PARABOLIC SUBGROUPS OF VERSHIK-KEROV’S GROUP

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Abstract. In this note we show that all parabolic subgroups of Vershik-Kerov’s group $GLB(R)$ (i.e., subgroups containing $T(\infty, R)$—the group of infinite dimensional upper triangular matrices) are net subgroups for a wide class of semilocal rings $R$.

1. Introduction

The classical result for finite dimensional general linear group over a field states that all “parabolic” subgroups, that is, containing the group of all upper triangular matrices, are “staircase groups” (see [1], p. 53, or [3] for more general result in the context of groups with $BN$-pair).

In [2] Borevich introduced a concept of a net of ideals $\sigma$ and a net subgroup $G(\sigma)$ (see definitions in the next section). Theorem 1 of [2] gives the following generalization:

Theorem 1.1. Let $R$ be a semilocal ring, in which 1 is a sum of two invertible elements. If $H$ is a parabolic subgroup of $GL_n(R)$, then there exists a unique $T$-net $\sigma = (\sigma_{ij})$ of two-sided ideals of $R$, such that $H = G(\sigma)$.

In our paper we extend this result to one infinite dimensional linear group. For $R$ an associative ring with 1 by $GL(\infty, R)$ we denote the group of all column-finite invertible infinite matrices over $R$ (indexed by positive integers $\mathbb{N}$). Let $T(\infty, R)$ denote a group of all infinite upper triangular matrices over $R$ (with invertible elements on the main diagonal). We define $GLB(R)$ as a subgroup of $GL(\infty, R)$ of all matrices which have a finite number of nonzero entries below the main diagonal (clearly, $T(\infty, R) < GLB(R) < GL(\infty, R)$). This group was considered in the case of finite field $k$ by Vershik and Kerov [6] and has applications in representation theory. $GLB(k)$ is infinite dimensional, locally compact, totally disconnected, and amenable in topological sense and unimodular group. The stable general linear group $GL_{\infty}(k)$, i.e., direct limit of $GL_n(k)$ under natural embeddings $g \to \text{diag}(g, 1)$, is its dense subgroup and the quotient group of $GLB(k)$ over the center is topologically simple.

In this paper we give a purely algebraic description of parabolic subgroups of $GLB(R)$ (i.e., containing $T(\infty, R)$). Our main result is the following theorem.
Theorem 1.2. Let $R$ be a semilocal ring, in which 1 is a sum of two invertible elements. If $H$ is a parabolic subgroup of $GL(B)$, then there exists a unique $T$--net $\sigma = (\sigma_{ij})$ of two-sided ideals of $R$, such that $H = G(\sigma)$.

Using this theorem we can prove the “standard properties” (see [3], §2) of parabolic subgroups in $GL(R)$.

Theorem 1.3. If $R$ is a semilocal ring, in which 1 is a sum of two invertible elements, then:
(i) If $P_1$, $P_2$ are two parabolic subgroups of $GL(R)$ and $gP_1g^{-1} \subseteq P_2$ for some $g \in GL(R)$, then $g \in P_2$ and $P_1 \subseteq P_2$.
(ii) Two different parabolic subgroups of $GL(R)$ are not conjugate.
(iii) Every parabolic subgroup of $GL(R)$ is self-normalized.

Let $S_{\text{fin}}(\mathbb{N})$ denote the regular matrix representation of all permutations of positive integers $\mathbb{N}$ with finite support. We have

Theorem 1.4 (Bruhat Decomposition Theorem). For any field $K$,

$$GLB(K) = T(\infty, K) \cdot S_{\text{fin}}(\mathbb{N}) \cdot T(\infty, K).$$

If $K = \mathbb{C}$ (complex numbers), then Theorem 1.4 follows from [6]. In [6] the Hecke algebra of double cosets of $GLB(\mathbb{C})$ over the Borel subgroup $B = T(\infty, \mathbb{C})$ was introduced, and because this Hecke algebra is isomorphic to the group algebra $\mathbb{C}(S_{\text{fin}}(\mathbb{N}))$, we obtain the Bruhat decomposition.

2. Proofs of main results

By $e_n$ we denote the unit matrix in $GL(\infty, R)$ ($GL_n(R)$) and by $e_{ij}$ a matrix with the only nontrivial element 1 in the $i$--th row and $j$--th column. We denote $t_{ij}(\zeta) = e + \zeta e_{ij}$, $\zeta \in R$, $i, j \in \mathbb{N}$, $d_i(\theta) = e + (\theta - 1)e_{ii}$, $\theta$--invertible, and $[x, y] = xyx^{-1}y^{-1}$.

Definition 2.1. A system $\sigma = (\sigma_{ij})$ $(i, j \in \mathbb{N})$ of two sided ideals $\sigma_{ij}$ of $R$ is called a net if

$$(*) \quad \sigma_{ir} \cdot \sigma_{rj} \subseteq \sigma_{ij} \quad \text{for all } i, j, r \in \mathbb{N}.$$ 

We call $\sigma$ a $T$--net if $\sigma_{ij} = R$ for $i \leq j$. If the set of indexes is $I = \{1, 2, \ldots, n\}$ we have the finite nets of ideals in $GL_n(R)$.

Let the set $M(\sigma)$ consist of all matrices $a$, such that $a_{ij} \in \sigma_{ij}$. If $\sigma$ satisfies $(*)$, then $\varepsilon + M(\sigma) = \{ \varepsilon + a : a \in M(\sigma) \}$ is closed under multiplication of matrices and by $G(\sigma)$ we denote its maximal subgroup. Let $G(m, \infty)$ denote the subgroup of $GL(\infty, R)$ of all matrices $a$ for which $a_{ij} = 0$ for $i > \max\{j, m\}$. It is clear that $GLB(R)$ is a direct limit of $G(m, \infty)$ under natural embeddings.

Proof of Theorem 1.2. For $H$, $T(\infty, R) < H < GLB(R)$, we define $T$--nets $\sigma$ and $\sigma(m)$ as follows:

$$\sigma_{ij} = \left\{ \begin{array} {l l} \{ \zeta \in R : t_{ij}(\zeta) \in H \} & \text{for } i > j, \\ R & \text{for } i \leq j, \end{array} \right.$$ 

and

$$\sigma(m)_{ij} = \left\{ \begin{array} {l l} 0 & \text{if } i > \max\{m, j\}, \\ \sigma_{ij} & \text{otherwise}. \end{array} \right.$$
We put $H(m) = H \cap G(m, \infty)$. It is clear that $H$ is a direct limit of $H(m)$ and $G(\sigma)$ is a direct limit of $G(\sigma(m))$. We now show that $H(m) = G(\sigma(m))$. If $g \in H(m)$, then $g = \left( \begin{array}{cc} g_1 & g_2 \\ g_0 & g_3 \end{array} \right)$ where $g_1 \in GL_m(R)$, $g_2 \in T(\infty, R)$. Since $H(m) \geq T(\infty, R)$, multiplying $g \in H(m)$ by matrices $\left( \begin{array}{cc} \epsilon_m & 0 \\ 0 & g_2^{-1} \end{array} \right)$ and $\left( \begin{array}{cc} \epsilon_m & g_3 \\ 0 & \epsilon \end{array} \right)$, we see that $g' = \left( \begin{array}{cc} g_1 & 0 \\ 0 & \epsilon \end{array} \right) \in H$. We denote by $\tilde{H}(m)$ the subgroup of $GL_m(R)$ generated by all such $g_1$. From Theorem 1.1 it follows that there exists a unique finite $T$-net $\tilde{T}(m)$ of ideals of $R$ such that $\tilde{H}(m) = G(\tilde{T}(m))$. From the construction of $\tilde{T}(m)$ and $\sigma(m)$ we deduce the equality $H(m) = G(\sigma(m))$ which implies $H = G(\sigma)$.

Proof of Theorem 1.3. We now prove (i). Then (ii) and (iii) follow easily from (i). If $g \cdot G(\sigma) \cdot g^{-1} \subset G(\sigma')$, then for some $m$ we have $g \in G(m, \infty)$. So

$$\left( \begin{array}{cc} g_1 & 0 \\ 0 & \epsilon \end{array} \right) \cdot G(\sigma) \cdot \left( \begin{array}{cc} g_1^{-1} & 0 \\ 0 & \epsilon \end{array} \right) \subset G(\sigma')$$

or equivalently $g_1 \cdot G(\tilde{T}(m)) \cdot g_1^{-1} \subset G(\tilde{T}(m))$ in the group $GL_m(R)$. We show that $g_1 \in G(\tilde{T}(m))$ which implies $g \in G(\sigma'(m))$. From decomposition $g_1 = uvdw$ where $u, w$ are upper unitriangular, $d$ is diagonal and $v$ is lower unitriangular ([2], Thm. 1) it suffices to show that $v \in G(\sigma'(m))$. We have $v = v_2 \cdot \ldots \cdot v_m$, where $v_i = \prod_{j=1}^{i-1} t_j(v_j)$. We proceed by induction. Assume that for some $r$, $2 \leq r \leq m$, we proved that $v_k \in G(\sigma'(m))$, $2 \leq k < r$. Thus $b \cdot G(\sigma(m)) \cdot b^{-1} \subset G(\sigma'(m))$, where $b = v_r \cdot \ldots \cdot v_m$. We have $c = [d_s(\theta^{-1})^{-1}] \in G(\sigma')$ and hence $c_{rs} = v_{rs}(\theta - 1) \in \sigma'_{rs}$. This implies $v_{rs} \in \sigma'_{rs}$ and $v_r \in G(\sigma')$.

Proof of Theorem 1.4. From [1], p. 45, for any field $K$ we have $GL_m(K) = T_m(K) \cdot S_m \cdot T_m(K)$, where $S_m$ is a regular matrix representation of symmetric group on $m$ elements. It means that $G(m, \infty) = T(\infty, K) \cdot S_m \cdot T(\infty, K)$ and since $S_{\text{fin}}(\mathbb{N})$ is direct limit of $S_m$ under natural embeddings Theorem 1.4 follows.

3. Remarks

a) In a semilocal ring $R$ the unit element 1 is a sum of two invertible elements if and only if every summand in the decomposition of a factor ring of $R$ over a Jacobson radical is different from two elements field ([2], Thm. 3).

b) As was claimed in [3] it is possible to extend Theorem 1.1 to rings $R$ such that $R$ is additively generated by all invertible elements and 1 is a sum of two invertible elements. It means that our results are also valid in this case.

c) Under the same assumption on $R$ as in the remark above, in [4] there is description of subgroups of $T(\infty, R)$ containing $D_{\text{fin}}(\infty, R)$ — the subgroup of finitary diagonal matrices. This result together with Theorem 1.2 give a description of two important intervals in the lattice of subgroups of $GLB(R)$.

References


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