ONE SIDED IDEALS AND CARLSON’S THEOREM

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Abstract. Using left ideals, right ideals, and the smallest two sided ideal in a compact right topological semigroup, we derive an extension of the Main Lemma to Carlson’s Theorem. This extension involves an infinite sequence of variable words over a finite alphabet, some of which are required to have the variable as the first letter and others of which are required to have the variable as the last letter.

1. Introduction

In 1988 T. Carlson published a Ramsey Theoretic result [2, Theorem 2] which has as corollaries many earlier results in Ramsey Theory. (See [8, Section 18.4] for a relatively short presentation of Carlson’s Theorem and some of its consequences.) Experience suggests that Carlson’s “Main Lemma” [2, Lemma 5.9] implies those Ramsey Theoretic corollaries of his theorem in which a finite collection of finite objects is partitioned into finitely many classes. We shall state this Main Lemma after introducing some necessary terminology.

For \( k \in \mathbb{N} \), the set of positive integers, let \( W_k \) be the free semigroup with identity \( e \) on the alphabet \( \{1, 2, \ldots, k\} \). That is, \( W_k \) consists of all “words with letters from \( \{1, 2, \ldots, k\} \)” (i.e. functions whose domain is an initial segment of \( \mathbb{N} \) and whose range is contained in \( \{1, 2, \ldots, k\} \)) together with the empty word, with the operation of concatenation. A variable word over \( W_k \) is a word on the alphabet \( \{1, 2, \ldots, k\} \cup \{v\} \) in which \( v \) occurs, where \( v \) is a “variable” not in \( \{1, 2, \ldots, k\} \). Given a variable word \( w \) over \( W_k \), and \( t \in \{1, 2, \ldots, k\} \), \( w(t) \) has its obvious meaning, namely the result of replacing all occurrences of \( v \) with \( t \). (There is a potential conflict here with the formal viewpoint which takes \( w \) to be a function. If we have occasion to need the value of the function \( w \) at \( t \) we will denote it as \( w_t \).)

1.1. Definition. Let \( k \in \mathbb{N} \) and let \( \langle w_n \rangle_{n=1}^{\infty} \) be a sequence of variable words over \( W_k \). The sequence \( \langle w_n \rangle_{n=1}^{\infty} \) is a variable reduction of \( \langle w_n \rangle_{n=1}^{\infty} \) if and only if there exist an increasing function \( g : \mathbb{N} \rightarrow \mathbb{N} \) and a function \( f : \mathbb{N} \rightarrow \{1, 2, \ldots, k\} \cup \{v\} \) such that:

\[
(1) \quad g(1) = 1,
\]
partitioned into finitely many classes, and let \( n > 0 \) for every \( \prod \) expressions of the form \( \langle \).

Similarly, a \( \beta S \) well as a basis for the closed sets) of \( L \).

There is a natural extension of the operation \( \cdot \) to \( \beta S \) making \( \beta S \) a compact right topological semigroup with \( S \) contained in its topological center. This says that for each \( p \in \beta S \) the function \( \rho_p : \beta S \to \beta S \) is continuous and for each \( x \in S \), the function \( \lambda_x : \beta S \to \beta S \) is continuous, where \( \rho_p(q) = q \cdot p \) and \( \lambda_x(q) = x \cdot q \). The operation is characterized by the fact that for any \( p \) and \( q \) in \( \beta S \) and any \( A \subseteq S \), \( A \in p \cdot q \) if and only if \( \{ x \in S : x^{-1}A \subseteq q \} \subseteq p \), where \( x^{-1}A = \{ y \in S : x \cdot y \in A \} \).

See [8] for an elementary introduction to the semigroup \( \beta S \).

Any compact Hausdorff right topological semigroup \( (T, \cdot) \) has a smallest two sided ideal \( K(T) \) which is the union of all of the minimal left ideals of \( T \), each of which is closed, and is also the union of all of the minimal right ideals of \( T \) [8, Theorem 2.8]. Given any minimal left ideal \( L \) and any minimal right ideal \( R \), \( L \cap R \) is a group (and in particular has an idempotent) [8, Theorem 2.7]. There is a partial ordering of the idempotents of \( T \) determined by \( p \leq q \) if and only if \( p \cdot q = q \cdot p \). An idempotent \( p \) is minimal with respect to this order if and only if \( p \in K(T) \) [8, Theorem 1.59]. Such an idempotent is called simply “minimal”.

Members of minimal idempotents in \( \beta S \) are the central subsets of \( S \). Central sets were introduced by H. Furstenberg in [3] and defined in terms of notions of
topological dynamics. These sets enjoy very strong combinatorial properties. (See [4 Proposition 8.21] or [8 Chapter 14].) See [8 Theorem 19.27] for a proof of the equivalence of the definition of “central” in terms of $\beta S$ with the original dynamical definition.

The basic algebraic fact used in [1] (as well as several other papers) is that any two sided ideal in a compact right topological semigroup $T$ contains $K(T)$. While variable words over $W_k$ yield a two sided ideal in an appropriately chosen compact right topological semigroup, left variable words and right variable words do not.

Left variable words do, however, correspond naturally to a right ideal (and right variable words correspond to a left ideal). In [7] we were able to use these natural left and right ideals to obtain a generalization of Theorem 1.3 involving both left and right variable words. In the current paper we use similar left and right ideals to obtain a generalization of Theorem 1.2 which involves both left and right variable words.

2. Extending Carlson’s Main Lemma

We shall have throughout this section a fixed $k \in \mathbb{N}$. We begin with the following simple lemma from semigroup theory.

2.1. Lemma. Let $S$ be a semigroup which has a minimal left ideal which contains an idempotent. Let $L$ be a left ideal of $S$, let $R$ be a right ideal of $S$, and let $e$ be any idempotent in $S$. There is a minimal idempotent $m \in Le \cap eR$. Necessarily $m \leq e$.

Proof. Notice that if $m \in Le$, then $me = m$ and if $m \in eR$, then $em = m$. Thus any idempotent $m \in Le \cap eR$ satisfies $m \leq e$.

By [8 Lemma 1.57] $S$ has a minimal right ideal which contains an idempotent. Thus, by [8 Corollary 1.47] we may presume that $L$ is a minimal left ideal and $R$ is a minimal right ideal. By [8 Theorem 1.46] $Le$ is a minimal left ideal and $eR$ is a minimal right ideal and so, by [8 Theorem 1.61], $Le \cap eR$ is a group. Let $m$ be the identity of $Le \cap eR$. \qed

Given a set $X$, we denote the set of finite nonempty subsets of $X$ by $\mathcal{P}_f(X)$.

2.2. Lemma. Let $S$ be a discrete semigroup, let $A \subseteq \bigtimes_{j=1}^{k+1} S$, and let $Z = \bigtimes_{j=1}^{k+1} \beta S$. Assume that $(y_1, y_2, \ldots, y_{k+1}) \in \text{cl}_Z(A)$, $C \subseteq S$, $W \in P_f(S)$, and $L \in P_f(\beta S)$. Then there exists $(a_1, a_2, \ldots, a_{k+1}) \in A$ such that, for all $l \in L$, all $u \in W$, and all $i \in \{1, 2, \ldots, k+1\}$,

$$(u a_i)^{-1}C \in l \iff u^{-1}C \in y_i l.$$

Proof. For $l \in L$, $u \in W$, and $i \in \{1, 2, \ldots, k+1\}$, let

$$C_{i, u, l} = \begin{cases} C & \text{if } C \in u y_i l, \\ S \setminus C & \text{if } C \notin u y_i l. \end{cases}$$

Then $\overline{C_{i, u, l}}$ is a neighborhood of $uy_i l = \lambda_u (\rho_l(y_i))$, so pick a member $U_{i, u, l}$ of $y_i$ such that $\lambda_u [\rho_l(U_{i, u, l})] \subseteq \overline{C_{i, u, l}}$.

For $i \in \{1, 2, \ldots, k+1\}$, let $N_i = \bigcap_{l \in L} \bigcap_{u \in W} U_{i, u, l}$. Pick $(a_1, a_2, \ldots, a_{k+1}) \in A \cap \bigtimes_{i=1}^{k+1} N_i$. \qed
2.3. Definition. (a) \( Y = (\times_{j=1}^{k} \beta W_k) \times \beta W_{k+1} \).
(b) \( Z = \times_{j=1}^{k+1} \beta W_{k+1} \).
(c) \( I = \{(w(1), w(2), \ldots, w(k+1)) : w \) is a variable word over \( W_k \} \).
(d) \( J = \{(w(1), w(2), \ldots, w(k+1)) : w \) is a left variable word over \( W_k \} \).
(e) \( H = \{(w(1), w(2), \ldots, w(k+1)) : w \) is a right variable word over \( W_k \} \).
(f) \( E = I \cup \{(w, w, \ldots, w) : w \in W_k \} \).

Notice that each of \( I, J, H, \) and \( E \) are contained in \((\times_{j=1}^{k} W_k) \times W_{k+1}\), and consequently their closures in \( Y \) and in \( Z \) are identical.

2.4. Lemma. \( \mathcal{E} \) is a subsemigroup of \( Y \), \( \mathcal{T} \) is an ideal of \( \mathcal{E} \), \( \mathcal{F} \) is a right ideal of \( \mathcal{E} \), and \( \mathcal{K} \) is a left ideal of \( \mathcal{E} \).

Proof. \[\text{[8] Theorems 2.22 and 4.17} \] \( \square \)

The proof of the following lemma uses an idea from [1]. This lemma is needed so that in Theorem 2.9 the cell of the partition \( \mathcal{F} \) can be guaranteed to be central in \( W_{k+1} \). This fact is significant because, as we have noted, central sets are guaranteed to contain substantial combinatorial structures.

2.5. Lemma. \( \mathcal{E} \cap K(Y) \neq \emptyset \) and so
\[ K(\mathcal{E}) = \mathcal{E} \cap \left( (\times_{j=1}^{k} K(\beta W_k)) \times K(\beta W_{k+1}) \right) = \mathcal{E} \cap K(Y) \).

Proof. Let \( s \) be a minimal idempotent in \( \beta W_k \) and pick by \[\text{[8] Theorem 1.60 and Corollary 2.6} \] an idempotent \( p \) in \( W_{k+1} \) such that \( p \leq s \). Let \( \vec{p} = (s, s, \ldots, s, p) \).

We claim that it suffices to show that \( \vec{p} \in \mathcal{T} \). Indeed, assume that we have done so. We have that \( \vec{p} \in (\times_{j=1}^{k} K(\beta W_k)) \times K(\beta W_{k+1}) \) and by \[\text{[8] Theorem 2.23} \] \( K(Y) = (\times_{j=1}^{k} K(\beta W_k)) \times K(\beta W_{k+1}) \). Thus we have that \( \vec{p} \in \mathcal{T} \cap K(Y) \subseteq \mathcal{E} \cap K(Y) \). It then follows from \[\text{[8] Theorem 1.65} \] that \( K(\mathcal{E}) = \mathcal{E} \cap K(Y) \).

To see that \( \vec{p} \in \mathcal{T} \), let \( A \in s \) and \( B \in p \) be given. We need to show that \( (A^k \times B) \cap I \neq \emptyset \). For each \( t \in \{1, 2, \ldots, k\} \) define \( g_t : W_{k+1} \rightarrow W_k \) by letting \( g_t(w) \) be the word obtained by replacing each occurrence of \( k+1 \) by an occurrence of \( t \) and let \( g_t : \beta W_{k+1} \rightarrow \beta W_k \) be its continuous extension. Let \( t \in \{1, 2, \ldots, k\} \) be given. By \[\text{[8] Corollary 4.22} \], \( g_t \) is a homomorphism and so \( \bar{g}_t(p) \leq \bar{g}_t(s) \). Since \( g_t \) is the identity on \( W_k \), we have that \( \bar{g}_t(s) = s \), and thus \( \bar{g}_t(p) \leq s \). Since \( s \) is minimal, we therefore have that \( \bar{g}_t(p) = s \).

Since \( A \) is a neighborhood of \( \bar{g}_t(p) \) for each \( t \in \{1, 2, \ldots, k\} \), we have that \( \bigcap_{t=1}^{k} g_t^{-1}[A] \in p \). Also, since \( \beta W_{k+1} \setminus \beta W_k \) is an ideal of \( \beta W_{k+1} \) and \( p \) is minimal in \( \beta W_{k+1} \), we have that \( W_{k+1} \setminus W_k \in \beta W_{k+1} \). Pick
\[ u \in (W_{k+1} \setminus W_k) \cap B \cap \bigcap_{t=1}^{k} g_t^{-1}[A] \]
and let \( w \) be the variable word over \( W_k \) which results from replacing each occurrence of \( k+1 \) by \( v \). Then \( w(k+1) = u \) and for each \( t \in \{1, 2, \ldots, k\} \) we have that \( g_t(w) = w(t) \). Consequently \( (w(1), w(2), \ldots, w(k+1)) \in I \cap (A^k \times B) \) as required. \( \square \)

The proof of the following lemma uses an old idea of H. Furstenberg and Y. Katznelson in [3]. There is some redundancy in this lemma. For example, (5) and (13) both tell us that \( p \cdot q_{k+1} = p \).
2.6. Lemma. Let $s$ be any minimal idempotent in $\beta W_k$. There exist minimal idempotents $p, q_{k+1}, r_{k+1} \in \beta W_{k+1}$ and $q_1, q_2, \ldots, q_k, r_1, r_2, \ldots, r_k \in \beta W_k$ such that:

\begin{itemize}
  \item[(1)] $\vec{p} = (s, s, \ldots, s, p) \in \mathbf{T}$;
  \item[(2)] $\vec{q} = (q_1, q_2, \ldots, q_{k+1}) \in \mathbf{J}$;
  \item[(3)] $\vec{r} = (r_1, r_2, \ldots, r_{k+1}) \in \mathbf{H}$;
  \item[(4)] $p = p \cdot s = s \cdot p$;
  \item[(5)] $\vec{p} \cdot \vec{q} = \vec{p}$ and $\vec{q} \cdot \vec{p} = \vec{q}$;
  \item[(6)] $\vec{p} \cdot \vec{r} = \vec{r}$ and $\vec{r} \cdot \vec{p} = \vec{p}$;
  \item[(7)] for $i \in \{1, 2, \ldots, k+1\}$ and $j \in \{1, 2, \ldots, k\}$, $q_i \cdot q_j = q_j$;
  \item[(8)] for $i \in \{1, 2, \ldots, k+1\}$ and $j \in \{1, 2, \ldots, k\}$, $r_j \cdot r_i = r_i$;
  \item[(9)] $s \cdot q_{k+1} = p$;
  \item[(10)] $r_{k+1} \cdot s = p$;
  \item[(11)] $q_{k+1} \cdot s = q_{k+1}$;
  \item[(12)] $s \cdot r_{k+1} = r_{k+1}$;
  \item[(13)] for $j \in \{1, 2, \ldots, k+1\}$, $p \cdot q_j = p$;
  \item[(14)] for $j \in \{1, 2, \ldots, k+1\}$, $r_j \cdot p = p$;
  \item[(15)] for $j \in \{1, 2, \ldots, k\}$, $q_j \cdot q_{k+1} = q_j \cdot p$ and
  \item[(16)] for $j \in \{1, 2, \ldots, k\}$, $r_{k+1} \cdot r_j = p \cdot r_j$.
\end{itemize}

Proof. Let $\vec{s} = (s, s, \ldots, s)$. Then $\vec{s} \in \mathbf{E}$. By [8] Corollary 2.6 and Theorem 2.7, every left ideal of $\mathbf{E}$ contains a minimal left ideal and every right ideal of $\mathbf{E}$ contains a minimal right ideal. Pick a minimal left ideal $L$ of $\mathbf{E}$ with $L \subseteq \mathbf{H}$ and a minimal right ideal $R$ of $\mathbf{E}$ with $R \subseteq \mathbf{J}$. Pick by Lemma 2.1 a minimal idempotent $\vec{p} \leq \vec{s}$ in $\mathbf{E}$ with $\vec{p} \in L \mathbf{E} \cap \mathbf{R} R$.

Now $\vec{p} = (p_1, p_2, \ldots, p_{k+1})$. Since $\vec{p} \leq \vec{s}$, we have that for each $t \in \{1, 2, \ldots, k+1\}$, $p_t \leq s$. Since $s$ is minimal in $\beta W_k$, we have in particular that $p_t = s$ for $t \in \{1, 2, \ldots, k\}$. Let $p = p_{k+1}$. Then we have $\vec{p} = (s, s, \ldots, s, p)$. Since $\vec{p} \in K(\mathbf{E})$ and, by Lemma 2.3, $K(\mathbf{E}) = \mathbf{E} \cap (\times_{j=1}^{k+1} K(\beta W_k) \times K(\beta W_{k+1}))$, we have that $p$ is minimal in $\beta W_{k+1}$. Since $p \leq s$, we have that (4) holds. Since $\vec{p}$ is minimal in $\mathbf{E}$ and $\mathbf{T}$ is an ideal of $\mathbf{E}$, we have that $\vec{p} \in \mathbf{T}$. That is, (1) holds.

Since $\vec{p} \in L \mathbf{E} \cap \mathbf{R} R$, pick $\vec{m} \in L$ and $\vec{n} \in R$ such that $\vec{p} = \vec{m} \vec{s} = \vec{s} \vec{n}$. Then $p = m_{k+1} s = s n_{k+1}$. And for $j \in \{1, 2, \ldots, k\}$, $s = m_j s = s n_j$. Also $m_{k+1} p = m_{k+1} s p = p p = p$ and $p n_{k+1} = p$. For $j \in \{1, 2, \ldots, k\}$, we have that $m_j p = m_j s p = s p = p$ and $p n_j = p$. Let $\vec{q} = \vec{n} \vec{p}$ and let $\vec{r} = \vec{p} \vec{m}$. Now $q_{k+1} q_{k+1} n_{k+1} p m_{k+1} = n_{k+1} p m_{k+1} = n_{k+1} p = q_{k+1}$. The fact that $q_j$ is an idempotent for each $j \in \{1, 2, \ldots, k\}$ follows from (7) which we shall verify below. Likewise, for each $j \in \{1, 2, \ldots, k+1\}$, $r_{j} r_{j} = r_{j}$.

Since $\vec{n} \in R$ and $\vec{m} \in L$ we have $\vec{q} \in R \subseteq \mathbf{J}$ and $\vec{r} \in R \subseteq \mathbf{H}$. Thus (2) and (3) hold. Also $\vec{q} \in R \subseteq K(\mathbf{E}) \subseteq \times_{j=1}^{k+1} K(\beta W_k) \times K(\beta W_{k+1})$, so we have that $q_{k+1}$ is minimal in $\beta W_{k+1}$ and $q_j$ is minimal in $\beta W_k$ for each $j \in \{1, 2, \ldots, k\}$. Similarly, $r_{k+1}$ is minimal in $\beta W_{k+1}$ and $r_j$ is minimal in $\beta W_k$ for each $j \in \{1, 2, \ldots, k\}$.

We proceed to verify the odd numbered statements from (5) through (15), the corresponding even numbered statements being analogously.

Since $\vec{p} = \vec{s} \vec{n}$ and $p_{k+1} = p$, we have that $\vec{p} = \vec{p} \vec{n}$. Therefore we have $\vec{p} \vec{q} = \vec{p} \vec{n} \vec{p} = \vec{p} \vec{n} \vec{p} = \vec{p} \vec{n} \vec{p} = \vec{q}$. Thus (5) holds.

To verify statement (7), first let $i, j \in \{1, 2, \ldots, k\}$. Then $q_i q_j = n_i s q_j = n_i s = q_i$. Now let $j \in \{1, 2, \ldots, k\}$. Then $q_{k+1} q_j = n_{k+1} m_j s = n_{k+1} p s = n_{k+1} p = q_{k+1}$.
Since \( sq_{k+1} = sn_{k+1}p = pp = p \) we have that (9) holds. For statement (11), we have \( q_{k+1} = n_{k+1}p = n_{k+1}p = q_{k+1} \). To verify (13), let \( j \in \{1, 2, \ldots, k\} \). Then \( pq_j = p_{j} = ps = p \). From (5) we know that \( p_{q_{k+1}} = p \). To verify (15), let \( j \in \{1, 2, \ldots, k\} \). Then \( q_{j}q_{k+1} = n_{j}n_{k+1}p = n_{j}pp = n_{j}p = n_{j}sp = q_{j}p \).

In Theorem 2.9 we shall be choosing a sequence of variable words \( \langle w_n \rangle_{n=1}^\infty \) such that whenever \( n \equiv 1 (\mod 3) \), \( w_n \) is a right variable word and whenever \( n \equiv 0 (\mod 3) \), \( w_n \) is a left variable word. We shall expect certain products of these words to lie in specified cells of finite partitions of \( \mathcal{W}_k \) and of \( \mathcal{W}_{k+1}\setminus \mathcal{W}_k \). We clearly cannot have the first word of such a product be a left variable word nor can we have the last word be a right variable word, since one may divide \( \mathcal{W}_k \) according to the first or last letter. Nor can we allow a right variable word to be followed by a left variable word, since that allows manipulation of adjacent occurrences of letters. (See [7] for a fuller discussion of these points.)

**2.7. Definition.** A set \( F \in \mathcal{P}_f(\mathbb{N}) \) is allowable if and only if max \( F \equiv 1 (\mod 3) \), \( \min F \neq 0 (\mod 3) \), and for all \( i < j \in F \), if \( i \equiv 1 (\mod 3) \) and \( j \equiv 0 (\mod 3) \), then there exists \( h \in F \) such that \( i < h < j \) and \( h \equiv 2 (\mod 3) \).

Notice that if in the following lemma \( F \) is allowable, then conclusion (1) or (2) applies.

**2.8. Lemma.** Let \( F \in \mathcal{P}_f(\mathbb{N}) \) and assume that max \( F \equiv 1 (\mod 3) \) and for all \( i < j \in F \), if \( i \equiv 1 (\mod 3) \) and \( j \equiv 0 (\mod 3) \), then there exists \( h \in F \) such that \( i < h < j \) and \( h \equiv 2 (\mod 3) \). Let \( s, p, \tilde{p} = (p_1, p_2, \ldots, p_{k+1}) \), \( q = (q_1, q_2, \ldots, q_{k+1}) \), and \( r = (r_1, r_2, \ldots, r_{k+1}) \) be as in Lemma 2.6 (so that \( p_{k+1} = p \) and \( p_j = s \) for \( j \in \{1, 2, \ldots, k\} \)). For \( f : F \to \{1, 2, \ldots, k + 1 \} \) and \( n \in F \), define

\[
\phi(f, n) = \begin{cases} 
q_{f(n)} & \text{if } n \equiv 0 (\mod 3), \\
r_{f(n)} & \text{if } n \equiv 1 (\mod 3), \\
p_{f(n)} & \text{if } n \equiv 2 (\mod 3).
\end{cases}
\]

(1) If \( \min F \equiv 0 (\mod 3) \) and \( k + 1 \notin \text{range}(f) \), then \( \prod_{n \in F} \phi(f, n) = s \).

(2) If \( \min F \equiv 0 (\mod 3) \) and \( k + 1 \in \text{range}(f) \), then \( \prod_{n \in F} \phi(f, n) = p \).

(3) If \( \min F \equiv 0 (\mod 3) \) and \( k + 1 \notin \text{range}(f) \), then \( \prod_{n \in F} \phi(f, n) = \{q_1, q_2, \ldots, q_k\} \).

(4) If \( \min F \equiv 0 (\mod 3) \) and \( k + 1 \in \text{range}(f) \), then \( \prod_{n \in F} \phi(f, n) \in \{q_{k+1}\} \cup \{q_1p, q_2p, \ldots, q_kp\} \).

**Proof.** We proceed by induction on \( |F| \). If \( F = \{m\} \), then \( m \equiv 0 (\mod 3) \) or \( m \equiv 2 (\mod 3) \). If \( m \equiv 0 (\mod 3) \), then \( \phi(f, m) = q_{f(m)} \) and if \( m \equiv 2 (\mod 3) \), then \( \phi(f, m) = p_{f(m)} \).

Now assume that \( |F| > 1 \), let \( m = \min F \), and let \( G = F \setminus \{m\} \). Let \( l = \min G \). Then the value of \( \prod_{n \in G} \phi(f, n) \) is determined by \( f_G \) and the congruence class of \( l \) using the induction hypothesis. The conclusions then follow from Lemma 2.6 and the fact that one cannot have both \( m \equiv 1 (\mod 3) \) and \( l \equiv 0 (\mod 3) \).
\[ \mathcal{W}_k. \] Indeed, let \( R = (k + 1)\mathcal{W}_{k+1} \). Then \( R \) is a right ideal of \( \mathcal{W}_{k+1} \) and so by [8, Corollary 4.18] \( R \) is a right ideal of \( \beta \mathcal{W}_{k+1} \), so that \( R \) is central. Given the sequence \( \langle w_n \rangle_{n=1}^{\infty} \) of variable words over \( \mathcal{W}_k \) as chosen in Theorem 2.9 one cannot have \( w_1(1)w_2(k+1) \in R \).

2.9. Theorem. Let \( A \) be a central subset of \( \mathcal{W}_k \) and let \( \mathcal{F} \) be a finite partition of \( \mathcal{W}_{k+1} \setminus \mathcal{W}_k \). Then there exist \( B \in \mathcal{F} \) such that \( B \) is central in \( \mathcal{W}_{k+1} \) and a sequence \( \langle w_n \rangle_{n=1}^{\infty} \) of variable words over \( \mathcal{W}_k \) such that:

(1) for each \( n \in \mathbb{N} \), if \( n \equiv 1 \) (mod 3), then \( w_n \) is a right variable word;

(2) for each \( n \in \mathbb{N} \), if \( n \equiv 0 \) (mod 3), then \( w_n \) is a left variable word; and

(3) for every allowable \( F \in \mathcal{P}_f(\mathbb{N}) \) and every \( h : F \to \{1, 2, \ldots, k+1\} \), if \( k+1 \in h[F] \), then \( \prod_{i \in F} w_n(h(i)) \in B \), and if \( k+1 \notin h[F] \), then \( \prod_{i \in F} w_n(h(i)) \in A \).

Proof. Pick a minimal idempotent \( s \in \beta \mathcal{W}_k \) such that \( A \in p \). Pick \( \vec{p} = (p_1, p_2, \ldots, p_{k+1}) \), \( \vec{q} = (q_1, q_2, \ldots, q_{k+1}) \), and \( \vec{r} = (r_1, r_2, \ldots, r_{k+1}) \) as guaranteed by Lemma 2.8. Notice that, with the obvious extension of the definition when \( \vec{p} = (0, 0, \ldots, 0) \), \( \vec{q} = (0, 0, \ldots, 0) \), and \( \vec{r} = (0, 0, \ldots, 0) \), \( \vec{r} \in c_{\mathbb{Z}}(\prod_{j=1}^{k+1} C_j) \). Let \( L = \{e, p, s, q_1, q_2, \ldots, q_{k+1}, q_1p, q_2p, \ldots, q_{k+1}p\} \). For notational convenience, let \( p_0 = q_0 = r_0 = e \). Also, if \( w \) is a variable word, we let \( w(0) = e \). If \( n \in \mathbb{N} \), \( f : \{1, 2, \ldots, 3n\} \to \{0, 1, \ldots, k+1\} \), and \( i \in \{1, 2, \ldots, 3n\} \), define \( \phi(f, i) \) as in Lemma 2.8 (with the obvious extension of the definition when \( f(i) = 0 \)). Given \( \theta \in L \), we say that the pair \( (f, \theta) \) is admissible if and only if for every \( j \in \{1, 2, \ldots, 3n\} \), we have \( \prod_{i=1}^{3n} \phi(f(j), i) \theta \in L \). Notice that, if \( n > 1 \), \( (f, \theta) \) is admissible, and \( g \) is the restriction of \( f \) to \( \{1, 2, \ldots, 3n-3\} \), then \( (g, r_{f(3n-2)}p_{f(3n-1)}q_{f(3n)}\theta) \) is also admissible.

We construct the sequence \( \langle w_n \rangle_{n=1}^{\infty} \) inductively, three terms at a time. Let \( W_0 = \{e\} \) and for each \( i \in \mathbb{N} \), as soon as we have chosen \( w_i \), let \( W_i = \{\prod_{j=1}^{3n} w_j(f(j)) : f : \{1, 2, \ldots, i\} \to \{0, 1, \ldots, k+1\}\} \).

Let \( n \in \mathbb{N} \cup \{0\} \) and assume that we have chosen \( w_i \) for all \( i \in \mathbb{N} \) with \( i \leq 3n \) (if any) so that, if \( n \geq 1 \), \( \theta \in L \), \( f : \{1, 2, \ldots, 3n\} \to \{0, 1, \ldots, k+1\} \), and \( (f, \theta) \) is admissible, then

\[(*) \quad \left( \prod_{i=1}^{3n} w_i(f(i)) \right)^{-1} C \in \theta \iff C \in \left( \prod_{i=0}^{n-1} r_{f(3i+1)} p_{f(3i+2)} q_{f(3i+3)} \right) \theta.
\]

Now \( F \in \mathcal{F} \), so pick by Lemma 2.2 a right variable word \( w_{3n+1} \) over \( \mathcal{W}_k \) such that for all \( l \in L \), all \( u \in W_{3n} \), and all \( t \in \{1, 2, \ldots, k+1\} \),

(a) \( (uw_{3n+1}(t))^{-1} C \in l \iff u^{-1} C \in r_l \).

Since \( w_{3n+1}(0) = e = r_0 \) we have also that (a) holds for \( t = 0 \).

Since \( \vec{p} \in \vec{F} \), pick by Lemma 2.2 (and the observation above about the case \( t = 0 \)) a variable word \( w_{3n+2} \) over \( \mathcal{W}_k \) such that for all \( l \in L \), all \( u \in W_{3n+1} \), and all \( t \in \{0, 1, \ldots, k+1\} \),

(b) \( (uw_{3n+2}(t))^{-1} C \in l \iff u^{-1} C \in p_l \).
Since $q \in \mathcal{J}$, pick by Lemma 2.2 a left variable word $w_{3n+3}$ over $W_k$ such that for all $l \in L$, all $u \in W_{3n+2}$, and all $t \in \{0,1,\ldots,k+1\}$,

(c) \quad (uw_{3n+3}(t))^{-1}C \in l \Leftrightarrow u^{-1}C \in q_l.

Now let $\theta \in L$, let $f : \{1,2,\ldots,3n+3\} \to \{0,1,\ldots,k+1\}$, and assume that $(f,\theta)$ is admissible. We verify that (a) holds. Notice that, since $(f,\theta)$ is admissible, we have that

\[ \{q_{f(3n+3)}\theta,p_{f(3n+3)}q_{f(3n+3)}\theta,r_{f(3n+3)}p_{f(3n+3)}q_{f(3n+3)}\theta\} \subseteq L. \]

(In the following, if $n = 0$ we interpret $\prod_{i=1}^{3n} w_i(f(i))$ as $e$.)

\[
\left( \prod_{i=1}^{3n+3} w_i(f(i)) \right)^{-1} C \in \theta \\
\Leftrightarrow \left( \prod_{i=1}^{3n} w_i(f(i)) \right)^{-1} w_{3n+1}(f(3n+1))w_{3n+2}(f(3n+2)) C \in q_{f(3n+3)} \theta \\
\Leftrightarrow \left( \prod_{i=1}^{3n} w_i(f(i)) \right)^{-1} w_{3n+1}(f(3n+1)) C \in p_{f(3n+3)}q_{f(3n+3)} \theta \\
\Leftrightarrow \left( \prod_{i=1}^{3n} w_i(f(i)) \right)^{-1} C \in r_{f(3n+3)}p_{f(3n+3)}q_{f(3n+3)} \theta \\
\Leftrightarrow C \in \left( \prod_{i=0}^{n} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)} \right) \theta.
\]

Here the first three double implications hold by (c), (b), and (a) respectively. If $n = 0$, the last double implication is a tautology. Otherwise, it is a consequence of the induction hypothesis.

The construction being complete, we claim that the sequence $(w_n)_{n=1}^{\infty}$ is as required. The first two conclusions are immediate. So let $F \in \mathcal{P}(\mathbb{N})$ be allowable and let $h : F \to \{1,2,\ldots,k+1\}$. Choose the least $n \in \mathbb{N}$ such that $F \subseteq \{1,2,\ldots,3n\}$. Define $f : \{1,2,\ldots,3n\} \to \{0,1,\ldots,k+1\}$ by

\[ f(i) = \begin{cases} h(i) & \text{if } i \in F, \\ 0 & \text{if } i \in \{1,2,\ldots,3n\} \setminus F. \end{cases} \]

Then we have immediately that $\prod_{i \in F} w_i(h(i)) = \prod_{i=1}^{3n} w_i(f(i))$.

By Lemma 2.8 we have that the pair $(f,e)$ is admissible. Recall that we identify $e$ with the principal ultrafilter generated by $e$. Thus by (a), we have that

\[ \prod_{i=1}^{3n} w_i(f(i)) \in C \Leftrightarrow C \in \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}. \]

Again by Lemma 2.8 we have that

\[ \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)} = \begin{cases} s & \text{if } k+1 \notin \text{range}(f), \\ p & \text{if } k+1 \in \text{range}(f). \end{cases} \]

In either case, we have that $C \subseteq \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}$, and so $\prod_{i=1}^{3n} w_i(f(i)) \in C$. If $k+1 \notin \text{range}(h)$, then $\prod_{i=1}^{3n} w_i(f(i)) \in C \cap W_k = A$. If $k+1 \in \text{range}(h)$, then $\prod_{i=1}^{3n} w_i(f(i)) \in C \setminus W_k = B$. \hfill \Box
We conclude by showing that Carlson’s Main Lemma is a consequence of Theorem 2.9.

2.10. Corollary (Carlson). Let \( k \in \mathbb{N} \), let the set of variable words over \( W_k \) be partitioned into finitely many classes, and let \( \langle w_n \rangle_{n=1}^{\infty} \) be a sequence of variable words. Then there exists a variable reduction \( \langle t_n \rangle_{n=1}^{\infty} \) of \( \langle w_n \rangle_{n=1}^{\infty} \) such that all expressions of the form \( \prod_{i=1}^{n} t_i(f(i)) \), where \( n \in \mathbb{N} \), \( f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\} \cup \{v\} \), and \( v \in \text{range}(f) \), lie in the same cell of the partition.

Proof. Let \( W_{k,v} \) be the set of variable words over \( W_k \), let \( F \) be a finite partition of \( W_{k,v} \) and \( \langle w_n \rangle_{n=1}^{\infty} \) be a sequence of variable words. Define a function \( \varphi : W_{k+1} \setminus W_k \to W_{k,v} \) as follows. If \( u = a_1 a_2 \cdots a_t \), where each \( a_i \in \{1, 2, \ldots, k+1\} \), let \( b_i = a_i \) if \( a_i \neq k+1 \) and \( b_i = v \) if \( a_i = k+1 \); then let \( \varphi(u) = \prod_{i=1}^{t} w(b_i) \).

Let \( F = \{ \varphi^{-1}[H] : H \in F \} \). Then \( F \) is a finite partition of \( W_{k+1} \setminus W_k \). Let \( A \) be any central subset of \( W_k \) and pick \( B \in F \) and a sequence \( \langle w'_n \rangle_{n=1}^{\infty} \) of variable words as guaranteed by Theorem 2.9. Pick \( H \in F \) such that \( B = \varphi^{-1}[H] \).

For each \( n \in \mathbb{N} \), let \( t_n \) be the length of \( w'_{3n-1} \) and write \( w'_{3n-1} = a_{n,0} a_{n,2} \cdots a_{n,t_n} \) where each \( a_{n,i} \in \{1, 2, \ldots, k\} \cup \{v\} \). (For this corollary, we are avoiding both left and right variable words.) Let \( \alpha_0 = 0 \) and for \( n \in \mathbb{N} \), let \( \alpha_n = \sum_{m=1}^{n} t_m \). For each \( n \in \mathbb{N} \), let \( t_n = \prod_{m=1}^{t_n} w_{\alpha_{n-1}+m}(a_{n,m}) \). Then \( \langle t_n \rangle_{n=1}^{\infty} \) is a variable reduction of \( \langle w'_n \rangle_{n=1}^{\infty} \).

To see that \( \langle t_n \rangle_{n=1}^{\infty} \) is as required, let \( n \in \mathbb{N} \) and let \( f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\} \cup \{v\} \) with \( v \in \text{range}(f) \). Define \( g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k+1\} \) by \( g(i) = f(i) \) if \( f(i) \neq v \) and \( g(i) = k+1 \) if \( f(i) = v \). Then \( \prod_{i=1}^{n} w_{3i-1}(g(i)) \) is \( t_n \) and so \( \prod_{i=1}^{n} t_i(f(i)) = \varphi(\prod_{i=1}^{n} w'_{3i-1}(g(i))) \in H \). \( \square \)

References

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