

## ONE SIDED IDEALS AND CARLSON'S THEOREM

NEIL HINDMAN AND RANDALL MCCUTCHEON

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ABSTRACT. Using left ideals, right ideals, and the smallest two sided ideal in a compact right topological semigroup, we derive an extension of the Main Lemma to Carlson's Theorem. This extension involves an infinite sequence of variable words over a finite alphabet, some of which are required to have the variable as the first letter and others of which are required to have the variable as the last letter.

### 1. INTRODUCTION

In 1988 T. Carlson published a Ramsey Theoretic result [2, Theorem 2] which has as corollaries many earlier results in Ramsey Theory. (See [8, Section 18.4] for a relatively short presentation of Carlson's Theorem and some of its consequences.) Experience suggests that Carlson's "Main Lemma" [2, Lemma 5.9] implies those Ramsey Theoretic corollaries of his theorem in which a finite collection of finite objects is partitioned into finitely many classes. We shall state this Main Lemma after introducing some necessary terminology.

For  $k \in \mathbb{N}$ , the set of positive integers, let  $\mathcal{W}_k$  be the free semigroup with identity  $e$  on the alphabet  $\{1, 2, \dots, k\}$ . That is,  $\mathcal{W}_k$  consists of all "words with letters from  $\{1, 2, \dots, k\}$ " (i.e. functions whose domain is an initial segment of  $\mathbb{N}$  and whose range is contained in  $\{1, 2, \dots, k\}$ ) together with the empty word, with the operation of concatenation. A *variable word* over  $\mathcal{W}_k$  is a word on the alphabet  $\{1, 2, \dots, k\} \cup \{v\}$  in which  $v$  occurs, where  $v$  is a "variable" not in  $\{1, 2, \dots, k\}$ . Given a variable word  $w$  over  $\mathcal{W}_k$ , and  $t \in \{1, 2, \dots, k\}$ ,  $w(t)$  has its obvious meaning, namely the result of replacing all occurrences of  $v$  with  $t$ . (There is a potential conflict here with the formal viewpoint which takes  $w$  to be a function. If we have occasion to need the value of the function  $w$  at  $t$  we will denote it as  $w_t$ .)

**1.1. Definition.** Let  $k \in \mathbb{N}$  and let  $\langle w_n \rangle_{n=1}^\infty$  be a sequence of variable words over  $\mathcal{W}_k$ . The sequence  $\langle t_n \rangle_{n=1}^\infty$  is a *variable reduction* of  $\langle w_n \rangle_{n=1}^\infty$  if and only if there exist an increasing function  $g : \mathbb{N} \rightarrow \mathbb{N}$  and a function  $f : \mathbb{N} \rightarrow \{1, 2, \dots, k\} \cup \{v\}$  such that:

$$(1) \quad g(1) = 1,$$

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- (2) for each  $n \in \mathbb{N}$ ,  $v \in f[\{g(n), g(n) + 1, \dots, g(n+1) - 1\}]$ , and  
 (3) for each  $n \in \mathbb{N}$ ,  $t_n = \prod_{i=g(n)}^{g(n+1)-1} w_i(f(i))$ , where the product is in increasing order of indices.

**1.2. Theorem** (Carlson). *Let  $k \in \mathbb{N}$ , let the set of variable words over  $\mathcal{W}_k$  be partitioned into finitely many classes, and let  $\langle w_n \rangle_{n=1}^\infty$  be a sequence of variable words. Then there exists a variable reduction  $\langle t_n \rangle_{n=1}^\infty$  of  $\langle w_n \rangle_{n=1}^\infty$  such that all expressions of the form  $\prod_{i=1}^n t_i(f(i))$ , where  $n \in \mathbb{N}$ ,  $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, k\} \cup \{v\}$ , and  $v \in \text{range}(f)$ , lie in the same cell of the partition.*

*Proof.* [2, Lemma 5.9]. □

Somewhat earlier, T. Carlson and S. Simpson had established a similar result which partitioned  $\mathcal{W}_k$  rather than the variable words over  $\mathcal{W}_k$  and required that most of the variable words used must have  $v$  as the leftmost letter. Such words are *left variable* words. Similarly, a *right variable* word must have the variable  $v$  as its rightmost letter.

**1.3. Theorem** (Carlson-Simpson). *Let  $k \in \mathbb{N}$  and let  $\mathcal{W}_k$  be partitioned into finitely many classes. Then there exists a sequence  $\langle w_n \rangle_{n=1}^\infty$  of variable words over  $\mathcal{W}_k$  such that for every  $n > 1$ ,  $w_n$  is a left variable word, and all expressions of the form  $\prod_{i=1}^n w_i(f(i))$ , where  $n \in \mathbb{N}$  and  $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, k\}$ , lie in the same cell of the partition.*

*Proof.* [3, Theorem 6.3]. □

In [1], V. Bergelson, A. Blass, and the first author of the current paper established a generalization of Theorem 1.2 by utilizing the algebraic structure of the Stone-Ćech compactification of a discrete semigroup. They could not, however, extend by these methods Theorem 1.3. The reasons involve the algebraic constructs used in the proof, which we pause now to introduce.

Given a discrete semigroup  $(S, \cdot)$ , we take the points of  $\beta S$  to be the ultrafilters on  $S$ , the principal ultrafilters being identified with the points of  $S$ . Given a set  $A \subseteq S$ ,  $\bar{A} = \{p \in \beta S : A \in p\}$ . The set  $\{\bar{A} : A \subseteq S\}$  is a basis for the open sets (as well as a basis for the closed sets) of  $\beta S$ .

There is a natural extension of the operation  $\cdot$  of  $S$  to  $\beta S$  making  $\beta S$  a compact right topological semigroup with  $S$  contained in its topological center. This says that for each  $p \in \beta S$  the function  $\rho_p : \beta S \rightarrow \beta S$  is continuous and for each  $x \in S$ , the function  $\lambda_x : \beta S \rightarrow \beta S$  is continuous, where  $\rho_p(q) = q \cdot p$  and  $\lambda_x(q) = x \cdot q$ . The operation is characterized by the fact that for any  $p$  and  $q$  in  $\beta S$  and any  $A \subseteq S$ ,  $A \in p \cdot q$  if and only if  $\{x \in S : x^{-1}A \in q\} \in p$ , where  $x^{-1}A = \{y \in S : x \cdot y \in A\}$ . See [8] for an elementary introduction to the semigroup  $\beta S$ .

Any compact Hausdorff right topological semigroup  $(T, \cdot)$  has a smallest two sided ideal  $K(T)$  which is the union of all of the minimal left ideals of  $T$ , each of which is closed, and is also the union of all of the minimal right ideals of  $T$  [8, Theorem 2.8]. Given any minimal left ideal  $L$  and any minimal right ideal  $R$ ,  $L \cap R$  is a group (and in particular has an idempotent) [8, Theorem 2.7]. There is a partial ordering of the idempotents of  $T$  determined by  $p \leq q$  if and only if  $p = p \cdot q = q \cdot p$ . An idempotent  $p$  is minimal with respect to this order if and only if  $p \in K(T)$  [8, Theorem 1.59]. Such an idempotent is called simply “minimal”.

Members of minimal idempotents in  $\beta S$  are the *central* subsets of  $S$ . Central sets were introduced by H. Furstenberg in [4] and defined in terms of notions of

topological dynamics. These sets enjoy very strong combinatorial properties. (See [4, Proposition 8.21] or [8, Chapter 14].) See [8, Theorem 19.27] for a proof of the equivalence of the definition of “central” in terms of  $\beta S$  with the original dynamical definition.

The basic algebraic fact used in [1] (as well as several other papers) is that any two sided ideal in a compact right topological semigroup  $T$  contains  $K(T)$ . While variable words over  $\mathcal{W}_k$  yield a two sided ideal in an appropriately chosen compact right topological semigroup, left variable words and right variable words do not.

Left variable words do, however, correspond naturally to a right ideal (and right variable words correspond to a left ideal). In [7] we were able to use these natural left and right ideals to obtain a generalization of Theorem 1.3 involving both left and right variable words. In the current paper we use similar left and right ideals to obtain a generalization of Theorem 1.2 which involves both left and right variable words.

2. EXTENDING CARLSON'S MAIN LEMMA

We shall have throughout this section a fixed  $k \in \mathbb{N}$ . We begin with the following simple lemma from semigroup theory.

**2.1. Lemma.** *Let  $S$  be a semigroup which has a minimal left ideal which contains an idempotent. Let  $L$  be a left ideal of  $S$ , let  $R$  be a right ideal of  $S$ , and let  $e$  be any idempotent in  $S$ . There is a minimal idempotent  $m \in Le \cap eR$ . Necessarily  $m \leq e$ .*

*Proof.* Notice that if  $m \in Le$ , then  $me = m$  and if  $m \in eR$ , then  $em = m$ . Thus any idempotent  $m \in Le \cap eR$  satisfies  $m \leq e$ .

By [8, Lemma 1.57]  $S$  has a minimal right ideal which contains an idempotent. Thus, by [8, Corollary 1.47] we may presume that  $L$  is a minimal left ideal and  $R$  is a minimal right ideal. By [8, Theorem 1.46]  $Le$  is a minimal left ideal and  $eR$  is a minimal right ideal and so, by [8, Theorem 1.61],  $Le \cap eR$  is a group. Let  $m$  be the identity of  $Le \cap eR$ . □

Given a set  $X$ , we denote the set of finite nonempty subsets of  $X$  by  $\mathcal{P}_f(X)$ .

**2.2. Lemma.** *Let  $S$  be a discrete semigroup, let  $A \subseteq \times_{j=1}^{k+1} S$ , and let  $Z = \times_{j=1}^{k+1} \beta S$ . Assume that  $(y_1, y_2, \dots, y_{k+1}) \in cl_Z(A)$ ,  $C \subseteq S$ ,  $W \in \mathcal{P}_f(S)$ , and  $L \in \mathcal{P}_f(\beta S)$ . Then there exists  $(a_1, a_2, \dots, a_{k+1}) \in A$  such that, for all  $l \in L$ , all  $u \in W$ , and all  $i \in \{1, 2, \dots, k + 1\}$ ,*

$$(ua_i)^{-1}C \in l \Leftrightarrow u^{-1}C \in y_{i,l}.$$

*Proof.* For  $l \in L$ ,  $u \in W$ , and  $i \in \{1, 2, \dots, k + 1\}$ , let

$$C_{i,u,l} = \begin{cases} C & \text{if } C \in uy_{i,l}, \\ S \setminus C & \text{if } C \notin uy_{i,l}. \end{cases}$$

Then  $\overline{C_{i,u,l}}$  is a neighborhood of  $uy_{i,l} = \lambda_u(\rho_l(y_i))$ , so pick a member  $U_{i,u,l}$  of  $y_i$  such that  $\lambda_u[\rho_l[\overline{U_{i,u,l}}]] \subseteq \overline{C_{i,u,l}}$ .

For  $i \in \{1, 2, \dots, k + 1\}$ , let  $N_i = \bigcap_{l \in L} \bigcap_{u \in W} U_{i,u,l}$ . Pick  $(a_1, a_2, \dots, a_{k+1}) \in A \cap \times_{i=1}^{k+1} N_i$ . □

- 2.3. Definition.** (a)  $Y = (\prod_{j=1}^k \beta\mathcal{W}_k) \times \beta\mathcal{W}_{k+1}$ .  
 (b)  $Z = \prod_{j=1}^{k+1} \beta\mathcal{W}_{k+1}$ .  
 (c)  $I = \{(w(1), w(2), \dots, w(k+1)) : w \text{ is a variable word over } \mathcal{W}_k\}$ .  
 (d)  $J = \{(w(1), w(2), \dots, w(k+1)) : w \text{ is a left variable word over } \mathcal{W}_k\}$ .  
 (e)  $H = \{(w(1), w(2), \dots, w(k+1)) : w \text{ is a right variable word over } \mathcal{W}_k\}$ .  
 (f)  $E = I \cup \{(w, w, \dots, w) : w \in \mathcal{W}_k\}$ .

Notice that each of  $I, J, H,$  and  $E$  are contained in  $(\prod_{j=1}^k \mathcal{W}_k) \times \mathcal{W}_{k+1}$ , and consequently their closures in  $Y$  and in  $Z$  are identical.

**2.4. Lemma.**  $\overline{E}$  is a subsemigroup of  $Y, \overline{I}$  is an ideal of  $\overline{E}, \overline{J}$  is a right ideal of  $\overline{E},$  and  $\overline{H}$  is a left ideal of  $\overline{E}$ .

*Proof.* [8, Theorems 2.22 and 4.17]. □

The proof of the following lemma uses an idea from [1]. This lemma is needed so that in Theorem 2.9 the cell of the partition  $\mathcal{F}$  can be guaranteed to be central in  $\mathcal{W}_{k+1}$ . This fact is significant because, as we have noted, central sets are guaranteed to contain substantial combinatorial structures.

**2.5. Lemma.**  $\overline{E} \cap K(Y) \neq \emptyset$  and so

$$K(\overline{E}) = \overline{E} \cap \left( \left( \prod_{j=1}^k K(\beta\mathcal{W}_k) \right) \times K(\beta\mathcal{W}_{k+1}) \right) = \overline{E} \cap K(Y).$$

*Proof.* Let  $s$  be a minimal idempotent in  $\beta\mathcal{W}_k$  and pick by [8, Theorem 1.60 and Corollary 2.6] an idempotent  $p$  in  $\mathcal{W}_{k+1}$  such that  $p \leq s$ . Let  $\vec{p} = (s, s, \dots, s, p)$ .

We claim that it suffices to show that  $\vec{p} \in \overline{I}$ . Indeed, assume that we have done so. We have that  $\vec{p} \in (\prod_{j=1}^k K(\beta\mathcal{W}_k)) \times K(\beta\mathcal{W}_{k+1})$  and by [8, Theorem 2.23]  $K(Y) = (\prod_{j=1}^k K(\beta\mathcal{W}_k)) \times K(\beta\mathcal{W}_{k+1})$ . Thus we have that  $\vec{p} \in \overline{I} \cap K(Y) \subseteq \overline{E} \cap K(Y)$ . It then follows from [8, Theorem 1.65] that  $K(\overline{E}) = \overline{E} \cap K(Y)$ .

To see that  $\vec{p} \in \overline{I}$ , let  $A \in s$  and  $B \in p$  be given. We need to show that  $(\overline{A}^k \times \overline{B}) \cap I \neq \emptyset$ . For each  $t \in \{1, 2, \dots, k\}$  define  $g_t : \mathcal{W}_{k+1} \rightarrow \mathcal{W}_k$  by letting  $g_t(w)$  be the word obtained by replacing each occurrence of  $k+1$  by an occurrence of  $t$  and let  $\tilde{g}_t : \beta\mathcal{W}_{k+1} \rightarrow \beta\mathcal{W}_k$  be its continuous extension. Let  $t \in \{1, 2, \dots, k\}$  be given. By [8, Corollary 4.22],  $\tilde{g}_t$  is a homomorphism and so  $\tilde{g}_t(p) \leq \tilde{g}_t(s)$ . Since  $g_t$  is the identity on  $\mathcal{W}_k$ , we have that  $\tilde{g}_t(s) = s$ , and thus  $\tilde{g}_t(p) \leq s$ . Since  $s$  is minimal, we therefore have that  $\tilde{g}_t(p) = s$ .

Since  $\overline{A}$  is a neighborhood of  $\tilde{g}_t(p)$  for each  $t \in \{1, 2, \dots, k\}$ , we have that  $\bigcap_{t=1}^k g_t^{-1}[A] \in p$ . Also, since  $\beta\mathcal{W}_{k+1} \setminus \beta\mathcal{W}_k$  is an ideal of  $\beta\mathcal{W}_{k+1}$  and  $p$  is minimal in  $\beta\mathcal{W}_{k+1}$ , we have that  $\mathcal{W}_{k+1} \setminus \mathcal{W}_k \in p$ . Pick

$$u \in (\mathcal{W}_{k+1} \setminus \mathcal{W}_k) \cap B \cap \bigcap_{t=1}^k g_t^{-1}[A]$$

and let  $w$  be the variable word over  $\mathcal{W}_k$  which results from replacing each occurrence of  $k+1$  by  $v$ . Then  $w(k+1) = u$  and for each  $t \in \{1, 2, \dots, k\}$  we have that  $g_t(u) = w(t)$ . Consequently  $(w(1), w(2), \dots, w(k+1)) \in I \cap (\overline{A}^k \times B)$  as required. □

The proof of the following lemma uses an old idea of H. Furstenberg and Y. Katznelson in [5]. There is some redundancy in this lemma. For example, (5) and (13) both tell us that  $p \cdot q_{k+1} = p$ .

**2.6. Lemma.** *Let  $s$  be any minimal idempotent in  $\beta\mathcal{W}_k$ . There exist minimal idempotents  $p, q_{k+1}, r_{k+1} \in \beta\mathcal{W}_{k+1}$  and  $q_1, q_2, \dots, q_k, r_1, r_2, \dots, r_k \in \beta\mathcal{W}_k$  such that:*

- (1)  $\bar{p} = (s, s, \dots, s, p) \in \bar{I}$ ;
- (2)  $\bar{q} = (q_1, q_2, \dots, q_{k+1}) \in \bar{J}$ ;
- (3)  $\bar{r} = (r_1, r_2, \dots, r_{k+1}) \in \bar{H}$ ;
- (4)  $p = p \cdot s = s \cdot p$ ;
- (5)  $\bar{p} \cdot \bar{q} = \bar{p}$  and  $\bar{q} \cdot \bar{p} = \bar{q}$ ;
- (6)  $\bar{p} \cdot \bar{r} = \bar{r}$  and  $\bar{r} \cdot \bar{p} = \bar{p}$ ;
- (7) for  $i \in \{1, 2, \dots, k+1\}$  and  $j \in \{1, 2, \dots, k\}$ ,  $q_i \cdot q_j = q_i$ ;
- (8) for  $i \in \{1, 2, \dots, k+1\}$  and  $j \in \{1, 2, \dots, k\}$ ,  $r_j \cdot r_i = r_i$ ;
- (9)  $s \cdot q_{k+1} = p$ ;
- (10)  $r_{k+1} \cdot s = p$ ;
- (11)  $q_{k+1} \cdot s = q_{k+1}$ ;
- (12)  $s \cdot r_{k+1} = r_{k+1}$ ;
- (13) for  $j \in \{1, 2, \dots, k+1\}$ ,  $p \cdot q_j = p$ ;
- (14) for  $j \in \{1, 2, \dots, k+1\}$ ,  $r_j \cdot p = p$ ;
- (15) for  $j \in \{1, 2, \dots, k\}$ ,  $q_j \cdot q_{k+1} = q_j \cdot p$ ; and
- (16) for  $j \in \{1, 2, \dots, k\}$ ,  $r_{k+1} \cdot r_j = p \cdot r_j$ .

*Proof.* Let  $\bar{s} = (s, s, \dots, s)$ . Then  $\bar{s} \in \bar{E}$ . By [8, Corollary 2.6 and Theorem 2.7], every left ideal of  $\bar{E}$  contains a minimal left ideal and every right ideal of  $\bar{E}$  contains a minimal right ideal. Pick a minimal left ideal  $L$  of  $\bar{E}$  with  $L \subseteq \bar{H}$  and a minimal right ideal  $R$  of  $\bar{E}$  with  $R \subseteq \bar{J}$ . Pick by Lemma 2.1 a minimal idempotent  $\bar{p} \leq \bar{s}$  in  $\bar{E}$  with  $\bar{p} \in L\bar{s} \cap \bar{s}R$ .

Now  $\bar{p} = (p_1, p_2, \dots, p_{k+1})$ . Since  $\bar{p} \leq \bar{s}$ , we have that for each  $t \in \{1, 2, \dots, k+1\}$ ,  $p_t \leq s$ . Since  $s$  is minimal in  $\beta\mathcal{W}_k$ , we have in particular that  $p_t = s$  for  $t \in \{1, 2, \dots, k\}$ . Let  $p = p_{k+1}$ . Then we have  $\bar{p} = (s, s, \dots, s, p)$ . Since  $\bar{p} \in K(\bar{E})$  and, by Lemma 2.5  $K(\bar{E}) = \bar{E} \cap (\times_{j=1}^k K(\beta\mathcal{W}_k) \times K(\beta\mathcal{W}_{k+1}))$ , we have that  $p$  is minimal in  $\beta\mathcal{W}_{k+1}$ . Since  $p \leq s$ , we have that (4) holds. Since  $\bar{p}$  is minimal in  $\bar{E}$  and  $\bar{I}$  is an ideal of  $\bar{E}$ , we have that  $\bar{p} \in \bar{I}$ . That is, (1) holds.

Since  $\bar{p} \in L\bar{s} \cap \bar{s}R$ , pick  $\bar{m} \in L$  and  $\bar{n} \in R$  such that  $\bar{p} = \bar{m}\bar{s} = \bar{s}\bar{n}$ . Then  $p = m_{k+1}s = sn_{k+1}$  and for  $j \in \{1, 2, \dots, k\}$ ,  $s = m_j s = sn_j$ . Also  $m_{k+1}p = m_{k+1}sp = pp = p$  and  $pn_{k+1} = p$ . For  $j \in \{1, 2, \dots, k\}$ , we have that  $m_j p = m_j sp = sp = p$  and  $pn_j = p$ . Let  $\bar{q} = \bar{n}\bar{p}$  and let  $\bar{r} = \bar{p}\bar{m}$ . Now  $q_{k+1}q_{k+1} = n_{k+1}pn_{k+1}p = n_{k+1}pp = n_{k+1}p = q_{k+1}$ . The fact that  $q_j$  is an idempotent for each  $j \in \{1, 2, \dots, k\}$  follows from (7) which we shall verify below. Likewise, for each  $j \in \{1, 2, \dots, k+1\}$ ,  $r_j r_j = r_j$ .

Since  $\bar{n} \in R$  and  $\bar{m} \in L$  we have  $\bar{q} \in R \subseteq \bar{J}$  and  $\bar{r} \in R \subseteq \bar{H}$ . Thus (2) and (3) hold. Also  $\bar{q} \in R \subseteq K(\bar{E}) \subseteq \times_{j=1}^k K(\beta\mathcal{W}_k) \times K(\beta\mathcal{W}_{k+1})$ , so we have that  $q_{k+1}$  is minimal in  $\beta\mathcal{W}_{k+1}$  and  $q_j$  is minimal in  $\beta\mathcal{W}_k$  for each  $j \in \{1, 2, \dots, k\}$ . Similarly,  $r_{k+1}$  is minimal in  $\beta\mathcal{W}_{k+1}$  and  $r_j$  is minimal in  $\beta\mathcal{W}_k$  for each  $j \in \{1, 2, \dots, k\}$ .

We proceed to verify the odd numbered statements from (5) through (15), the corresponding even numbered statements being analogous.

Since  $\bar{p} = \bar{s}\bar{n}$  and  $pn_{k+1} = p$ , we have that  $\bar{p} = \bar{p}\bar{n}$ . Therefore we have  $\bar{p}\bar{q} = \bar{p}\bar{n}\bar{p} = \bar{p}\bar{p} = \bar{p}$  and  $\bar{q}\bar{p} = \bar{n}\bar{p}\bar{p} = \bar{n}\bar{p} = \bar{q}$ . Thus (5) holds.

To verify statement (7), first let  $i, j \in \{1, 2, \dots, k\}$ . Then  $q_i q_j = n_i s q_j = n_i s = q_i$ . Now let  $j \in \{1, 2, \dots, k\}$ . Then  $q_{k+1} q_j = n_{k+1} p n_j s = n_{k+1} p s = n_{k+1} p = q_{k+1}$ .

Since  $sq_{k+1} = sn_{k+1}p = pp = p$  we have that (9) holds. For statement (11), we have  $q_{k+1}s = n_{k+1}ps = n_{k+1}p = q_{k+1}$ . To verify (13), let  $j \in \{1, 2, \dots, k\}$ . Then  $pq_j = pn_js = ps = p$ . From (5) we know that  $pq_{k+1} = p$ . To verify (15), let  $j \in \{1, 2, \dots, k\}$ . Then  $q_jq_{k+1} = n_jsn_{k+1}p = n_jpp = n_jp = n_jsp = q_jp$ .  $\square$

In Theorem 2.9 we shall be choosing a sequence of variable words  $\langle w_n \rangle_{n=1}^\infty$  such that whenever  $n \equiv 1 \pmod{3}$ ,  $w_n$  is a right variable word and whenever  $n \equiv 0 \pmod{3}$ ,  $w_n$  is a left variable word. We shall expect certain products of these words to lie in specified cells of finite partitions of  $\mathcal{W}_k$  and of  $\mathcal{W}_{k+1} \setminus \mathcal{W}_k$ . We clearly cannot have the first word of such a product be a left variable word nor can we have the last word be a right variable word, since one may divide  $\mathcal{W}_k$  according to the first or last letter. Nor can we allow a right variable word to be followed by a left variable word, since that allows manipulation of adjacent occurrences of letters. (See [7] for a fuller discussion of these points.)

**2.7. Definition.** A set  $F \in \mathcal{P}_f(\mathbb{N})$  is *allowable* if and only if  $\max F \not\equiv 1 \pmod{3}$ ,  $\min F \not\equiv 0 \pmod{3}$ , and for all  $i < j$  in  $F$ , if  $i \equiv 1 \pmod{3}$  and  $j \equiv 0 \pmod{3}$ , then there exists  $h \in F$  such that  $i < h < j$  and  $h \equiv 2 \pmod{3}$ .

Notice that if in the following lemma  $F$  is allowable, then conclusion (1) or (2) applies.

**2.8. Lemma.** Let  $F \in \mathcal{P}_f(\mathbb{N})$  and assume that  $\max F \not\equiv 1 \pmod{3}$  and for all  $i < j$  in  $F$ , if  $i \equiv 1 \pmod{3}$  and  $j \equiv 0 \pmod{3}$ , then there exists  $h \in F$  such that  $i < h < j$  and  $h \equiv 2 \pmod{3}$ . Let  $s, p, \vec{p} = (p_1, p_2, \dots, p_{k+1}), \vec{q} = (q_1, q_2, \dots, q_{k+1})$ , and  $\vec{r} = (r_1, r_2, \dots, r_{k+1})$  be as in Lemma 2.6 (so that  $p_{k+1} = p$  and  $p_j = s$  for  $j \in \{1, 2, \dots, k\}$ ). For  $f : F \rightarrow \{1, 2, \dots, k + 1\}$  and  $n \in F$ , define

$$\phi(f, n) = \begin{cases} q_{f(n)} & \text{if } n \equiv 0 \pmod{3}, \\ r_{f(n)} & \text{if } n \equiv 1 \pmod{3}, \\ p_{f(n)} & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

- (1) If  $\min F \not\equiv 0 \pmod{3}$  and  $k + 1 \notin \text{range}(f)$ , then  $\prod_{n \in F} \phi(f, n) = s$ .
- (2) If  $\min F \not\equiv 0 \pmod{3}$  and  $k + 1 \in \text{range}(f)$ , then  $\prod_{n \in F} \phi(f, n) = p$ .
- (3) If  $\min F \equiv 0 \pmod{3}$  and  $k + 1 \notin \text{range}(f)$ , then  $\prod_{n \in F} \phi(f, n) \in \{q_1, q_2, \dots, q_k\}$ .
- (4) If  $\min F \equiv 0 \pmod{3}$  and  $k + 1 \in \text{range}(f)$ , then  $\prod_{n \in F} \phi(f, n) \in \{q_{k+1}\} \cup \{q_1p, q_2p, \dots, q_kp\}$ .

*Proof.* We proceed by induction on  $|F|$ . If  $F = \{m\}$ , then  $m \equiv 0 \pmod{3}$  or  $m \equiv 2 \pmod{3}$ . If  $m \equiv 0 \pmod{3}$ , then  $\phi(f, m)$  is  $q_{f(m)}$  and if  $m \equiv 2 \pmod{3}$ , then  $\phi(f, m)$  is  $p_{f(m)}$ .

Now assume that  $|F| > 1$ , let  $m = \min F$ , and let  $G = F \setminus \{m\}$ . Let  $l = \min G$ . Then the value of  $\prod_{n \in G} \phi(f, n)$  is determined by  $f|_G$  and the congruence class of  $l$  using the induction hypothesis. The conclusions then follow from Lemma 2.6 and the fact that one cannot have both  $m \equiv 1 \pmod{3}$  and  $l \equiv 0 \pmod{3}$ .  $\square$

Notice that in the following theorem, which is our main result, the fact that  $A$  can be any central subset of  $\mathcal{W}_k$  yields a stronger result than choosing some  $A$  out of a given finite partition of  $\mathcal{W}_k$ , because for any finite partition, some cell must be central. Notice also that one cannot reverse the roles of  $A$  and  $B$  by taking  $B$  to be an arbitrary central set in  $\mathcal{W}_{k+1}$  and picking  $A$  out of a finite partition of

$\mathcal{W}_k$ . Indeed, let  $R = (k + 1)\mathcal{W}_{k+1}$ . Then  $R$  is a right ideal of  $\mathcal{W}_{k+1}$  and so by [8, Corollary 4.18]  $\overline{R}$  is a right ideal of  $\beta\mathcal{W}_{k+1}$ , so that  $R$  is central. Given the sequence  $\langle w_n \rangle_{n=1}^\infty$  of variable words over  $\mathcal{W}_k$  as chosen in Theorem 2.9 one cannot have  $w_1(1)w_2(k + 1) \in R$ .

**2.9. Theorem.** *Let  $A$  be a central subset of  $\mathcal{W}_k$  and let  $\mathcal{F}$  be a finite partition of  $\mathcal{W}_{k+1} \setminus \mathcal{W}_k$ . Then there exist  $B \in \mathcal{F}$  such that  $B$  is central in  $\mathcal{W}_{k+1}$  and a sequence  $\langle w_n \rangle_{n=1}^\infty$  of variable words over  $\mathcal{W}_k$  such that:*

- (1) *for each  $n \in \mathbb{N}$ , if  $n \equiv 1 \pmod{3}$ , then  $w_n$  is a right variable word;*
- (2) *for each  $n \in \mathbb{N}$ , if  $n \equiv 0 \pmod{3}$ , then  $w_n$  is a left variable word; and*
- (3) *for every allowable  $F \in \mathcal{P}_f(\mathbb{N})$  and every  $h : F \rightarrow \{1, 2, \dots, k + 1\}$ , if  $k + 1 \in h[F]$ , then  $\prod_{n \in F} w_n(h(n)) \in B$ , and if  $k + 1 \notin h[F]$ , then  $\prod_{n \in F} w_n(h(n)) \in A$ .*

*Proof.* Pick a minimal idempotent  $s \in \beta\mathcal{W}_k$  such that  $A \in p$ . Pick  $\vec{p} = (p_1, p_2, \dots, p_{k+1})$ ,  $\vec{q} = (q_1, q_2, \dots, q_{k+1})$ , and  $\vec{r} = (r_1, r_2, \dots, r_{k+1})$  as guaranteed by Lemma 2.6. Notice that, with  $p$  as in Lemma 2.6 we have  $p_{k+1} = p$  and for  $j \in \{1, 2, \dots, k\}$ ,  $p_j = s$ . Since  $p$  is minimal in  $\mathcal{W}_{k+1}$  and  $\mathcal{W}_{k+1} \setminus \mathcal{W}_k$  is an ideal of  $\mathcal{W}_{k+1}$ , we have that  $\mathcal{W}_{k+1} \setminus \mathcal{W}_k \in p$ . Pick  $B \in \mathcal{F}$  such that  $B \in p_{k+1}$ . Let  $C = A \cup B$  and notice that  $\vec{p} \in cl_Z(\times_{j=1}^{k+1} C)$ .

Let  $L = \{e, p, s, q_1, q_2, \dots, q_{k+1}, q_1p, q_2p, \dots, q_{k+1}p\}$ . For notational convenience, let  $p_0 = q_0 = r_0 = e$ . Also, if  $w$  is a variable word, we let  $w(0) = e$ . If  $n \in \mathbb{N}$ ,  $f : \{1, 2, \dots, 3n\} \rightarrow \{0, 1, \dots, k + 1\}$ , and  $i \in \{1, 2, \dots, 3n\}$ , define  $\phi(f, i)$  as in Lemma 2.8 (with the obvious extension of the definition when  $f(i) = 0$ ). Given  $\theta \in L$ , we say that the pair  $(f, \theta)$  is *admissible* if and only if for every  $j \in \{1, 2, \dots, 3n\}$ , we have  $(\prod_{i=j}^{3n} \phi(f, i))\theta \in L$ . Notice that, if  $n > 1$ ,  $(f, \theta)$  is admissible, and  $g$  is the restriction of  $f$  to  $\{1, 2, \dots, 3n - 3\}$ , then  $(g, r_{f(3n-2)}p_{f(3n-1)}q_{f(3n)}\theta)$  is also admissible.

We construct the sequence  $\langle w_n \rangle_{n=1}^\infty$  inductively, three terms at a time. Let  $W_0 = \{e\}$  and for each  $i \in \mathbb{N}$ , as soon as we have chosen  $w_i$ , let  $W_i = \{\prod_{j=1}^i w_j(f(j)) : f : \{1, 2, \dots, i\} \rightarrow \{0, 1, \dots, k + 1\}\}$ .

Let  $n \in \mathbb{N} \cup \{0\}$  and assume that we have chosen  $w_i$  for all  $i \in \mathbb{N}$  with  $i \leq 3n$  (if any) so that, if  $n \geq 1$ ,  $\theta \in L$ ,  $f : \{1, 2, \dots, 3n\} \rightarrow \{0, 1, \dots, k + 1\}$ , and  $(f, \theta)$  is admissible, then

$$(*) \quad \left( \prod_{i=1}^{3n} w_i(f(i)) \right)^{-1} C \in \theta \Leftrightarrow C \in \left( \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)} \right) \theta.$$

Now  $\vec{r} \in \overline{H}$ , so pick by Lemma 2.2 a right variable word  $w_{3n+1}$  over  $\mathcal{W}_k$  such that for all  $l \in L$ , all  $u \in W_{3n}$ , and all  $t \in \{1, 2, \dots, k + 1\}$ ,

$$(a) \quad (uw_{3n+1}(t))^{-1}C \in l \Leftrightarrow u^{-1}C \in r_t l.$$

Since  $w_{3n+1}(0) = e = r_0$  we have also that (a) holds for  $t = 0$ .

Since  $\vec{p} \in \overline{I}$ , pick by Lemma 2.2 (and the observation above about the case  $t = 0$ ) a variable word  $w_{3n+2}$  over  $\mathcal{W}_k$  such that for all  $l \in L$ , all  $u \in W_{3n+1}$ , and all  $t \in \{0, 1, \dots, k + 1\}$ ,

$$(b) \quad (uw_{3n+2}(t))^{-1}C \in l \Leftrightarrow u^{-1}C \in p_t l.$$

Since  $\vec{q} \in \overline{\mathcal{J}}$ , pick by Lemma 2.2 a left variable word  $w_{3n+3}$  over  $\mathcal{W}_k$  such that for all  $l \in L$ , all  $u \in W_{3n+2}$ , and all  $t \in \{0, 1, \dots, k + 1\}$ ,

$$(c) \quad (uw_{3n+3}(t))^{-1}C \in l \Leftrightarrow u^{-1}C \in q_t l.$$

Now let  $\theta \in L$ , let  $f : \{1, 2, \dots, 3n + 3\} \rightarrow \{0, 1, \dots, k + 1\}$ , and assume that  $(f, \theta)$  is admissible. We verify that  $(*)$  holds. Notice that, since  $(f, \theta)$  is admissible, we have that

$$\{q_{f(3n+3)}\theta, p_{f(3n+2)}q_{f(3n+3)}\theta, r_{f(3n+1)}p_{f(3n+2)}q_{f(3n+3)}\theta\} \subseteq L.$$

(In the following, if  $n = 0$  we interpret  $\prod_{i=1}^{3n} w_i(f(i))$  as  $e$ .)

$$\begin{aligned} & \left( \prod_{i=1}^{3n+3} w_i(f(i)) \right)^{-1} C \in \theta \\ \Leftrightarrow & \left( \left( \prod_{i=1}^{3n} w_i(f(i)) \right) w_{3n+1}(f(3n + 1))w_{3n+2}(f(3n + 2)) \right)^{-1} C \in q_{f(3n+3)}\theta \\ \Leftrightarrow & \left( \left( \prod_{i=1}^{3n} w_i(f(i)) \right) w_{3n+1}(f(3n + 1)) \right)^{-1} C \in p_{f(3n+2)}q_{f(3n+3)}\theta \\ \Leftrightarrow & \left( \prod_{i=1}^{3n} w_i(f(i)) \right)^{-1} C \in r_{f(3n+1)}p_{f(3n+2)}q_{f(3n+3)}\theta \\ \Leftrightarrow & C \in \left( \prod_{i=0}^n r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)} \right) \theta. \end{aligned}$$

Here the first three double implications hold by (c), (b), and (a) respectively. If  $n = 0$ , the last double implication is a tautology. Otherwise, it is a consequence of the induction hypothesis.

The construction being complete, we claim that the sequence  $\langle w_n \rangle_{n=1}^\infty$  is as required. The first two conclusions are immediate. So let  $F \in \mathcal{P}_f(\mathbb{N})$  be allowable and let  $h : F \rightarrow \{1, 2, \dots, k + 1\}$ . Choose the least  $n \in \mathbb{N}$  such that  $F \subseteq \{1, 2, \dots, 3n\}$ . Define  $f : \{1, 2, \dots, 3n\} \rightarrow \{0, 1, \dots, k + 1\}$  by

$$f(i) = \begin{cases} h(i) & \text{if } i \in F, \\ 0 & \text{if } i \in \{1, 2, \dots, 3n\} \setminus F. \end{cases}$$

Then we have immediately that  $\prod_{i \in F} w_i(h(i)) = \prod_{i=1}^{3n} w_i(f(i))$ .

By Lemma 2.8 we have that the pair  $(f, e)$  is admissible. Recall that we identify  $e$  with the principal ultrafilter generated by  $e$ . Thus by  $(*)$ , we have that

$$\prod_{i=1}^{3n} w_i(f(i)) \in C \Leftrightarrow C \in \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}.$$

Again by Lemma 2.8 we have that

$$\prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)} = \begin{cases} s & \text{if } k + 1 \notin \text{range}(f), \\ p & \text{if } k + 1 \in \text{range}(f). \end{cases}$$

In either case, we have that  $C \in \prod_{i=0}^{n-1} r_{f(3i+1)}p_{f(3i+2)}q_{f(3i+3)}$ , and so  $\prod_{i=1}^{3n} w_i(f(i)) \in C$ . If  $k + 1 \notin \text{range}(h)$ , then  $\prod_{i=1}^{3n} w_i(f(i)) \in C \cap \mathcal{W}_k = A$ . If  $k + 1 \in \text{range}(h)$ , then  $\prod_{i=1}^{3n} w_i(f(i)) \in C \setminus \mathcal{W}_k = B$ . □

We conclude by showing that Carlson's Main Lemma is a consequence of Theorem 2.9.

**2.10. Corollary** (Carlson). *Let  $k \in \mathbb{N}$ , let the set of variable words over  $\mathcal{W}_k$  be partitioned into finitely many classes, and let  $\langle w_n \rangle_{n=1}^\infty$  be a sequence of variable words. Then there exists a variable reduction  $\langle t_n \rangle_{n=1}^\infty$  of  $\langle w_n \rangle_{n=1}^\infty$  such that all expressions of the form  $\prod_{i=1}^n t_i(f(i))$ , where  $n \in \mathbb{N}$ ,  $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, k\} \cup \{v\}$ , and  $v \in \text{range}(f)$ , lie in the same cell of the partition.*

*Proof.* Let  $\mathcal{W}_{k,v}$  be the set of variable words over  $\mathcal{W}_k$ , let  $\mathcal{F}$  be a finite partition of  $\mathcal{W}_{k,v}$ , and let  $\langle w_n \rangle_{n=1}^\infty$  be a sequence of variable words. Define a function  $\varphi : \mathcal{W}_{k+1} \setminus \mathcal{W}_k \rightarrow \mathcal{W}_{k,v}$  as follows. If  $u = a_1 a_2 \cdots a_l$ , where each  $a_i \in \{1, 2, \dots, k+1\}$ , let  $b_i = a_i$  if  $a_i \neq k+1$  and  $b_i = v$  if  $a_i = k+1$ ; then let  $\varphi(u) = \prod_{i=1}^l w(b_i)$ .

Let  $\mathcal{F} = \{\varphi^{-1}[H] : H \in \mathcal{F}\}$ . Then  $\mathcal{F}$  is a finite partition of  $\mathcal{W}_{k+1} \setminus \mathcal{W}_k$ . Let  $A$  be any central subset of  $\mathcal{W}_k$  and pick  $B \in \mathcal{F}$  and a sequence  $\langle w'_n \rangle_{n=1}^\infty$  of variable words as guaranteed by Theorem 2.9. Pick  $H \in \mathcal{F}$  such that  $B = \varphi^{-1}[H]$ .

For each  $n \in \mathbb{N}$ , let  $l_n$  be the length of  $w'_{3n-1}$  and write  $w'_{3n-1} = a_{n,1} a_{n,2} \cdots a_{n,l_n}$  where each  $a_{n,i} \in \{1, 2, \dots, k\} \cup \{v\}$ . (For this corollary, we are avoiding both left and right variable words.) Let  $\alpha_0 = 0$  and for  $n \in \mathbb{N}$ , let  $\alpha_n = \sum_{m=1}^n l_m$ . For each  $n \in \mathbb{N}$ , let  $t_n = \prod_{m=1}^{l_n} w_{\alpha_{n-1}+m}(a_{n,m})$ . Then  $\langle t_n \rangle_{n=1}^\infty$  is a variable reduction of  $\langle w_n \rangle_{n=1}^\infty$ .

To see that  $\langle t_n \rangle_{n=1}^\infty$  is as required, let  $n \in \mathbb{N}$  and let  $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, k\} \cup \{v\}$  with  $v \in \text{range}(f)$ . Define  $g : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, k+1\}$  by  $g(i) = f(i)$ , if  $f(i) \neq v$  and  $g(i) = k+1$  if  $f(i) = v$ . Then  $\prod_{i=1}^n w'_{3i-1}(g(i)) \in B$  and so  $\prod_{i=1}^n t_i(f(i)) = \varphi(\prod_{i=1}^n w'_{3i-1}(g(i))) \in H$ .  $\square$

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DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, WASHINGTON, DC 20059

*E-mail address:* [nhindman@fac.howard.edu](mailto:nhindman@fac.howard.edu)

*E-mail address:* [nhindman@aol.com](mailto:nhindman@aol.com)

*URL:* <http://members.aol.com/nhindman/>

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MEMPHIS, MEMPHIS, TENNESSEE 38152

*E-mail address:* [randall@msci.memphis.edu](mailto:randall@msci.memphis.edu)