TOROIDAL SURGERIES ON HYPERBOLIC KNOTS

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Abstract. For a hyperbolic knot \( K \) in \( S^3 \), a toroidal surgery is Dehn surgery which yields a 3-manifold containing an incompressible torus. It is known that a toroidal surgery on \( K \) is an integer or a half-integer. In this paper, we prove that all integers occur among the toroidal slopes of hyperbolic knots with bridge index at most three and tunnel number one.

1. Introduction

Let \( K \) be a knot in the 3-sphere \( S^3 \), and let \( E(K) = S^3 - \text{Int}N(K) \) be its exterior. A slope on \( \partial E(K) \) is the isotopy class of an essential unoriented simple loop. As usual \[11\], the set of slopes on \( \partial E(K) \) is parameterized by \( \mathbb{Q} \cup \{\infty\} \) so that \( 1/0 \) is the meridian slope and \( 0/1 \) is the longitude slope. For a slope \( r \) on \( \partial E(K) \), \( K(r) \) denotes the closed orientable 3-manifold obtained by \( r \)-Dehn surgery on \( K \). Thus \( K(r) = E(K) \cup V \), where \( V \) is a solid torus glued to \( E(K) \) along their boundaries in such a way that \( r \) bounds a disk in \( V \).

Now suppose that \( K \) is a hyperbolic knot, i.e. the interior of \( E(K) \) has a complete hyperbolic structure. If \( K(r) \) is not hyperbolic, the surgery and the slope \( r \) are said to be exceptional. By the hyperbolic Dehn surgery theorem \[12\], \( K \) has only finitely many exceptional surgeries. A closed 3-manifold is toroidal if it contains an incompressible torus. If \( K(r) \) is toroidal, the surgery is said to be toroidal. Clearly, a toroidal surgery is exceptional.

There are some results on toroidal surgeries on hyperbolic knots. Gordon and Luecke \[8\] showed that if \( K(m/n) \) is toroidal, then \( |n| \leq 2 \). Hence a toroidal slope on a hyperbolic knot is either an integer or a half-integer. For hyperbolic alternating knots, toroidal slopes are integers divisible by four \[11\] (see also \[10\]). In this paper, we show that all integers can occur among the toroidal slopes of hyperbolic knots.

Theorem 1.1. For any integer \( r \), there exists a hyperbolic knot \( K \) in \( S^3 \) such that \( K(r) \) is toroidal. Furthermore, \( K \) has bridge index at most three and tunnel number one.

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2. Doubly Seifert-fibered knots

In this section, we will describe a construction of knots in $S^3$ that have toroidal surgeries done by Dean [4].

Let $H$ be a standardly embedded handlebody of genus two in $S^3$. Then $H' = S^3 - \text{Int}H$ is also a handlebody of genus two. Let $F = \partial H = \partial H'$. If a knot $K$ is embedded in $F$, then $\partial N(K) \cap F$ defines a slope on $\partial E(K)$, which is called the surface slope of $K$ with respect to $F$. Note that a surface slope is always integral.

**Lemma 2.1.** Let $K$ be a knot in $F$, and let $r$ be the surface slope of $K$ with respect to $F$. Assume that $K$ is non-separating in $F$. Then $K(r) \cong W \cup W'$, where $W$ ($W'$) is obtained from $H$ ($H'$ resp.) by attaching a 2-handle along $K$, and they are glued along their boundaries $\partial W$ and $\partial W'$, which are tori.

**Proof.** This is a special case of [4, Lemma 2.1.1]. Let $c_1$ and $c_2$ be the curves $F \cap \partial N(K)$. Then $c_i$ bounds a meridian disk $D_i$ of the attached solid torus $V$ in $K(r)$ for $i = 1, 2$. Let $\tilde{F} = (F - N(K)) \cup D_1 \cup D_2$. Since $K$ is non-separating in $F$, $\tilde{F}$ is a torus. We split $K(r)$ along $\tilde{F}$ into $W$ and $W'$. Then $W$ and $W'$ are homeomorphic to the described ones. \hfill \Box

For non-zero integers $m$ and $n$, let $G_{m,n}$ denote the group $\langle x, y \mid x^m y^n = 1 \rangle$. An element in a free group is primitive if it is a part of a basis. An element $w$ in the free group $\langle x, y \rangle$ is $(m, n)$ Seifert-fibered if $\langle x, y \mid w = 1 \rangle \cong G_{m,n}$. If $|m| = 1$ or $|n| = 1$, then $G_{m,n} \cong \mathbb{Z}$.

If a knot $K$ in $F$ represents a Seifert-fibered element of $\pi_1(H)$, then we say that $K$ is Seifert-fibered with respect to $H$. In particular, if $K$ represents a primitive element of $\pi_1(H)$, then $K$ is said to be primitive with respect to $H$. Also, if $K$ is Seifert-fibered with respect to both of $H$ and $H'$, then it is said to be doubly Seifert-fibered with respect to $F$. Note that the abelianization of $G_{m,n}$ is $\mathbb{Z} \oplus \mathbb{Z}_{(m,n)}$, where $(m,n)$ is the greatest common divisor of $m$ and $n$. Therefore if $K$ is Seifert-fibered with respect to $H$, say, then $K$ is non-separating on $F = \partial H$.

**Lemma 2.2.** If a knot $K$ on $\partial H$ is $(m, n)$ Seifert-fibered with respect to $H$ for $|m|, |n| \geq 2$, then the manifold $W$ obtained by adding a 2-handle to $H$ along $K$ is a Seifert fibered manifold with incompressible boundary.

**Proof.** Note that $W$ has Heegaard genus two. By additivity of Heegaard genus (see [3]), $W$ is irreducible, since $\pi_1(W) = G_{m,n}$. Hence $W$ is Haken. Then $W$ is a Seifert fibered manifold by [13], since $G_{m,n}$ has a nontrivial center. The last part follows from the fact that the only Seifert fibered manifold with non-empty compressible boundary is a solid torus. See also [4, Lemma 2.2.1]. \hfill \Box

**Lemma 2.3.** Let $K$ be a doubly Seifert-fibered knot in $F$ with surface slope $r$. Then $K(r)$ is toroidal.

**Proof.** This immediately follows from Lemmas 2.1 and 2.2. \hfill \Box

3. Proof of Theorem 1.1

Let $r$ be an integer. If $r$ is a toroidal surgery on a knot $K$, then $-r$ is one on the mirror image of $K$. Therefore we may assume that $r \geq 0$. 

We will divide the proof into three cases.

Case 1. $r \equiv 0 \pmod{4}$.

Let $K(b_1, b_2)$ be the 2-bridge knot corresponding to a continued fraction $[b_1, b_2]$. Then $K(2, -2)$ is the figure-eight knot, and 0 and 4 are toroidal surgeries [12]. If $r \geq 8$, $K(3, r/2)$ is hyperbolic, and $r$ gives a toroidal surgery [2, Theorem 1.1(2)]. Note that any 2-bridge knot has tunnel number one.

Case 2. $r \equiv \pm 1 \pmod{4}$.

Let $k \geq 1$, $k \not\equiv 0 \pmod{3}$, and $n \leq -2$. Let $K(4k + 3, 3, 2, n)$ be the twisted torus knot lying on $H$. It is obtained from the torus knot of type $(4k + 3, 3)$ by adding $n$-full twists on two strands that are parallel in the standard torus knot picture [4]. See Figure 1, where the two ends are glued to form $H$. We see that the surface slope with respect to $\partial H$ is $3(4k + 3) + 4n$. A knot in $S^3$, it is isotopic to $K(3, 4k + 3, 2, n)$, and hence it has bridge index at most three. Also, it is clear that the arc $\gamma$ shown in Figure 1 is an unknotting tunnel.

Let $\langle x, y \rangle$ and $\{a, b\}$ be the bases of $\pi_1(H)$ and $\pi_1(H')$, respectively, as in Figure 1. The following two lemmas are checked straightforwardly.

**Lemma 3.1.** In $\pi_1(H)$,

$$K(4k + 3, 3, 2, n) \text{ represents } \left\{ \begin{array}{ll} x^{\frac{4k+7}{3}} y x^{\frac{4k+2}{3}} y & \text{if } k \equiv 1 \pmod{3}, \\ x^{\frac{4k+5}{3}} y x^{\frac{4k+4}{3}} y & \text{if } k \equiv 2 \pmod{3}. \end{array} \right.$$  

**Lemma 3.2.** In $\pi_1(H')$, $K(4k + 3, 3, 2, n)$ represents $a^{2b^{-n}}ab^{-n}$.

**Lemma 3.3.** With respect to $H$,

$$K(4k + 3, 3, 2, n) \text{ is } \left\{ \begin{array}{ll} \text{Seifert-fibered} & \text{if } k \equiv 1 \pmod{3}, \\ \left(\frac{4k+5}{3}, 2\right) & \text{Seifert-fibered} \text{ if } k \equiv 2 \pmod{3}. \end{array} \right.$$  

**Proof.** We prove the case where $k \equiv 1 \pmod{3}$. The other case is similar.

$$\langle x, y \mid x^{\frac{4k+7}{3}} y x^{\frac{4k+2}{3}} y = 1 \rangle = \langle x, y \mid x^{\frac{4k+5}{3}} (x^{\frac{4k+2}{3}} y)^2 = 1 \rangle = \langle x, y, z \mid z^{\frac{4k+7}{3}} z^2 = 1, z = x^{\frac{4k+2}{3}} y \rangle = \langle x, z \mid x^{\frac{4k+5}{3}} z^2 = 1 \rangle.$$
Lemma 3.4. $K(4k+3,3,2,n)$ is $(3,n)$ Seifert-fibered with respect to $H$.

Proof. 

\[ \langle a, b \mid a^2 b^{-n} a b^{-n} = 1 \rangle = \langle a, b \mid a(ab^{-n})^2 = 1 \rangle = \langle a, b, c \mid ac^2 = 1, c = ab^{-n} \rangle = \langle b, c \mid c^3 b^n = 1 \rangle. \]

Proposition 3.5. For $K(4k+3,3,2,n)$, the surface slope $3(4k+3)+4n$ with respect to $F$ is toroidal. Therefore the knot is hyperbolic.

Proof. The knot $K(4k+3,3,2,n)$ is doubly Seifert-fibered by Lemmas 3.3 and 3.4. Hence the surface slope is toroidal by Lemma 2.3. Since it has bridge index at most three, it is either a torus knot or a hyperbolic knot. But a torus knot has no non-zero toroidal surgery. (In fact, 0-surgery on the trefoil is the only toroidal surgery for torus knots. See [9].) Thus the knot is hyperbolic.

For a given integer $r > 0$ such that $r \equiv 1 \pmod{4}$, we choose $k$ so that $r \leq 12k+1, k \not\equiv 0 \pmod{3}$. Then the knot $K(4k+3,3,2,n)$, $n = -2 - \frac{(12k+1)-r}{4}$, has the surface slope $r$ exactly. If $r \equiv -1 \pmod{4}$, then consider the knot $K(7,3,2,n)$ with $n = -6 + \frac{1}{r}$. Then it has the surface slope $-r$. Hence its mirror image is a desired one.

As stated before, if a hyperbolic 2-bridge knot has a toroidal slope, then the slope is an integer divisible by 4 [1]. Therefore our knots are 3-bridge.

Case 3. $r \equiv 2 \pmod{4}$.

Let $k \geq 1, k \not\equiv 0 \pmod{3}$ and $n \leq -2$. Let $K(4k+6,3,2,n)$ be the twisted torus knot lying on $H$. See Figure 2. We see that the surface slope with respect to $\partial H$ is $3(4k+6)+4n$.

As in Case 2, the knot has bridge index at most three, and tunnel number one. We use the same bases of $\pi_1(H)$ and $\pi_1(H')$ as in Case 1.

The following two lemmas are checked straightforwardly.
Lemma 3.6. In $\pi_1(H)$, $K(4k + 6, 3, 2, n)$ represents $x^{\frac{4k+11}{2}} y x^{\frac{4k+9}{2}} y$ if $k \equiv 1 \pmod{3}$, $x^{\frac{4k+11}{2}} y x^{\frac{4k+7}{2}} y$ if $k \equiv 2 \pmod{3}$.

Lemma 3.7. In $\pi_1(H')$, $K(4k + 6, 3, 2, n)$ represents $a^2b^n ab^{-n}$.

The next lemmas are proved by the same way as in the proofs of Lemmas 3.3 and 3.4.

Lemma 3.8. With respect to $H$, $K(4k + 6, 3, 2, n)$ is $(3, n)$ Seifert-fibered with respect to $H$.

Lemma 3.9. $K(4k + 6, 3, 2, n)$ is $(3, n)$ Seifert-fibered with respect to $H'$.

Proposition 3.10. For $K(4k + 6, 3, 2, n)$, the surface slope $3(4k + 6) + 4n$ with respect to $F$ is toroidal. Therefore the knot is hyperbolic.

Proof. The arguments in the proof of Proposition 3.5 work well.

Thus we have proved Theorem 1.1.

Remark 3.11. Eudave-Muñoz [5] gave infinitely many hyperbolic knots with half-integer toroidal surgeries. For example, the $(-2,3,7)$-pretzel knot has a toroidal slope $37/2$. Among his knots, $k(l, m, n, 0)$ (in his notation) has a non-integral toroidal slope $-\frac{1}{2} - l + l^2m + 2lm - 2l^2m^2 + (2lm - 1)^2n$.

Indeed, $k(3,1,1,0)$ is the $(-2,3,7)$-pretzel knot. Also, $k(l, m, 0, p)$ has a non-integral toroidal slope $-\frac{1}{2} - l + l^2m + 2lm - 2l^2m^2 + (2lm - 1 - l)^2p$.

See also [6] Propositions 5.3, 5.4. Since his knots are expected to give all hyperbolic knots with non-integer toroidal surgeries [7], it seems to be reasonable to conjecture that not all $n/2$ can be realized as toroidal slopes of hyperbolic knots. In fact, we may conjecture that $|n/2| \geq 37/2$.

REFERENCES


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