ON THE CHROMATIC NUMBER OF KNESER HYPERGRAPHS

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Abstract. We give a simple and elementary proof of Kříž’s lower bound on the chromatic number of the Kneser r-hypergraph of a set system $\mathcal{S}$.

1. Introduction

Let $\mathcal{S}$ be a system of subsets of a finite set $X$. The Kneser r-hypergraph $KG_r(\mathcal{S})$ has $\mathcal{S}$ as the vertex set, and an $r$-tuple $(S_1, S_2, \ldots, S_r)$ of sets in $\mathcal{S}$ forms an edge if $S_i \cap S_j = \emptyset$ for all $i \neq j$. In particular, $KG(\mathcal{S}) = KG_2(\mathcal{S})$ is the Kneser graph of $\mathcal{S}$. Kneser [8] conjectured in 1955 that

$$\chi\left(\frac{n!}{k!}\right) \geq n - 2k + 2, \quad n \geq 2k,$$

where $\left(\begin{array}{c} n \\ k \end{array}\right)$ denotes the system of all $k$-element subsets of the set $\{1, 2, \ldots, n\}$, and $\chi$ denotes the chromatic number. This was proved in 1978 by Lovász [12], as one of the earliest and most spectacular applications of topological methods in combinatorics. Several other proofs have been published since then, all of them topological; among them, at least those of Bárany [2], Dol’nikov [6] (also see [5] and [7]), and Sarkaria [14] can be regarded as substantially different from each other and from Lovász’ original proof. Erdős’ generalization of Kneser’s conjecture to hypergraphs, dealing with the chromatic number of $KG_r\left(\left(\begin{array}{c} n \\ k \end{array}\right)\right)$, was established by Alon, Frankl, and Lovász [1].

Kříž [10], [11] proved a remarkable lower bound for the chromatic number of $KG_r(\mathcal{S})$ for an arbitrary set system $\mathcal{S}$, which easily implies the correct bound in the case when $\mathcal{S} = \left(\begin{array}{c} n \\ k \end{array}\right)$ considered by Alon et al. (for $r = 2$, the result was obtained earlier by Dol’nikov [2]).

To state this result, we first recall that a mapping $c: V \rightarrow [m]$ is a (proper) coloring of a hypergraph $H = (V, E)$ if none of the edges $e \in E$ is monochromatic under $c$. The chromatic number $\chi(H)$ of $H$ is the smallest $m$ such that a proper coloring $c: V \rightarrow [m]$ exists. We define the r-colorability defect of $H = (V, E)$ as the smallest number of vertices that must be removed so that the edges living completely on the remaining points form an r-colorable hypergraph, i.e.

$$cd_r(\mathcal{H}) = \min\left\{|Y| : \chi(\{V \setminus Y, \{e \in E : e \cap Y = \emptyset\}\}) \leq r\right\}.$$
Kříž’s result can be stated as follows.

**Theorem 1.1** (Dol’nikov for $r = 2$; Kříž). For any finite set system $(X, S)$ and any $r \geq 2$, we have

$$
\chi(KG_r(S)) \geq \frac{1}{r-1} \cdot \text{cd}_r((X, S)).
$$

The proof in [10] does not work in the generality stated there (as was pointed out by Živaljević) but Theorem 1.1 remains valid [11]. We remark that $KG_r(S)$ is denoted by $[S, r]$ in [10], and $\text{cd}_r(S)$ is denoted by $w(S, r)$ there and called the $r$-width.

In this paper, we present another proof of Theorem 1.1. The basic approach is similar to that of Kříž, but our proof is somewhat simpler and, hopefully, more accessible to non-specialists in topology.

We only assume the reader’s familiarity with a few basic topological notions (such as simplicial complex and its geometric realization); more special topological notions are reviewed in Section 2 in very concrete form just suitable for our purposes. We refer to Björner [3] or Živaljević [16], [17] for wider background and for nice recent overviews of topological methods in combinatorics.

After this paper has been submitted for publication, the author obtained a “de-topologized” (combinatorial) proof of Kneser’s conjecture [13], by directly connecting some of the ideas of the present paper to a combinatorial lemma (Tucker’s lemma) in one of the proofs of the Borsuk–Ulam theorem. This result was further extended by Ziegler [15], who proved a common generalization of Theorem 1.1 and of theorems of Alon et al. [11] and of Sarkaria [14]. The proof is based on topological ideas but uses no “continuous” structure.

2. Preliminaries

**Simplicial complexes.** For our purposes, a (finite) simplicial complex $K$ is a hereditary family of subsets of a finite set (i.e. if $F \in K$ and $F' \subseteq F$ then $F' \in K$); the sets in $K$ are called simplices. The *dimension* of a simplex is the number of its vertices minus 1. The vertex set of $K$ is denoted by $V(K)$, and the polyhedron of a geometric realization of $K$ is denoted by $||K||$.

Let $(V, \leq)$ be a partially ordered set. The *order complex* of $(V, \leq)$ is the simplicial complex with vertex set $V$ and with all chains under $\leq$ (i.e. subsets of $V$ linearly ordered by $\leq$) as simplices. The *first barycentric subdivision* of a simplicial complex $K$, denoted by $\text{sd}(K)$, is the order complex of the set of all nonempty simplices of $K$ ordered by inclusion. The polyhedra of $K$ and of $\text{sd}(K)$ are canonically homeomorphic.

Let $K, L$ be simplicial complexes. A *simplicial map* $f : K \to L$ is a map $V(K) \to V(L)$ such that the image of any simplex of $K$ is contained in a simplex of $L$. A simplicial map induces a map $||K|| \to ||L||$ of topological spaces.

The *join* $K \ast L$ of simplicial complexes $K$ and $L$ with $V(K) \cap V(L) = \emptyset$ is the simplicial complex with vertex set $V(K) \cup V(L)$ and with simplices $F \cup G$ for all $F \in K$ and $G \in L$. If $V(K)$ and $V(L)$ are not disjoint, then $K \ast L$ is the join of $K$ with an isomorphic copy of $L$ whose vertex set is disjoint from $V(K)$. If $K_1, K_2, L_1, L_2$ are simplicial complexes, $V(K_1) \cap V(L_1) = \emptyset$, and $f : K_1 \to K_2$ and $g : L_1 \to L_2$ are simplicial maps, then $f \ast g : K_1 \ast L_1 \to K_2 \ast L_2$ is the simplicial map given by $(f \ast g)(v) = f(v)$ for $v \in V(K_1)$ and $(f \ast g)(v) = g(v)$ for $v \in V(L_1)$. 
Connectivity. Let $X, Y$ be topological spaces, and $k \geq 0$ an integer. All mappings between topological spaces are implicitly assumed to be continuous. $X$ is $k$-connected if for any $j = 0, 1, \ldots, k$, any mapping $f$ of the $j$-dimensional sphere $S^j$ into $X$ can be extended to a mapping of the $(j+1)$-dimensional ball into $X$.

$Z_p$-spaces. A $Z_p$-space is a pair $(X, \nu)$, where $\nu : X \to X$ is a homeomorphism $X \to X$ with $\nu^p = \text{id}_X$; $\nu$ is called a $Z_p$-action on $X$ (here $Z_p$ denotes the group $\mathbb{Z}/p\mathbb{Z}$, i.e. integers modulo $p$ with addition). The $Z_p$-action $\nu$ is called free if for each $x \in X$ the points $x, \nu(x), \nu^2(x), \ldots, \nu^{p-1}(x)$ are pairwise distinct. For prime $p$, it suffices to require $\nu(x) \neq x$ for all $x$. A simplicial $Z_p$-complex is a simplicial complex $X$ with a $Z_p$-action on $|X|$ given by a simplicial map $X \to X$.

For $Z_p$-spaces $(X, \nu), (Y, \omega)$, a $Z_p$-mapping $f : (X, \nu) \to (Y, \omega)$ is a mapping of $X$ into $Y$ that commutes with the $Z_p$-actions, i.e. $f \circ \nu = \omega \circ f$.

The $Z_p$-index. For integers $k$ and $p$, we define the simplicial complex $E_{k,p}$ whose maximal simplices are the edges of the complete $(k+1)$-uniform $(k+1)$-partite hypergraph with classes of size $p$. More formally, the vertex set is $[k+1] \times [p]$ and the simplices have the form $\{(j_1, i_1), (j_2, i_2), \ldots, (j_q, i_q)\}$, $1 \leq j_1 < j_2 < \cdots < j_q \leq k+1$ and $i_t \in [p]$, $t = 1, 2, \ldots, q$. The mapping $\omega : V(E_{k,p}) \to V(E_{k,p})$ given by $(j, i) \mapsto (j, i+1)$, where $p+1$ means 1, is a free simplicial $Z_p$-action on $E_{k,p}$. In particular, $E_{k,1}$ is the $k$-dimensional sphere represented as the unit sphere of the $L_1$-norm, and the $Z_2$-action is the antipodality $x \mapsto -x$. The important property of $E_{k,p}$ is that its polyhedron is a $k$-dimensional, $(k-1)$-connected free $Z_p$-space; any $k$-dimensional $(k-1)$-connected free simplicial $Z_p$-complex (or $Z_p$-CW-complex) would do equivalently in the definition below.

For a free $Z_p$-space $(X, \nu)$, the $Z_p$-index is defined by

$$\text{ind}_{Z_p}(X) = \min \{k : \text{there is a } Z_p\text{-map } (X, \nu) \to (|E_{k,p}|, \omega)\}$$

(the action $\nu$ is not shown in the notation $\text{ind}_{Z_p}$, but is understood from context). This kind of index, under the name genus, was introduced by Krasnosel’skiĭ [9] (for $Z_2$-spaces); our presentation follows [16]. Let us remark that, while various definitions of indices and deep theories related to them have been developed in algebraic topology, the index just introduced is mainly a convenient notational shorthand.

The key fact about the $Z_p$-index is $\text{ind}_{Z_p}(|E|_{k,p}) = k$, i.e. there is no $Z_p$-map $|E_{k,p}| \to |E_{k-1,p}|$. For $p = 2$, this is one of the versions of the well-known Borsuk–Ulam theorem, and for larger $p$, it is a particular case of a theorem of Dold [4]; see e.g. [17] for a sketch of a proof using only basic homotopy theory.

Clearly, if there is a $Z_p$-map $(X, \nu) \to (Y, \omega)$, then $\text{ind}_{Z_p}(X) \leq \text{ind}_{Z_p}(Y)$. For a free simplicial $Z_p$-complex, we have $\text{ind}_{Z_p}(|K|) \leq \dim(K)$ (this can be shown easily using the $(k-1)$-connectedness of $E_{k,p}$; see, for example, [17]). For free simplicial $Z_p$-complexes $K$ and $L$, we have

$$\text{ind}_{Z_p}(K \ast L) \leq \text{ind}_{Z_p}(K) + \text{ind}_{Z_p}(L) + 1,$$

where the $Z_p$-action on $K \ast L$ is the join of the $Z_p$-actions on $K$ and on $L$. This is easily derived from the isomorphism of $E_{k,p} \ast E_{\ell,p}$ with $E_{k+\ell+1,p}$.

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1 The $(k-1)$-connectedness can be derived in several ways, for example by representing $E_{k,p}$ as the $(k+1)$-fold join $[p]^{k+1}$, where $[p]$ is the $p$-point discrete space, and using the fact that the join of a $j$-connected simplicial complex and of an $\ell$-connected simplicial complex is $(j + \ell + 2)$-connected (see e.g. [9]).
3. Proof of Theorem

First, let \( r = p \) be a prime number. Let \( X = [n] \), and let \( S \) be a set system on \( X \) with \( \text{cd}_p((X, S)) > \ell \).

We define a partial ordering \( \leq \) on the set of all ordered \( p \)-tuples \((A_1, A_2, \ldots, A_p)\) of subsets of \( X \) by letting \((A_1, \ldots, A_p) \leq (A'_1, \ldots, A'_p)\) iff \( A_i \subseteq A'_i \) for all \( i = 1, 2, \ldots, p \).

Consider the set of all ordered \( p \)-tuples \((A_1, A_2, \ldots, A_p)\) such that the \( A_i \) are pairwise disjoint subsets of \( X \) whose union covers all but at most \( \ell \) points of \( X \), and let \( K = K(X, p, \ell) \) be the order complex of this set with the ordering \( \leq \) defined above. A simplicial free \( Z_p \)-action \( \nu \) is defined on \( K \) by the cyclic shift:

\[
\nu: (A_1, \ldots, A_p) \mapsto (A_2, A_3, \ldots, A_p, A_1).
\]

Suppose that \( c: S \to [m] \) is a proper \( m \)-coloring of the Kneser \( p \)-hypergraph \( \text{KG}_p(S) \). This time we consider the set of all ordered \( p \)-tuples \((C_1, \ldots, C_p)\) of subsets of \([m]\) with \( \bigcup_{i=1}^p C_i \neq \emptyset \) and \( \bigcap_{i=1}^p C_i = \emptyset \). Let \( L \) be the order complex of this set with the componentwise inclusion ordering \( \leq \) as above. The simplicial \( Z_p \)-action on \( L \), again given by the cyclic shift of coordinates (i.e. \((C_1, \ldots, C_p) \mapsto (C_2, \ldots, C_p, C_1)\)), is free—here we use that \( p \) is a prime.

Using the \( m \)-coloring \( c \), we are going to define a simplicial \( Z_p \)-map \( f: K \to L \). For a subset \( A \subseteq X \), let

\[
g(A) = \{c(S) : S \subseteq A, S \in S\},
\]

and for a vertex \((A_1, A_2, \ldots, A_p)\) of \( K \), put

\[
f((A_1, A_2, \ldots, A_p)) = (g(A_1), g(A_2), \ldots, g(A_p)).
\]

If \( c \) is a proper coloring, then no \( p \) pairwise disjoint sets of \( S \) can have the same color, and it follows that \( \bigcap_{i=1}^p g(A_i) = \emptyset \). Since we assume \( \text{cd}_p((X, S)) > \ell \), for any ordered \( p \)-tuple \((A_1, \ldots, A_p)\) in \( V(K) \), there are \( i \in [p] \) and \( S \in S \) with \( S \subseteq A_i \).

Therefore, \( \bigcup_{i=1}^p g(A_i) \neq \emptyset \), so \( f((A_1, \ldots, A_p)) \in V(L) \), and it is easy to see that \( f \) is a simplicial \( Z_p \)-map \( K \to L \).

It remains to bound the indices \( \text{ind}_{Z_p}(K) \) and \( \text{ind}_{Z_p}(L) \). As for the latter, we have \( \text{ind}_{Z_p}(L) \leq \text{dim}(L) = m(p-1) \). Indeed, supposing that \((C_1, \ldots, C_p)\) is the largest element in a chain of vertices of \( L \), each \( j \in [m] \) is in at most \( p-1 \) of the \( C_i \), and each time we pass to a smaller element of the chain, some \( j \in [m] \) is omitted from at least one of the sets; thus, the chain has at most \( m(p-1)+1 \) elements.

The \( Z_p \)-index of \( K \) can be bounded from below in several ways (homology computation, inductive argument showing an appropriate connectivity, shelling argument); we use a simple approach inspired by Sarkaria’s papers.

First we consider the larger complex \( K_0 = K(X, p, n-1) \), with all \( p \)-tuples of pairwise disjoint subsets of \( X \), not all of them empty, as vertices. It is well-known that \( \text{ind}_{Z_p}(K_0) = n-1 \) (for those familiar with deleted joins, we remark that \( K_0 \) is the first barycentric subdivision of the \( p \)-fold 2-wise deleted join of the \((n-1)\)-simplex—see e.g. [14]). In fact, \( K_0 \) is \( Z_p \)-isomorphic to \( \text{sd}(E_{n-1,p}) \): the isomorphism \( \varphi: V(\text{sd}(E_{n-1,p})) \to V(K_0) \) is given by \( \{(j_1, i_1), (j_2, i_2), \ldots, (j_q, i_q)\} \mapsto (A_1, A_2, \ldots, A_p) \), where \( A_i = \{j_t : i_t = i, t = 1, 2, \ldots, q\} \).

Let \( K_1 \) be the subcomplex of \( K_0 \) with \( V(K_1) = V(K_0) \setminus V(K) \) and with simplices inherited from \( K_0 \), i.e. the simplices are the \( F \in K_0 \) with \( F \subseteq V(K_1) \). The
vertices of $K_1$ are $p$-tuples $(A_1, \ldots, A_p)$ of disjoint sets with $|\bigcup_{i=1}^p A_i| \leq n-\ell-1$, and $\text{ind}_{Z_p}(K_1) \leq \dim(K_1) = n-\ell-2$. We have $K_0 \subseteq K * K_1$, and so by [1]

$$\text{ind}_{Z_p}(K) \geq \text{ind}_{Z_p}(K_0) - \text{ind}_{Z_p}(K_1) - 1 = n - 1 - (n-\ell-2) - 1 = \ell.$$ 

Since we have constructed the $Z_p$-map $f: K \to L$, we have $\ell \leq \text{ind}_{Z_p}(K) \leq \text{ind}_{Z_p}(L) \leq m(p-1)$. This proves Theorem [1] for all prime $r$.

The non-prime case is handled by a short combinatorial argument, which is given in [11] and which we omit.

**Remark.** As we have seen, the simplicial complex $K_0$ is the subdivision of $E_{n-1,p}$; in particular, for $p = 2$, it is an $(n-1)$-sphere. The subcomplex $K_1$ is the subdivision of the $(n-\ell-2)$-skeleton of $E_{n-1,p}$. For $p = 2$, the simplicial complex $K$ also has a nice interpretation (noted by G. Ziegler): it can be regarded as the subdivision of the $\ell$-skeleton of the cube $[0,1]^n$ (interpreted as a cell complex, with faces being the usual faces of the cube, i.e. cubes of various dimensions). Indeed, a vertex $(A,B)$ of $K$ can be encoded by a sequence $v \in \{0,1,*\}^X$, where $v_x = 0$ if $x \in A$, $v_x = 1$ if $x \in B$, and $v_x = *$ otherwise. Each such $v$ specifies a face of the $n$-cube.

**References**


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