

CHARACTER DEGREE SETS THAT DO NOT BOUND THE CLASS OF A p -GROUP

I. M. ISAACS AND M. C. SLATTERY

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ABSTRACT. Suppose that we are given a set \mathcal{S} of powers of a prime p and that $1 \in \mathcal{S}$. A technique is presented that enables the construction of a p -group of specified nilpotence class n such that its set of irreducible character degrees is exactly \mathcal{S} . If $|\mathcal{S}| \geq 2$, then this can be done for $2 \leq n \leq p$ and if $p \in \mathcal{S}$, then the only requirement is $2 \leq n$.

1. INTRODUCTION

If G is a finite group, then as usual, we write $\text{cd}(G)$ to denote the set of degrees of the irreducible characters of G . Now fix some prime p and let \mathcal{S} be a finite set of powers of p , subject only to the condition that $1 \in \mathcal{S}$. It is known that there necessarily exists some p -group P such that $\text{cd}(P) = \mathcal{S}$, and in fact, it is always possible to choose P so that its nilpotence class $c(P)$ is at most 2. (This is the main result of [2].)

For some choices of the set \mathcal{S} , there exists an upper bound on the nilpotence classes of those p -groups P for which $\text{cd}(P) = \mathcal{S}$, while for other sets \mathcal{S} , there is no such bound. For example, it was shown in [1] that if $\text{cd}(P) = \{1, p^e\}$, then the class of P is at most p if $e > 1$, but that the class of P can be unboundedly large if $e = 1$. (The problem of determining which sets of powers of p imply an upper bound on the nilpotence class of a p -group having that degree set was first suggested by the second author in his paper [6].)

We shall say that a set \mathcal{S} of powers of p (containing 1) is **class bounding** if there exists an upper bound (depending on \mathcal{S} , of course) for the nilpotence classes of all p -groups P such that $\text{cd}(P) = \mathcal{S}$. By the result mentioned in the previous paragraph, for example, we see that if $|\mathcal{S}| = 2$, then $|\mathcal{S}|$ is class bounding if and only if $p \notin \mathcal{S}$. It was proved in [5] that the corresponding result also holds when $|\mathcal{S}| = 3$. In this case too, the set \mathcal{S} is class bounding if and only if $p \notin \mathcal{S}$. Other class-bounding sets were constructed in [5], including some having arbitrarily large cardinality, but all known class-bounding sets of powers of p fail to contain the number p .

Of course, these results suggest the general conjecture that an arbitrary set \mathcal{S} of powers of p (containing 1) is class bounding if and only if $p \notin \mathcal{S}$. The main result of this paper establishes the “only if” part of this conjecture.

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Theorem A. *Let p be any prime, let $n \geq 2$ be an integer and suppose that \mathcal{S} is a finite set of powers of p containing both 1 and p . Then there exists a p -group P with nilpotence class n and $\text{cd}(P) = \mathcal{S}$.*

Our proof of Theorem A uses a variation of the technique used in [5] to prove the corresponding result when $|\mathcal{S}| = 3$.

2. THE CONSTRUCTION

Our principal construction tool is the well known “pullback”, as is described in the following lemma.

(2.1) Lemma. *Let $A \triangleleft X$ and $B \triangleleft Y$ be groups such that $X/A \cong Y/B$. Then there exists a group P having normal subgroups R and S with the following properties:*

- (a) *There exist isomorphisms $\alpha : P/R \rightarrow X$ and $\beta : P/S \rightarrow Y$.*
- (b) *$R \cap S = 1$.*
- (c) *$\alpha(RS/R) = A$ and $\beta(RS/S) = B$.*

Proof. Write $\bar{X} = X/A$ and $\bar{Y} = Y/B$ and let $\theta : \bar{X} \rightarrow \bar{Y}$ be the given isomorphism. Let $\Gamma = X \times Y$ be the external direct product and take P to be the subgroup of Γ consisting of all pairs (x, y) such that $\theta(\bar{x}) = \bar{y}$. It is clear that for each element $x \in X$, there exists an element $y \in Y$ such that (x, y) lies in P , and since θ is surjective, we see that the reverse is true too: For each element $y \in Y$, there exists $x \in X$ such that $(x, y) \in P$. In other words, the projection homomorphisms from P to X and from P to Y are surjective. We let R and S be respectively the kernels of these homomorphisms and we let α and β be the corresponding isomorphisms from P/R onto X and from P/S onto Y . We have now established (a).

The first component of an arbitrary element of R is 1, and hence the second component can be any element $y \in Y$ such that $\theta(\bar{1}) = \bar{y}$. In other words, the second component of an element of R is an arbitrary element of B , and thus $R = 1 \times B$. Also, since θ is injective, we see that $S = A \times 1$, and in particular, it is clear that $R \cap S = 1$, as required.

Now $\alpha(RS/R)$ is the image of $RS = A \times B$ under the projection to X , and it follows that $\alpha(RS/R) = A$. Similarly, $\beta(RS/S) = B$, and the proof is complete. \square

In the notation of Lemma 2.1, it is easy to see that if X and Y are p -groups, then P is also a p -group. Furthermore, it follows from the facts that $P/R \cong X$, $P/S \cong Y$ and $R \cap S = 1$ that the nilpotence class $c(P) = \max\{c(X), c(Y)\}$.

(2.2) Definition. Let $a \geq 0$ be an integer. We will say that a p -group P is **suitable** with respect to a if P has an abelian normal subgroup A such that the following hold:

- (1) P/A is elementary abelian of order p^a .
- (2) Every linear character of A extends to its stabilizer in P .

Note that if P is suitable with respect to a , then since the subgroup A is abelian and has index p^a , it follows that every irreducible character of P has degree at most $|P : A| = p^a$. Note also that every abelian p -group is suitable with respect to 0 and that every p -group having an abelian subgroup of index p is suitable with respect to 1. (To check this last assertion, it suffices to observe that an invariant irreducible character of a normal subgroup of prime index in any finite group is guaranteed to be extendible. This is Corollary 6.20 of [3].) The key to the proof of Theorem A is the following “one-new-degree” theorem.

(2.3) Theorem. *Let $b > a \geq 0$ be integers and suppose that Q is a p -group that is suitable with respect to a . Then there exists a p -group P that is suitable with respect to b and such that $\text{cd}(P) = \text{cd}(Q) \cup \{p^b\}$. Furthermore, P can be chosen so that its nilpotence class $c(P)$ is the maximum of 2 and the class $c(Q)$.*

Proof. Let $X = Q \times V$, where V is elementary abelian of order p^{b-a} and view Q and V as subgroups of X . Since Q is suitable with respect to a , we know that Q has an abelian normal subgroup A such that Q/A is elementary abelian of order p^a , and it follows that $A \triangleleft X$ and that X/A is elementary abelian of order p^b .

Now let Y be extraspecial of order p^{2b+1} and let B be a maximal abelian subgroup of Y . Then as is well known, $|Y : B| = p^b$ and Y/B is elementary abelian, and thus $X/A \cong Y/B$. We can therefore construct the “pullback” P , as in Lemma 2.1, and we use the notation of that lemma. As we remarked earlier, P is a p -group and $c(P)$ is the maximum of $c(X) = c(Q)$ and $c(Y) = 2$, as required.

Now $RS \cong R \times S \cong A \times B$, which is abelian, and $P/RS \cong X/A$, which is elementary abelian of order p^b . To prove that P is suitable with respect to b , therefore, it suffices to show that every linear character of the subgroup RS extends to its stabilizer in P . Every such linear character is uniquely of the form $\lambda\mu$, where $R \subseteq \ker(\lambda)$ and $S \subseteq \ker(\mu)$, and because of the uniqueness, we see that the stabilizer of $\lambda\mu$ in P is the intersection of the stabilizers in P of λ and μ .

There are just two possibilities for the stabilizer of μ in P . To see why this is so, observe that μ can be identified with a linear character μ_0 of $B \cong RS/S$. (Recall that B is a maximum abelian subgroup of the extraspecial group $Y \cong P/S$.) If $Y' \subseteq \ker(\mu_0)$, then μ_0 is extendible to Y and hence is invariant in Y . Otherwise, μ_0 induces irreducibly to Y , and hence B is its stabilizer in Y . We deduce that either μ is invariant in (and extends to) P or else the stabilizer of μ in P is the subgroup RS . In the latter case, we see that the stabilizer of $\lambda\mu$ in P is RS , and in this situation it is trivially true that $\lambda\mu$ extends to its stabilizer, as required. We can thus suppose that μ is invariant in P and we let T be the stabilizer of λ in P . Then T is also the stabilizer of $\lambda\mu$ in P and we must show that $\lambda\mu$ extends to T . But since μ extends to P in this situation, it suffices to show that λ extends to T .

Now λ corresponds to a linear character λ_0 of $A \cong RS/R$. Also, the stabilizer T_0 of λ_0 in $X = Q \times V$ is the subgroup that corresponds to T/R under the isomorphism $X \cong P/R$. Since Q is suitable, we know that λ_0 extends to its stabilizer in Q , and thus it also extends to its stabilizer T_0 in $X = Q \times V$. It follows from this that λ extends to its stabilizer T in P , and this completes the proof that P is suitable with respect to b , as desired.

Finally, we must show that $\text{cd}(P) = \text{cd}(Q) \cup \{p^b\}$. Since $P/R \cong X = Q \times V$, we see that Q is a homomorphic image of P , and thus $\text{cd}(Q) \subseteq \text{cd}(P)$. Similarly, the extraspecial group $Y \cong P/S$ is a homomorphic image of P and we know that $\text{cd}(Y) = \{1, p^b\}$, and thus $p^b \in \text{cd}(P)$, as desired. To complete the proof, we must consider an arbitrary character $\chi \in \text{Irr}(P)$ and show that either $\chi(1) = p^b$ or $\chi(1) \in \text{cd}(Q)$.

Let $\lambda\mu$ be a linear constituent of χ_{RS} , where, as before, $R \subseteq \ker(\lambda)$ and $S \subseteq \ker(\mu)$. We saw previously that the stabilizer of μ is either RS or P . If the stabilizer of μ is RS , we know that the stabilizer of $\lambda\mu$ is also RS , and thus $(\lambda\mu)^P$ is irreducible. In this case, $\chi = (\lambda\mu)^P$ has degree $|P : RS| = p^b$, and there is nothing further to prove. In the remaining case, we know that $\lambda\mu$ extends to its stabilizer T in P , and thus since T/RS is abelian, it follows that every irreducible character

of T that lies over $\lambda\mu$ is linear. We conclude from this that $\chi(1) = |P : T|$. But we know in this case that T corresponds to the stabilizer T_0 of some linear character λ_0 of A in the group $X = Q \times V$, and we also know that λ_0 extends to T_0 . It follows that the number $|P : T| = |X : T_0|$ lies in the set $\text{cd}(X) = \text{cd}(Q)$. This completes the proof. \square

Our principal application of Theorem 2.3 is the following, from which Theorem A is immediate.

(2.4) Corollary. *Suppose that Q is a p -group that is suitable with respect to e , where p^e is the largest member of $\text{cd}(Q)$, and let \mathcal{B} be a set of powers of p all of which exceed p^e . Then there exists a p -group P such that $\text{cd}(P) = \text{cd}(Q) \cup \mathcal{B}$. Furthermore, we can choose P so that $c(P) = c(Q)$ except when $c(Q) = 1$ and \mathcal{B} is nonempty, in which case $c(P) = 2$.*

Proof. We can certainly assume that \mathcal{B} is nonempty and we write $\mathcal{B} = \{p^{e_1}, p^{e_2}, \dots, p^{e_r}\}$, where $e_1 < e_2 < \dots < e_r$. We set $P_0 = Q$ and we use Theorem 2.3 repeatedly to construct a sequence of p -groups P_i , where $1 \leq i \leq r$. We can do this so that $\text{cd}(P_i) = \text{cd}(P_{i-1}) \cup \{p^{e_i}\}$ and each of the groups P_i is suitable with respect to e_i . Furthermore, the nilpotence classes of the groups P_i will all be equal to $c(Q)$ except when $c(Q) = 1$, in which case $c(P_i) = 2$ for $i > 0$. The group $P = P_r$ has the desired properties. \square

Proof of Theorem A. By [1], we know that there exists a p -group Q such that $c(Q) = n$ and $\text{cd}(Q) = \{1, p\}$. Furthermore, Q has an abelian subgroup of index p , and thus Q is suitable with respect to 1. We can now apply Corollary 2.4 with this group Q and with $\mathcal{B} = \mathcal{S} - \{1, p\}$ to obtain a group P such that $c(P) = n$ and $\text{cd}(P) = \text{cd}(Q) \cup \mathcal{B} = \mathcal{S}$, as desired. \square

3. FURTHER APPLICATIONS

We can also use Corollary 2.4 to give a new proof of the main theorem of [2].

(3.1) Theorem. *Let \mathcal{S} be a set of powers of p and assume that $1 \in \mathcal{S}$. Then there exists a p -group P with nilpotence class 2 and $\text{cd}(P) = \mathcal{S}$.*

Proof. Let Q be any abelian p -group. Then Q is suitable with respect to 0 and $\text{cd}(Q) = \{1\}$. Now apply Corollary 2.4 with $\mathcal{B} = \mathcal{S} - \{1\}$ to obtain the desired group P . \square

If \mathcal{S} is a class-bounding set of powers of p , we ask how small the corresponding bound can be. We can use Corollary 2.4 to show that this bound can never be smaller than p , and in fact, we have a bit more.

(3.2) Theorem. *Let \mathcal{S} be a set of powers of p and assume that $1 \in \mathcal{S}$ and $|\mathcal{S}| > 1$. If n is an integer and $2 \leq n \leq p$, then there exists a p -group P with nilpotence class n and $\text{cd}(P) = \mathcal{S}$.*

Proof. Let 1 and p^e be the two smallest members of \mathcal{S} . We claim that there exists a p -group R such that $\text{cd}(R) = \{1, p^e\}$ and class $c(R) = p$, and that in addition, R can be chosen so that it has an abelian normal subgroup B such that R/B is elementary abelian of order p^e . If $e = 1$, it is easy to construct such a group: Let R be the wreath product of a cyclic group of order p with itself. If $e > 1$, then Example 3.9 of [1] shows that R exists.

Since $n \leq p$, there is some homomorphic image Q of R with $c(Q) = n$, and we know that $\text{cd}(Q) \subseteq \text{cd}(R) = \{1, p^e\}$. Also, because $n \geq 2$, we see that Q is nonabelian, and thus $\text{cd}(Q) = \{1, p^e\}$. Furthermore, the image $A \subseteq Q$ of B is abelian and $|Q : A| \leq |R : B| = p^e$. It follows that $|Q : A| = p^e$ and $Q/A \cong R/B$ is elementary abelian.

Now if λ is any linear character of A , we show that λ extends to its stabilizer in Q . If some linear character of Q lies over λ , this is obvious, and otherwise there exists $\chi \in \text{Irr}(Q)$ lying over λ with $\chi(1) = p^e$. Since $|P : A| = p^e$, it follows that $\lambda^G = \chi$, and thus the full stabilizer of λ in P is the subgroup A . In this case too, we see that λ extends to its stabilizer, and thus Q is suitable with respect to e .

We can now apply Corollary 2.4 with $\mathcal{B} = \mathcal{S} - \{1, p^e\}$ to obtain a group P with the desired properties. \square

4. SOME QUESTIONS

The big remaining question, of course, is the following.

Question 1. If \mathcal{S} is a set of powers of p that contains 1 but not p , must \mathcal{S} be class bounding?

It seems hard to find sets that one can prove to be class bounding, and so the answer to Question 1 may well be “no”. In fact, the assumption that \mathcal{S} is class bounding may turn out to be extremely restrictive. It may be, for example, that if \mathcal{S} is class bounding and $\text{cd}(P) = \mathcal{S}$, then the structure of P is very tightly constrained. For example, we do not know the answer to the following.

Question 2. Does there exist a p -group P such that $\text{cd}(P)$ is class bounding and P is not metabelian?

If the answer to Question 2 is “no”, then the answer to Question 1 must also be “no”. This is because there exist p -groups P of arbitrarily large derived length for which $p \notin \text{cd}(P)$. For example, we can take P to be a Sylow p -subgroup of $GL(n, p^e)$ where $e > 1$ and n is large. In this situation, it is known (see [4]) that $\text{cd}(P)$ consists of powers of p^e , and so does not contain p , but the nilpotence class and derived length of P are unboundedly large as n grows.

For every class-bounding set \mathcal{S} , there is some integer $b(\mathcal{S})$, which is the largest possible nilpotence class for a p -group P such that $\text{cd}(P) = \mathcal{S}$. By Theorem 3.2, we know that $b(\mathcal{S}) \geq p$ for every class-bounding set other than $\{1\}$, but is there an upper bound for $b(\mathcal{S})$?

Question 3. Can $b(\mathcal{S})$ be arbitrarily large for a class-bounding set \mathcal{S} ?

If the answer to Question 3 is “no”, then again, by considering a Sylow p -subgroup of $GL(n, p^e)$ with $e > 1$ and n large, we see that the answer to Question 1 must be “no”.

We know of just one class-bounding set \mathcal{S} for which we can show that $b(\mathcal{S})$ exceeds p : the set $\mathcal{S} = \{1, 4, 16\}$. (This set is class bounding because it has cardinality 3 and does not contain $p = 2$.) If P is a Sylow 2-subgroup of $GL(4, 4)$, then it is well known that P has class 3. Also, all members of $\text{cd}(P)$ are powers of 4 and it is easy to see that P has an abelian subgroup of index 16. It follows that $\text{cd}(P) \subseteq \{1, 4, 16\}$, and it is not too hard to show that all three of these degrees occur, and thus $\text{cd}(P) = \{1, 4, 16\}$. This suggests what is probably the least ambitious of our questions.

Question 4. If $p > 2$, does there exist a p -group P of class $p+1$ such that $\text{cd}(P) = \{1, p^2, p^4\}$?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, 480 LINCOLN DRIVE, MADISON, WISCONSIN 53706

E-mail address: isaacs@math.wisc.edu

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, MARQUETTE UNIVERSITY, P.O. BOX 1881, MILWAUKEE, WISCONSIN 53201

E-mail address: mikes@mscs.mu.edu