operators, has been developed, see for instance [10].

M.W. Wong analysed in [13] the case of Weyl quantization with symbols in $L^p(\mathbb{R}^{2n})$ and gave conditions for the operators to be continuous, compact and Hilbert-Schmidt. In this work we also consider symbols belonging to $L^p(\mathbb{R}^{2n})$ and study the behaviour of the corresponding anti-Wick operators. We obtain a classification of the operators according to the $L^p(\mathbb{R}^{2n})$ space of the symbols.

To be more precise, given a function $a(z)$, $z = (x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi = \mathbb{R}^{2n}_z$, consider the Weyl operator:

$$Op_w[a]u(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy đ\xi, \qquad đ\xi = (2\pi)^{-n} d\xi,$$

as well as the anti-Wick operator:

$$Op_{aw}[a]u(x) = \int_{\mathbb{R}^{2n}} a(z) P_z u \; dz,$$

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where $P_z u(x) = (u, \Phi_z)\Phi_z(x)$ are orthogonal projections in $L^2(\mathbb{R}^n)$ on the functions $\Phi_z(x) = \pi^{-\frac{n}{4}} e^{ix\eta} e^{-\frac{|x-y|^2}{2}}$ depending on the parameter $z = (y, \eta) \in \mathbb{R}^{2n}$. If $a(z)$ belongs to a Shubin class Γ_ρ^m, then both operators are continuous maps from $S(\mathbb{R}^n)$ to itself, extending to continuous maps on the duals $S'(\mathbb{R}^n)$. The adjoint operator is in both cases the operator associated with $\overline{a}(z)$, the complex conjugate of the symbol $a(z)$, so that, as we mentioned before, real symbols correspond to self-adjoint operators. Moreover the anti-Wick operator satisfies the additional property: $a(z) \geq 0$ $(a(z) > 0)$ for all $z \in \mathbb{R}^{2n}$ implies $Op_{aw}[a] \geq 0$ $(Op_{aw}[a] > 0)$ and, if $a(z) > 0$ and elliptic, the operator is a bijection of $S(\mathbb{R}^n)$ extending to a bijection of $S'(\mathbb{R}^n)$; see for reference [2], [3]. The relation between Weyl and anti-Wick operators is given by the equality $Op_{aw}[a] = Op_w[(2\pi)^{-n} a * \sigma]$; that means every anti-Wick operator is also a Weyl operator with Weyl symbol given by the convolution between its anti-Wick symbol and the Gaussian function $\sigma(z) = 2^n e^{-|z|^2}$. The converse is true only modulo regularizing operators; that is, every Weyl operator can be written as the sum of an anti-Wick operator plus a regularizing operator, for details see [10], [2]. If $a(z)$ is in the Shubin class Γ_ρ^m, then the convolution $(a * \sigma)(z)$ is also in the same class, so that in this case the properties of Weyl and anti-Wick operators are very similar. But, if we consider $a \in L^p(\mathbb{R}^n)$, then in general we only have $a * \sigma \in \Gamma_0^0$, that is, we are in the "limit" case $\rho = 0$ of Shubin classes (see in the next section the second proof of Theorem 1.2 and Lemma 1.1); however all the continuity properties mentioned above remain valid in view of the Calderon-Vaillancourt theorem; see [8], [4], [5]. We obtain in this case an interesting class of compact operators on $L^2(\mathbb{R}^n)$. Operators of this type, called also *localization operators*, were introduced by Daubechies as filters in signal analysis (see [6]); they were studied by Wong in [13], where the continuity of the map $a(z) \in L^p(\mathbb{R}^{2n}) \to Op_{aw}[a] \in B(L^2)$ is obtained by means of interpolation methods.

We follow here a direct approach, and in section 2 we present two different proofs of the continuity of this map, which we consider of interest for their particular simplicity; as a corollary we obtain the compactness property as in [13]. For Weyl symbols, on the contrary, we remark that the boundedness of the map $b(z) \in L^p(\mathbb{R}^{2n}) \to Op_w[b] \in B(L^2)$ fails if $p > 2$; see [11]. In section 3 we study Hilbert-Schmidt and trace class properties of these operators, pointing out the following differences between Weyl and anti-Wick symbols. An operator is Hilbert-Schmidt if and only if its Weyl symbol is in L^2, while we show that for anti-Wick symbols in L^p with $p \in [1, 2]$ one has always Hilbert-Schmidt operators; for $2 \leq p \leq \infty$ one can find operators with anti-Wick symbol in L^p which are not Hilbert-Schmidt; a sufficient condition for an operator to be trace class is that its anti-Wick symbol belongs to L^1, while the Weyl symbol needs to be in L^1 together with all its derivatives; see for instance Robert [9]. For $p > 1$, there exist operators with anti-Wick symbol in L^p which are not trace class.

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1. Theorems on boundedness and compactness

In this section we first give a direct proof of the continuity of the map $a \to Op_{aw}[a]$, for $a \in L^p(\mathbb{R}^{2n})$. Second, we prove the same result as a consequence of the Calderon-Vaillancourt theorem. Finally, we show the compactness of $Op_{aw}[a]$.

The following lemma, used in Theorem 1.2, clarifies the relation between L^p anti-Wick and Weyl symbols in the Γ^m_ρ classes.

Lemma 1.1. *Let $k \geq 0$ and $\langle z\rangle^k a(z) \in L^p(\mathbb{R}^{2n}), 1 \leq p < +\infty$. Then, for any multi-index α there exist positive functions $\Psi_\alpha(z)$ converging to zero as $|z| \to \infty$ such that $|\langle z\rangle^k \partial_z^\alpha (a * \sigma)(z)| \leq \|\langle z\rangle^k a(z)\|_{L^p}^{1/2} \Psi_\alpha(z)$. In particular, $a * \sigma \in \Gamma_0^{-k}$.*

Proof. Using Peetre's inequality $\langle z\rangle^k \leq 2^k \langle w\rangle^k \langle z-w\rangle^k$, we have, for suitable $M > 0$,

$$\begin{aligned}
&\langle z\rangle^k |\partial_z^\alpha (a * \sigma)(z)| \\
&\leq 2^k \int \langle z-w\rangle^{-M} |a(w)| \langle w\rangle^k \langle z-w\rangle^{M+k} \partial_z^\alpha \sigma(z-w) dw \\
&\leq 2^k \|\langle z-w\rangle^{-M}\|_{L^q} \Big(\int |a(w)|^p \langle w\rangle^{pk} \langle z-w\rangle^{p(M+k)} \Big(\partial_z^\alpha \sigma(z-w)\Big)^p dw \Big)^{\frac{1}{p}} \\
&\leq 2^k \|\langle z-w\rangle^{-M}\|_{L^q} \|a(w)\langle w\rangle^k\|_{L^p}^{1/2} \\
&\quad \times \Big(\int |a(w)|^p \langle w\rangle^{pk} \langle z-w\rangle^{2p(M+k)} \Big(\partial_z^\alpha \sigma(z-w)\Big)^{2p} dw \Big)^{\frac{1}{2p}},
\end{aligned}$$

where $\frac{1}{q} + \frac{1}{p} = 1$ and we have used Hölder's inequality. An application of the dominated convergence theorem shows that the last integral tends to zero as $|z| \to +\infty$. □

Theorem 1.2. *Let $a \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Then the linear map $T : a \in L^p(\mathbb{R}^{2n}) \to Op_{aw}[a] \in B(L^2(\mathbb{R}^n))$ is continuous.*

First proof. Letting $u \in S(\mathbb{R}^n)$, we consider two different estimates of $|(u, \Phi_z)|$:

$$\begin{aligned}
|(u,\Phi_z)| &= \pi^{-n/4} \Big| \int_{\mathbb{R}^n} e^{-i\eta x} e^{-\frac{|x-y|^2}{2}} u(x)\, dx \Big| \\
&\leq \pi^{-n/4} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2}} |u(x)|\, dx = (\Phi_0 * |u|)(y),
\end{aligned} \tag{1.1}$$

with $\Phi_0(x) = \pi^{-n/4} e^{-\frac{|x|^2}{2}}$;

$$\begin{aligned}
|(u,\Phi_z)| &= \Big| \mathcal{F}_{x\to\eta}[(\tau_y \Phi_0) u](\eta) \Big| = \Big| \mathcal{F}_{x\to\eta}[(\tau_y \Phi_0)(\eta)] * \widehat{u}(\eta) \Big| \\
&= (2\pi)^{n/2} \Big| \Phi_0(\eta) * \widehat{u}(\eta) \Big| \leq (2\pi)^{n/2} (\Phi_0 * |\widehat{u}|)(\eta),
\end{aligned} \tag{1.2}$$

where $\mathcal{F}$ denotes the Fourier transform and τ_y the translation by y.

Using (1.1) and (1.2), we can estimate the L^q norm of the orthogonal projection $P_z u$ onto the vector Φ_z uniformly with respect to z:

$$\begin{aligned}
\|(u,\Phi_z)\Phi_z(x)\|_{L^q} &= \pi^{-\frac{n}{4}} \Big(\int |(u,\Phi_z)|^q e^{-q\frac{|x-y|^2}{2}}\, dz \Big)^{\frac{1}{q}} \\
&\leq (2\pi)^{n/4} \Big(\int |(\Phi_0 * |u|)(y)|^{q/2} |(\Phi_0 * |\widehat{u}|)(\eta)|^{q/2}\, dy d\eta \Big)^{\frac{1}{q}} \\
&= (2\pi)^{n/4} \|\sqrt{\Phi_0 * |u|}\,\|_{L^q_y} \|\sqrt{\Phi_0 * |\widehat{u}|}\,\|_{L^q_\eta} := K(u).
\end{aligned} \tag{1.3}$$

Now, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of functions converging to zero in $L^p(\mathbb{R}^{2n})$ as $n \to \infty$, and such that $A_n := T a_n$ tends to an operator B in $B(L^2(\mathbb{R}^n))$; we will prove that $B = 0$, so that the closed graph theorem gives the continuity of the map T.

Using (1.3) and Hölder's inequality, we get, for $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}|A_n u(x)| &= \Big|\int_{\mathbb{R}^{2n}} a_n(z)P_z u\,dz\Big| \le \int |a_n(z)|\,|(u,\Phi_z)\Phi_z(x)|\,dz\\ &\le \|a_n\|_p\,\|(u,\Phi_z)\Phi_z(x)\|_q \le K(u)\|a_n\|_p.\end{aligned}$$

Therefore the sequence of functions $A_n u(x)$ converges to the null function uniformly with respect to x.

If $u, v \in S(\mathbb{R}^n)$, we get

$$|(v, A_n u)| \le \int_{\mathbb{R}^n} |v(x)|\,|A_n u(x)|\,dx \le \|v\|_{L^1}\,\|A_n u\|_{L^\infty};$$

hence $(v, A_n u) \to 0$ as $n \to \infty$. For $u, v \in S(\mathbb{R}^n)$ we have

$$|(v,(A_n - B)u)| \le \|v\|_{L^2}\,\|u\|_{L^2}\,\|A_n - B\|_{B(L^2)};$$

since $\|A_n - B\|_{B(L^2)} \to 0$ as $n \to \infty$, it follows $B = 0$. □

Second proof. In Γ^0_0 we consider the topology induced by the seminorms:

$$\sup_{\substack{z\in\mathbb{R}^{2n}\\|\alpha|\le k}} \left|D^\alpha_z b(z)\right|, \qquad k \in \mathbb{N},\ \alpha \in \mathbb{Z}^n_+.$$

From the Calderon-Vaillancourt theorem (see [8], [4], [5]) we get the continuity of the map $b \in \Gamma^0_0 \to Op_w[b] \in L^2(\mathbb{R}^n)$ with the norm estimate

$$\|Op_w[b]\| \le C \sup_{\substack{z\in\mathbb{R}^{2n}\\|\alpha|\le 2n+1}} \left|D^\alpha_z b(z)\right| \tag{1.4}$$

for suitable $C > 0$.

Using Lemma 1.1 with $k = 0$, we have the estimate $|D^\alpha_z b(z)| \le C\|a\|_{L^p}$ (which holds also for $p = +\infty$, with a slight modification of the argument of Lemma 1.1); therefore from (1.4) it follows that $\|Op_{aw}[a]\| \le C\|a\|_{L^p}$, for suitable constants $C > 0$. □

If $b \in L^p(\mathbb{R}^{2n})$ has compact support and for any $\phi \in S(\mathbb{R}^{2n})$ we have $\phi * b \in S(\mathbb{R}^{2n})$, then the following lemma holds:

Lemma 1.3. *Let $b \in L^p(\mathbb{R}^{2n}), 1 \le p \le +\infty$. If* supp b *is compact, then $B = Op_{aw}[b]$ is a regularizing operator, i.e. an operator with Weyl symbol in $S(\mathbb{R}^{2n})$.*

Corollary 1.4. *Let $a \in L^p(\mathbb{R}^{2n})$, $1 \le p < +\infty$. Then the operator $Op_{aw}[a] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.*

Proof. Let $\chi_M(z)$ the characteristic function of the set $\{z \in \mathbb{R}^{2n} : |z| \le M\}$. Then from Lemma 1.3 the operator $A_M = Op_{aw}[a\chi_M]$ is compact. By Theorem 1.2, $\|A - A_M\|_{B(L^2)} \le C\,\|a - a\chi_M\|_{L^p} \to 0$ as $M \to +\infty$; hence A is compact. □

More generally we can consider anti-Wick symbols in the sum of L^p spaces, setting by definition $Op_{aw}[a + b] = Op_{aw}[a] + Op_{aw}[b]$ if $a \in L^{p_1}$ and $b \in L^{p_2}$, $1 \le p_1,\ p_2 \le +\infty$. Then the following result holds:

Corollary 1.5. *Let $p \in [1, +\infty)$ and $a(z) \in L^p(\mathbb{R}^{2n})$, with $a(z) \ge 0$. Then, for every $\epsilon > 0$, $Op_{aw}[a + \epsilon]$ is a bicontinuous bijection on $L^2(\mathbb{R}^{2n})$.*

Proof. Since $a(z)+\epsilon>0$, we have $Op_{aw}[a+\epsilon]>0$; so it is injective on $L^2(\mathbb{R}^{2n})$. As $Op_{aw}[a]$ is compact, $Op_{aw}[a+\epsilon]=Op_{aw}[a]+\epsilon I$ is a Fredholm operator of index zero, so it is also surjective. The continuity of $Op_{aw}[a+\epsilon]^{-1}$ follows from the inverse mapping theorem. □

We notice that, extending to Γ_0^0 the argument in [10], Chapter 4, the compactness of Weyl operators also follows from Lemma 1.1.

2. Hilbert-Schmidt and trace class operators

Let $S_2(L^2(\mathbb{R}^n))$ be the space of Hilbert-Schmidt operators and $S_1(L^2(\mathbb{R}^n))$ the space of trace class operators on $L^2(\mathbb{R}^n)$. We give sufficient conditions for operators with anti-Wick symbols in $L^p(\mathbb{R}^{2n})$ to belong to these classes. Our results are the following.

Theorem 2.1. *Let $a\in L^p(\mathbb{R}^{2n})$, $p\in[1,2]$. Then $A=Op_{aw}[a]\in S_2(L^2(\mathbb{R}^n))$ and $\|A\|_2=\|b\|_{L^2(\mathbb{R}^{2n})}$, where $b=(2\pi)^{-n}a*\sigma\in L^2(\mathbb{R}^{2n})$ is the Weyl symbol of A.*

Proof. Using the same arguments as in [10], we have also for simbols b in the limit class Γ_0^0 that $b\in L^2(\mathbb{R}^{2n})$ if and only if $Op_w[b]\in S_2(L^2(\mathbb{R}^n))$, with $\|A\|_2=\|b\|_{L^2(\mathbb{R}^{2n})}$. Using Young's inequality $\|b\|_2\le(2\pi)^{-n}\|a\|_p\|\sigma\|_q$, $p\ge1$, $q\ge1$, $\frac{1}{p}=\frac{3}{2}-\frac{1}{q}$, we then have $b\in L^2(\mathbb{R}^{2n})$ under the condition $p\in[1,2]$. □

The previous result cannot be improved, as shown by the following:

Theorem 2.2. *For all $p>2$, there exist $a\in L^p(\mathbb{R}^{2n})$ such that $Op_{aw}[a]\notin S_2(L^2(\mathbb{R}^n))$.*

Proof. Assume, to the contrary, that there exists $p>2$ such that $Op_{aw}[a]\in S_2(L^2(\mathbb{R}^n))$ for all $a\in L^p$. Then, for all $a\in L^p$, the convolution $b=(2\pi)^{-n}a*\sigma$ should belong to $L^2(\mathbb{R}^{2n})$, since every symbol $b\in\Gamma_0^0$ satisfies $b\in L^2\Leftrightarrow Op_w[b]\in S_2(L^2(\mathbb{R}^n))$. Then the map $a\in L^p\to a*\sigma\in L^2,(p>2)$ is linear and continuous; moreover it commutes with translations, so by a property of the convolution transform it would be the null operator (see [12]), a contradiction. □

Now we give an explicit example:

Example 2.3. For all $p>2$ we determine anti-Wick symbols $a\in L^p(\mathbb{R}^{2n})$ such that $Op_w(a*\sigma)\notin S_2(L^2(\mathbb{R}^n))$. Using again the property $Op_w(a*\sigma)\in S_2(L^2(\mathbb{R}^n))$ $\Longleftrightarrow a*\sigma\in L^2(\mathbb{R}^{2n})$, we search for an a such that $|a*\sigma|^2\notin L^1(\mathbb{R}^{2n})$.

Let $\langle z\rangle^{-s}=(1+|z|^2)^{-s/2}$; since $\langle z\rangle^{-s}>0$, it follows that

$$|\langle z\rangle^{-s}*\sigma(z)|^2=2^n\left|\int_{\mathbb{R}^{2n}}e^{-|t|^2}\langle z-t\rangle^{-s}dt\right|^2=2^n\left(\int_{\mathbb{R}^{2n}}e^{-|t|^2}\langle z-t\rangle^{-s}dt\right)^2.$$

We want to determine s such that

$$\left(\int_{\mathbb{R}^{2n}}e^{-|t|^2}\langle z-t\rangle^{-s}dt\right)^2\ge\frac{C}{(1+|z|^2)^n}\notin L^1(\mathbb{R}^{2n}),$$

i.e. $\int_{\mathbb{R}^{2n}} e^{-|t|^2}\langle z-t\rangle^{-s}\langle z\rangle^n dt \geq C$. Using Peetre's inequality, we get

$$\begin{aligned}
&\int (1+|z|^2)^{(n-s)/2}(1+|z|^2)^{s/2}e^{-|t|^2}(1+|z-t|^2)^{-s/2}\,dt \\
&\geq 2^{-s}\int (1+|z|^2)^{(n-s)/2}(1+|t|^2)^{-s/2}e^{-|t|^2}\,dt \\
&= C'_s(1+|z|^2)^{(n-s)/2} \; > \; C
\end{aligned}$$

if $s \leq n$. On the other hand, since $\langle z\rangle^{-s} \in L^p(\mathbb{R}^{2n})$ we get $s > \frac{2n}{p}$. Therefore, $a(z) = \langle z\rangle^{-s}$ is the example we have searched for, under the condition $\frac{2n}{p} < s \leq n$, which we can satisfy for some s if $p > 2$.

Now we state our result about trace class operators.

Theorem 2.4. *Let $a \in L^1(\mathbb{R}^{2n})$. Then $Op_{aw}[a] \in S_1(L^2(\mathbb{R}^n))$ and*

$$Tr(Op_{aw}[a]) = (2\pi)^{-2n}\int a(y,\eta)\,\sigma(x-y,\xi-\eta)\,dxdyd\xi d\eta. \tag{2.1}$$

Proof. Let $b = (2\pi)^{-n} a * \sigma$. By Young's inequality $\|b\|_1 \leq (2\pi)^{-n}\,\|a\|_1\,\|\sigma\|_1$, and, for all $\alpha \in \mathbb{Z}^n_+$, $\|\partial^\alpha b\|_1 \leq (2\pi)^{-n}\,\|a\|_1\,\|\partial^\alpha\sigma\|_1$. Then $b \in L^1$ and $\partial^\alpha b \in L^1$, $\forall \alpha \in \mathbb{Z}^n_+$, and we have that $Op_w[b] \in S_1(L^2(\mathbb{R}^n))$ and trace's formula (2.1) holds (see Theorem II-53 of [9]). □

We notice that a similar result under different hypothesis is in Berezin [1]. We also remark that, with a minor modification of the same argument, Theorem 2.4 also holds for complex Borel measures as well as for distributions with compact support.

Remark 2.5. Let $a \in L^p(\mathbb{R}^{2n})$, $p > 1$. Then $Op_{aw}[a]$ is not, in general, a trace class operator. An easy example can be constructed from the harmonic oscillator (for simplicity in dimension 1) $H = -\frac{d^2}{dx^2} + |x|^2$. Denote by G^m_ρ the space of pseudo-differential operators with Weyl symbols in Γ^m_ρ, and consider the operator $A = H + I \in G^2_1$; its Weyl symbol is $h_0(x,\xi) = \xi^2 + x^2 + 1 = |z|^2 + 1$, with $z = (x,\xi)$. The operator $A^{-1} = (H+I)^{-1} \in G^{-2}_1$ has Weyl symbol $\sigma_w(A^{-1}) = (|z|^2+1)^{-1} + r(z)$, with $r(z) \in \Gamma^{-3}_1$.

Then $\sigma_w(A^{-1}) \in L^p(\mathbb{R}^2)$ for $p > 1$, but $\sigma_w(A^{-1}) \notin L^1(\mathbb{R}^2)$, and, therefore, $A^{-1} \in S_2(L^2(\mathbb{R}))$, but $A^{-1} \notin S_1(L^2(\mathbb{R}))$ because $\|A^{-1}\|_{tr} = \sum_{j\in\mathbb{N}} \frac{1}{2j+2} = +\infty$.

Every $K \in G^m_\rho$ has an anti-Wick symbol, if we consider it modulo regularizing operators (see, for instance, [2], [10]). Hence there exists $\tilde{A}^{-1} \in G^2_1$, such that $\tilde{A}^{-1} - A^{-1} \in G^{-\infty}$, and its anti-Wick simbol is $\tilde{a}(z) = (1+|z|^2)^{-1} + r(z) + \tilde{r}(z)$, with $\tilde{r}(z) \in \Gamma^{-4}_1$.

Then, $\tilde{a}(z) = (1+|z|^2)^{-1} + r(z)$, with $r(z) \in \Gamma^{-3}_1$. Therefore $\tilde{A}^{-1} \notin S_1(L^2(\mathbb{R}))$, and its anti-Wick simbol satisfies $\tilde{a}(z) \notin L^1(\mathbb{R}^2)$ but $\tilde{a}(z) \in L^p(\mathbb{R}^2)$, $\forall p > 1$.

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