PARABOLIC SUBGROUPS OF VERSHIK-KEROV’S GROUP

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Abstract. In this note we show that all parabolic subgroups of Vershik-Kerov’s group GLB(R) (i.e. subgroups containing T(∞, R)—the group of infinite dimensional upper triangular matrices) are net subgroups for a wide class of semilocal rings R.

1. Introduction

The classical result for finite dimensional general linear group over a field states that all “parabolic” subgroups, that is, containing the group of all upper triangular matrices, are “staircase groups” (see [1], p. 53, or [3] for more general result in the context of groups with BN−pair).

In [2] Borevich introduced a concept of a net of ideals σ and a net subgroup G(σ) (see definitions in the next section). Theorem 1 of [2] gives the following generalization:

Theorem 1.1. Let R be a semilocal ring, in which 1 is a sum of two invertible elements. If H is a parabolic subgroup of GLn(R), then there exists a unique T−net σ = (σij) of two-sided ideals of R, such that H = G(σ).

In our paper we extend this result to one infinite dimensional linear group. For R an associative ring with 1 by GL(∞, R) we denote the group of all column-finite invertible infinite matrices over R (indexed by positive integers N). Let T(∞, R) denote a group of all infinite upper triangular matrices over R (with invertible elements on the main diagonal). We define GLB(R) as a subgroup of GL(∞, R) of all matrices which have a finite number of nonzero entries below the main diagonal (clearly, T(∞, R) < GLB(R) < GL(∞, R)). This group was considered in the case of finite field k by Vershik and Kerov [6] and has applications in representation theory. GLB(k) is infinite dimensional, locally compact, totally disconnected, and amenable in topological sense and unimodular group. The stable general linear group GL∞(k), i.e. direct limit of GLn(k) under natural embeddings g → diag(g, 1), is its dense subgroup and the quotient group of GLB(k) over the center is topologically simple.

In this paper we give a purely algebraic description of parabolic subgroups of GLB(R) (i.e. containing T(∞, R)). Our main result is the following theorem.

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Theorem 1.2. Let $R$ be a semilocal ring, in which $1$ is a sum of two invertible elements. If $H$ is a parabolic subgroup of $GL(R)$, then there exists a unique $T$–net $\sigma = (\sigma_{ij})$ of two-sided ideals of $R$, such that $H = G(\sigma)$.

Using this theorem we can prove the “standard properties” (see [3], §2) of parabolic subgroups in $GL(R)$.

Theorem 1.3. If $R$ is a semilocal ring, in which $1$ is a sum of two invertible elements, then:

(i) If $P_1, P_2$ are two parabolic subgroups of $GL(R)$ and $gP_1g^{-1} \subset P_2$ for some $g \in GL(R)$, then $g \in P_2$ and $P_1 \subset P_2$.

(ii) Two different parabolic subgroups of $GL(R)$ are not conjugate.

(iii) Every parabolic subgroup of $GL(R)$ is self-normalized.

Let $S_{\text{fin}}(\mathbb{N})$ denote the regular matrix representation of all permutations of positive integers $\mathbb{N}$ with finite support. We have

Theorem 1.4 (Bruhat Decomposition Theorem). For any field $K$,

$$GLB(K) = T(\infty, K) \cdot S_{\text{fin}}(\mathbb{N}) \cdot T(\infty, K).$$

If $K = \mathbb{C}$ (complex numbers), then Theorem 1.4 follows from [6]. In [6] the Hecke algebra of double cosets of $GLB(\mathbb{C})$ over the Borel subgroup $B = T(\infty, \mathbb{C})$ was introduced, and because this Hecke algebra is isomorphic to the group algebra $\mathbb{C}(S_{\text{fin}}(\mathbb{N}))$, we obtain the Bruhat decomposition.

2. PROOFS OF MAIN RESULTS

By $e_i$ ($e_n$) we denote the unit matrix in $GL(\infty, R)$ ($GL_n(R)$) and by $e_{ij}$ a matrix with the only nontrivial element $1$ in the $i$–th row and $j$–th column. We denote $t_{ij}(\zeta) = e + \zeta e_{ij}, \zeta \in R$, $i, j \in \mathbb{N}$, $d_i(\theta) = e + (\theta - 1)e_{ii}$, $\theta$–invertible, and $[x, y] = xyx^{-1}y^{-1}$.

Definition 2.1. A system $\sigma = (\sigma_{ij})$ ($i, j \in \mathbb{N}$) of two sided ideals $\sigma_{ij}$ of $R$ is called a net if

$$(\ast) \quad \sigma_{ir} \cdot \sigma_{rj} \subset \sigma_{ij} \quad \text{for all} \quad i, j, r \in \mathbb{N}.$$

We call $\sigma$ a $T$–net if $\sigma_{ij} = R$ for $i \leq j$. If the set of indexes is $I = \{1, 2, \ldots, n\}$ we have the finite nets of ideals in $GL_n(R)$.

Let the set $M(\sigma)$ consist of all matrices $a$, such that $a_{ij} \in \sigma_{ij}$. If $\sigma$ satisfies $(\ast)$, then $e + M(\sigma) = \{e + a : a \in M(\sigma)\}$ is closed under multiplication of matrices and by $G(\sigma)$ we denote its maximal subgroup. Let $G(m, \infty)$ denote the subgroup of $GL(\infty, R)$ of all matrices $a$ for which $a_{ij} = 0$ for $i > \max\{j, m\}$. It is clear that $GL(R)$ is a direct limit of $G(m, \infty)$ under natural embeddings.

Proof of Theorem 1.2. For $H, T(\infty, R) < H < GLB(R)$, we define $T$–nets $\sigma$ and $\sigma(m)$ as follows:

$$\sigma_{ij} = \begin{cases} \{\zeta \in R : t_{ij}(\zeta) \in H\} & \text{for} \ i > j, \\ R & \text{for} \ i \leq j, \end{cases}$$

and

$$\sigma(m)_{ij} = \begin{cases} 0 & \text{if} \ i > \max\{m, j\}, \\ \sigma_{ij} & \text{otherwise}. \end{cases}$$
We put $H(m) = H \cap G(m, \infty)$. It is clear that $H$ is a direct limit of $H(m)$ and $G(\sigma)$ is a direct limit of $G(\sigma(m))$. We now show that $H(m) = G(\sigma(m))$. If $g \in H(m)$, then $g = \begin{pmatrix} g_1 & g_2 \\ 0 & g_2 \end{pmatrix}$ where $g_1 \in GL_{m}(R)$, $g_2 \in T(\infty, R)$. Since $H(m) \supset T(\infty, R)$, multiplying $g \in H(m)$ by matrices $\begin{pmatrix} e_m & 0 \\ 0 & g_2^{-1} \end{pmatrix}$ and $\begin{pmatrix} e_m & 0 \\ 0 & g_1 \end{pmatrix}$, we see that $g' = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in H$. We denote by $\tilde{H}(m)$ the subgroup of $GL_{m}(R)$ generated by all such $g_1$. From Theorem 1.1 it follows that there exists a unique finite $T$-net $\tilde{\sigma}(m)$ of ideals of $R$ such that $\tilde{H}(m) = G(\tilde{\sigma}(m))$. From the construction of $\tilde{\sigma}(m)$ and $\sigma(m)$ we deduce the equality $H(m) = G(\sigma(m))$ which implies $H = G(\sigma)$.

**Proof of Theorem 1.3.** We now prove (i). Then (ii) and (iii) follow easily from (i). If $g \cdot G(\sigma) \cdot g^{-1} \subset G(\sigma')$, then for some $m$ we have $g \in G(m, \infty)$. So

$$\left( \begin{pmatrix} g_1 & 0 \\ 0 & e \end{pmatrix} \right) \cdot G(\sigma) \cdot \left( \begin{pmatrix} g_1^{-1} & 0 \\ 0 & e \end{pmatrix} \right) \subset G(\sigma')$$

or equivalently $g_1 \cdot G(\tilde{\sigma}(m)) \cdot g_1^{-1} \subset G(\tilde{\sigma}'(m))$ in the group $GL_{m}(R)$. We show that $g_1 \in G(\tilde{\sigma}'(m))$ which implies $g \in G(\sigma'(m))$. From decomposition $g_1 = uvdw$ where $u, w$ are upper unitriangular, $d$ is diagonal and $v$ is lower unitriangular ([2], Thm. 1) it suffices to show that $v \in G(\sigma'(m))$. We have $v = v_2 \cdot \ldots \cdot v_m$, where $v_i = \prod_{j=1}^{i-1} t_{ij}(v_{ij})$. We proceed by induction. Assume that for some $r$, $2 \leq r \leq m$, we proved that $v_k \in G(\sigma'(m))$, $2 \leq k \leq r$. Thus $b \cdot G(\sigma(m)) \cdot b^{-1} \subset G(\sigma'(m))$, where $b = v_r \cdot \ldots \cdot v_m$. We have $c = [d_0(\theta)^{-1}, b] \in G(\sigma')$ and hence $c_{rs} = v_{rs}(\theta - 1) \in \sigma'_{rs}$. This implies $v_{rs} \in \sigma'_{rs}$ and $v_r \in G(\sigma')$.

**Proof of Theorem 1.4.** From [1], p. 45, for any field $K$ we have $GL_{m}(K) = T_{m}(K) \cdot S_{m} \cdot T_{m}(K)$, where $S_{m}$ is a regular matrix representation of symmetric group on $m$ elements. It means that $G(m, \infty) = T(\infty, K) \cdot S_{m} \cdot T(\infty, K)$ and since $S_{\text{fin}}(\mathbb{N})$ is direct limit of $S_{m}$ under natural embeddings Theorem 1.4 follows.

3. Remarks

a) In a semilocal ring $R$ the unit element 1 is a sum of two invertible elements if and only if every summand in the decomposition of a factor ring of $R$ over a Jacobson radical is different from two elements field ([2], Thm. 3).

b) As was claimed in [3] it is possible to extend Theorem 1.1 to rings $R$ such that $R$ is additively generated by all invertible elements and 1 is a sum of two invertible elements. It means that our results are also valid in this case.

c) Under the same assumption on $R$ as in the remark above, in [4] there is description of subgroups of $T(\infty, R)$ containing $D_{\text{fin}}(\infty, R)$ — the subgroup of finitary diagonal matrices. This result together with Theorem 1.2 give a description of two important intervals in the lattice of subgroups of $GL_{B}(R)$.

References


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