

GENERAL BLOW-UPS OF THE PROJECTIVE PLANE

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ABSTRACT. We study linear series on a projective plane blown up in a bunch of general points. Such series arise from plane curves of fixed degree with assigned fat base points. We give conditions (expressed as inequalities involving the number of points and the degree of the plane curves) on these series to be base point free, i.e. to define a morphism to a projective space. We also provide conditions for the morphism to be a higher order embedding. In the discussion of the optimality of obtained results we relate them to the Nagata Conjecture expressed in the language of Seshadri constants and we give a lower bound on these invariants.

0. INTRODUCTION

Rational surfaces obtained by blowing up \mathbb{P}^2 in several points have been studied in algebraic geometry for a long time. Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r points with exceptional divisors E_1, \dots, E_r and let $H = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$. Line bundles of the form

$$L = dH - \sum_{i=1}^r k_i E_i$$

have been addressed by many authors with respect to different properties. Here we are interested in properties of the morphism ϕ_L of X to a projective space, defined by the linear system $|L|$, provided that such a morphism exists (i.e. that $|L|$ is base point free). Even in the simplest case $k_1 = \dots = k_r = 1$ an optimal criterion for $|L|$ to define a morphism was obtained only recently by Coppens [8, section 3.3]. It states that L is globally generated if X is obtained from \mathbb{P}^2 by blowing up at most $r \leq h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 3$ general points and $d \geq 7$.

On the other hand, again under the assumption that all k_i are equal to 1, Kuchle [16] and independently Xu [22] showed that L is ample provided $L^2 > 0$ and $d \geq 3$. Recently Biran [6, Corollary 2.1.B] obtained a similar optimal result if all k_i are equal to 2. In this direction we prove the following (not optimal) result valid for all k .

Theorem 3. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let $k \geq 2$ and d be integers such that $d \geq 3k + 1$. If $r \leq \frac{d^2}{k^2} - 1$, then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is ample.*

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With ampleness of L established it is natural to ask when L is very ample. This problem also has a long history, e.g. [4], [5], the ultimate result (again in the case all $k_i = 1$) being proved by D'Almeida and Hirschowitz [9]. They showed that ϕ_L is an embedding if X is obtained from \mathbb{P}^2 by blowing up at most $r \leq h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 6$ general points and $d \geq 5$.

The notions of the global generation of a line bundle and the very ampleness were generalized in several ways in the past decade. Here we focus on the k -very ampleness (see Definition 1.1) which in a sense governs the geometry of secant varieties of X . Since a k -very ample line bundle has degree at least k on any curve, a necessary condition for L to be k -very ample is that $k_i \geq k$ for all $i = 1, \dots, r$ (for a slightly different approach see [20]). Thus fat points necessarily come into the picture. If $r \leq 8$, i.e. if X is a Del Pezzo surface, Di Rocco [11] obtained the complete characterization of k -very ample line bundles. It seems that for the greater number of points $r \geq 9$ no satisfactory generalizations of results of Coppens have been found. In this direction Theorem 4 provides a criterion for the global generation of L . The problem we study here was investigated by Ballico and Coppens [1], however their cohomological conditions seem difficult to verify in general. In this paper we restrict to the homogeneous case $k_1 = \dots = k_r = k$ and prove the following.

Theorem 1. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let k and d be non-negative integers such that $d \geq 4k + 1$. If $r \leq \frac{(d+3)^2}{(k+1)^2} - 4$, then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is k -very ample.*

Our approach is a combination of methods used by Di Rocco [11] and those introduced by Ein and Lazarsfeld [12] and used by Xu [22]. This result does not seem to be optimal. After the proof we state a conjectural bound and discuss its relation to the famous Nagata Conjecture. Towards the Nagata Conjecture we prove the following bound.

Theorem 2. *Let P_1, \dots, P_r be $r \geq 9$ general points in \mathbb{P}^2 and let k_1, \dots, k_r be fixed non-negative integers. If $C \subset \mathbb{P}^2$ is a curve of degree d such that $\text{mult}_{P_i} C \geq k_i$, then*

$$d \geq \frac{1}{\sqrt{r+1}} \sum_{i=1}^r k_i.$$

This bound improves the previous result of Xu [21] (see Proposition 2.2) except in the case when $r + 1$ is a square where the strong inequality follows from [22]. In roughly half of the cases the above bound also improves those following from results obtained by Harbourne [13].

Notation. We work throughout over the field \mathbb{C} of complex numbers. If X is a variety, by K_X we denote the canonical divisor of X , and by $H^i(X, \mathcal{F}) = H^i(\mathcal{F})$ the cohomology groups of a coherent sheaf \mathcal{F} on X . The numerical equivalence of divisors is denoted by \equiv . For a given real number α we denote by $\lceil \alpha \rceil$ its round-up, i.e. the least integer greater than or equal to α . By a very general point of an algebraic variety we mean a point lying away from a countable sum of Zariski closed subsets.

1. K -VERY AMPLENESS

Recently there has been considerable interest in generalizations of the notions of the global generation and the very ampleness of a line bundle. For the purpose of the present note we recall the following.

Definition 1.1. Let k be a non-negative integer. A line bundle L on a projective variety X is said to be k -very ample if the evaluation mapping

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_Z)$$

is surjective for every 0-dimensional subscheme $Z \subset X$ of length at most $k + 1$.

Thus a line bundle is globally generated if and only if it is 0-very ample and it is very ample if and only if it is 1-very ample. Geometrically, if X is embedded by a k -very ample line bundle, then there are no $(k + 1)$ -secant $(k - 1)$ -planes in the embedding. Equivalently, one obtains an embedding of the Hilbert scheme $S^{[k+1]}X$ into an appropriate Grassmanian; see [7] for details.

It follows at once from the definition that a k -very ample line bundle is l -very ample for every $l \leq k$. It is also easy to see that the tensor product $L_1 \otimes L_2$ of a k_1 -very ample line bundle L_1 with a k_2 -very ample line bundle L_2 is $(k_1 + k_2)$ -very ample. Thus a naive way to obtain a result along the lines of Theorem 1 would be to use the result of D’Almeida-Hirschowitz mentioned in the Introduction. This might be formulated as follows.

Proposition 1.2 (A naive criterion). *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let k be a positive integer. If $r \leq \frac{(|d/k|+2)(|d/k|+1)}{2} - 6$ and $d \geq 5k$, then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is k -very ample.*

Roughly speaking this naive result is weaker by a factor $\frac{1}{2}$ than the result proved in Theorem 3.

On surfaces there is the following useful criterion of Beltrametti and Sommese [3] which generalizes Reider’s theorem to k -very ample adjoint line bundles.

Proposition 1.3 (Beltrametti-Sommese). *Let X be a smooth surface and let N be a nef line bundle on X , such that $N^2 \geq 4k + 5$. Let $Z \subset X$ be a zero-dimensional subscheme of length $k + 1$. Then either $K_X + N$ is k -very ample at Z or there exists an effective divisor D containing Z such that $N - 2D$ is \mathbb{Q} -effective (i.e. there exists a positive integer m such that $|m(N - 2D)| \neq \emptyset$) and the following inequalities are satisfied:*

$$ND - k - 1 \leq D^2 < \frac{ND}{2} < k + 1.$$

2. NAGATA CONJECTURE AND SESHADRI CONSTANTS

The following conjecture was formulated by Nagata [18] over forty years ago.

Nagata Conjecture. *Let P_1, \dots, P_r be $r \geq 9$ general points in \mathbb{P}^2 and let k_1, \dots, k_r be fixed non-negative integers. If $C \subset \mathbb{P}^2$ is a curve of degree d such that $\text{mult}_{P_i} C \geq k_i$, then*

$$d \geq \frac{1}{\sqrt{r}} \sum_{i=1}^r k_i.$$

It still remains open, apart from the case when the number of blown up points r is a square, which was settled by Nagata himself. In fact if r is a square and $r \geq 16$ Nagata proved that the sharp inequality holds for very general points P_1, \dots, P_r (which was essential for his counterexample to the Hilbert's 14th problem). In the remaining cases (i.e. r not a square) both formulations are of course equivalent. For the purpose of this paper it is convenient to formulate the Nagata Conjecture in the language of Seshadri constants.

We recall briefly that Seshadri constants were introduced by Demailly [10] in order to measure the local positivity of a line bundle.

Definition 2.1. Let X be a smooth projective variety and P_1, \dots, P_r pairwise distinct points in X . The r -tuple Seshadri constant of L at P_1, \dots, P_r is the number

$$\varepsilon(L; P_1, \dots, P_r) = \sup\{\varepsilon \text{ such that } \pi^*L - \varepsilon \sum_{i=1}^r E_i \text{ is nef}\},$$

where π is the blowing up of X at P_1, \dots, P_r with exceptional divisors E_1, \dots, E_r .

The Nagata Conjecture now states (see [23], [2])

Nagata Conjecture, second formulation. Let P_1, \dots, P_r be $r \geq 9$ very general points in \mathbb{P}^2 . Then

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \dots, P_r) = \frac{1}{\sqrt{r}}.$$

Note that the inequality $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \dots, P_r) \leq \frac{1}{\sqrt{r}}$ is obvious, so only the other inequality is of interest. It might be also worthwhile here to observe that in the above formulation the case when the number of points r is a square $r = s^2$ is easy to prove. Indeed, let C be a smooth curve of degree s and let P_1, \dots, P_{s^2} be arbitrary points on C . Since C is in particular irreducible we check that the Seshadri constant $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \dots, P_{s^2}) = \frac{1}{s}$. In fact if it were smaller, say equal $\varepsilon < \frac{1}{s}$, then there would be a curve D passing through P_1, \dots, P_{s^2} such that its proper transform on the blow-up would spoil the nefness of $H - \frac{1}{s} \sum_{i=1}^{s^2} E_i$. But on \mathbb{P}^2 this would imply that D has a negative intersection with C , which is absurd. This shows that the Seshadri constant is maximal for the special choice of points P_1, \dots, P_{s^2} and then by the semi-continuity result in the spirit of [19] we conclude that it must be maximal for a very general set of s^2 points.

It seems that when r is not a square the best general bound known up to now is the following result of Xu [21, Theorem 1(a)].

Proposition 2.2 (Xu). Let P_1, \dots, P_r be $r \geq 9$ general points in \mathbb{P}^2 . Then

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \dots, P_r) \geq \frac{\sqrt{r-1}}{r}.$$

It should be pointed out that for special values of r there are various results e.g. [6], [13], [14] which imply better bounds (also often better than those of Theorem 2). However such results lack the uniformity of the above Proposition and of Theorem 2. Xu's approach is based on the following observation (which appears implicitly already in [12]) which is crucial also for our consideration.

Lemma 2.3 (Xu). Let P_1, \dots, P_s be general points in \mathbb{P}^2 and let C be a reduced and irreducible curve of degree p passing through the points P_i with the multiplicity

$\text{mult}_{P_i} C = m_i$, for $i = 1, \dots, s$. Then

$$p^2 \geq \sum_{i=1}^s m_i^2 - m_j$$

for arbitrary $j \in \{1, \dots, s\}$ with $m_j > 0$.

3. BLOW-UPS OF \mathbb{P}^2

We are now in a position to formulate and prove the main results of this paper.

Theorem 1. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let k and d be non-negative integers such that $d \geq 4k + 1$. If $r \leq \frac{(d+3)^2}{(k+1)^2} - 4$, then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is k -very ample.*

Proof. The idea is to look at L as an adjoint line bundle and with help of inequalities from Proposition 1.3 show that no curve D spoiling the k -very ampleness of L can exist. Since for $k = 0$ and $k = 1$ the Theorem is known we assume that $k \geq 2$.

Turning to the details, let $N = L - K_X = (d + 3)H - (k + 1) \sum_{i=1}^r E_i$. First we show that N satisfies assumptions of Proposition 1.3. First of all we have

$$N^2 = (d + 3)^2 - r(k + 1)^2 \geq 4(k + 1)^2 > 4k + 5.$$

Our next claim is that

$$(3.1) \quad \begin{aligned} N.C &\geq k + 1 \text{ for every irreducible and reduced curve } C \subset X \\ &\text{and the equality holds only if } C \text{ is one of the exceptional divisors } E_i. \end{aligned}$$

Taking this for granted for a while, the theorem follows immediately. Indeed, N is then obviously nef so that Proposition 1.3 applies. Assuming that $L = K_X + N$ fails to be k -very ample we are given a curve D in X satisfying

$$N.D - k - 1 \leq D^2 < \frac{N.D}{2} < k + 1.$$

Now (3.1), together with the first inequality, implies that $D^2 \geq 0$ and $D^2 = 0$ if and only if D consists of the exceptional divisors. Since these are disjoint this implies in turn $D^2 < 0$, a contradiction. So we are left with $D^2 \geq 1$ but then the Hodge Index Theorem gives

$$1 \cdot (4(k + 1)^2) \leq D^2 \cdot N^2 \leq (N.D)^2.$$

Hence $N.D \geq 2(k + 1)$ which contradicts the last of the above inequalities.

It remains to show (3.1). If C is one of the exceptional divisors the claim is obvious, so we may assume that C is of the form $C = \pi^*F - \sum_{i=1}^r m_i E_i$, where F is a reduced and irreducible plane curve of degree p . Renumbering the points if necessary we may assume that $m_1 \geq \dots \geq m_s \geq 1$ and $m_{s+1} = \dots = m_r = 0$. Then

$$N.C = (d + 3)p - (k + 1) \sum_{i=1}^s m_i$$

and we claim that

$$(3.2) \quad (d + 3)p > (k + 1) \left(\sum_{i=1}^s m_i + 1 \right).$$

Since by our assumption $\frac{d+3}{k+1} \geq \sqrt{r+4} \geq \sqrt{s+4}$ it suffices to show that

$$(3.3) \quad p\sqrt{s+4} > \sum_{i=1}^s m_i + 1.$$

Now we have to consider several cases.

Case 1a. We assume that $m_s = 1$ and $s \geq 6$. Under this assumption (3.3) is equivalent to

$$p\sqrt{s+4} > \sum_{i=1}^{s-1} m_i + 2.$$

Since $(\sum_{i=1}^{s-1} m_i)^2 \leq (s-1) \sum_{i=1}^{s-1} m_i^2$ it is enough to show

$$(s-1) \sum_{i=1}^{s-1} m_i^2 < p^2(s+4) + 4 - 4p\sqrt{s+4}.$$

From Lemma 2.3 we have $\sum_{i=1}^{s-1} m_i^2 + 1 \leq p^2 + 1$, hence our inequality follows from

$$5p^2 - 4p\sqrt{s+4} + 4 > 0$$

which is easily seen to hold for $p \geq \sqrt{s-1}$ (Lemma 2.3 again) and $s \geq 6$.

Case 1b. We assume that $m_s = 1$ and $s \leq 5$. Using our assumption $d \geq 4k+1$ and (3.2) it is enough to show

$$(3.4) \quad 4p > \sum_{i=1}^s m_i + 1.$$

Since C is irreducible either $s \leq 2$ and $m_1 = m_s = 1$ in which case we are done, or $p \geq m_1 + 1$. But then we have

$$4p \geq 4m_1 + 4 \geq \sum_{i=1}^{s-1} m_i + 4 > \sum_{i=1}^{s-1} m_i + 2 = \sum_{i=1}^s m_i + 1$$

and this shows (3.4).

Case 2a. We assume that $m_s \geq 2$ and $s \geq 4$. Since $(\sum_{i=1}^s m_i)^2 \leq s \sum_{i=1}^s m_i^2$ we see from (3.3) that it is enough to prove

$$s \sum_{i=1}^s m_i^2 < p^2(s+4) + 1 - 2p\sqrt{s+4}.$$

From Lemma 2.3 we have $\sum_{i=1}^s m_i^2 \leq p^2 + m_s$ hence we are reduced to showing that

$$sm_s + 2p\sqrt{s+4} < 4p^2 + 1.$$

Now looking at Lemma 2.3 in a slightly different way, we have

$$(3.5) \quad p^2 \geq \sum_{i=1}^s m_i^2 - m_s \geq (s-1)m_s$$

or equivalently $sm_s \leq s \frac{p^2}{s-1}$. Hence it is enough to show

$$2p\sqrt{s+4} < (4 - \frac{s}{s-1})p^2 + 1.$$

Since (3.5) implies in particular that $p \geq \sqrt{2(s-1)}$ the inequality in question reduces finally to

$$\sqrt{2(s+4)} < \frac{3s-4}{\sqrt{s-1}}$$

which is easy to check for $s \geq 4$.

Case 2b. Finally we assume that $m_s \geq 2$ and $s \leq 3$. This is easy and goes e.g. as Case 1b. \square

As already mentioned in the Introduction we expect that the above Theorem is not optimal. A natural generalization of results of Coppens and D’Almeida and Hirschowitz would be the following.

Conjecture. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let k and d be non-negative integers such that $d \geq 4k + 1$. If r satisfies*

$$(3.6) \quad \binom{d+2}{2} - r \binom{k+1}{2} \geq 3(k+1),$$

then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is k -very ample.

This means that given a fixed k for d sufficiently large the line bundle L is k -very ample as soon as it has enough sections (the dimension of the $(k+1)$ -st secant variety of X must be the expected one under a k -very ample embedding). The referee informed us that Alexander had a similar conjecture but we were not able to track it down in the literature.

Remark 3.1. The above Conjecture implies the Nagata Conjecture (for $r \geq 17$).

Indeed, let $r \geq 17$ be fixed. Then given k we compute the minimal d_k satisfying (3.6) and $d \geq 4k + 1$ (the bound $r \geq 17$ is in fact caused by the latter condition).

We have $d_k = \left\lceil \frac{-3 + \sqrt{1 + 4(6 + rk)(k + 1)}}{2} \right\rceil$. Of course a k -very ample line bundle is nef, so $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \dots, P_r) \geq \frac{k}{d_k}$ for every k . Passing with k to infinity we get $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \dots, P_r) \geq \frac{1}{\sqrt{r}}$.

Using similar methods as in the proof of Theorem 1 we show a criterion for the ampleness of a line bundle of the form $L = dH - k \sum_{i=1}^r E_i$ which in turn is used in the proof of Theorem 2.

Theorem 3. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let $k \geq 2$ and d be integers such that $d \geq 3k + 1$. If $r \leq \frac{d^2}{k^2} - 1$, then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is ample.*

Proof. The idea is to apply the Nakai-Moishezon criterion. First of all we have

$$N^2 = d^2 - rk^2 \geq k^2 > 0.$$

Next we claim that $N.C > 0$ for all irreducible and reduced curves $C \subset X$. This is obvious if C is one of the exceptional curves so we may assume that $C = \pi^*F - \sum_{i=1}^r m_i E_i$ where F is a reduced and irreducible plane curve of degree p . Furthermore without loss of generality we may assume that $m_1 \geq \dots \geq m_s \geq 1$ and $m_{s+1} = \dots = m_r = 0$. Computing $N.C$ we see that our claim is equivalent to

$$(3.7) \quad dp > k \sum_{i=1}^s m_i.$$

Since $\frac{d}{k} \geq \sqrt{r+1} \geq \sqrt{s+1}$ and $(\sum_{i=1}^s m_i)^2 \leq s \sum_{i=1}^s m_i^2$ it suffices to show that

$$s \sum_{i=1}^s m_i^2 < p^2(s+1).$$

From Lemma 2.3 we have

$$(3.8) \quad \sum_{i=1}^s m_i^2 \leq p^2 + m_s$$

so that we are done if

$$(3.9) \quad sm_s < p^2$$

holds. In fact (3.8) implies (3.9) unless $m_1 = \dots = m_s = 1$. In the latter case in (3.7) we have to show

$$p \frac{d}{k} > s.$$

If $s \leq 5$, then using the assumption $d \geq 3k + 1$ it is enough to show $3p \geq s$, which follows by the assumption that the points P_1, \dots, P_r are general. So we assume that $s \geq 6$. Since $\frac{d}{k} \geq \sqrt{s+1}$ it is enough to prove that

$$p > \frac{s}{\sqrt{s+1}}.$$

From the generality assumption again we have $\binom{p+2}{2} - s \geq 1$ or equivalently $p \geq \frac{\sqrt{8s+9}-3}{2}$. Now the claim follows from the simple observation that the real valued function

$$f(s) = \frac{\sqrt{8s+9}-3}{2} - \frac{s}{\sqrt{s+1}}$$

is positive for $s \geq 6$. □

Now we pass to the proof of our global generation result. In the case of fat points, i.e. $k \geq 2$, this result seems to be new. Moreover it implies Theorem 2. As pointed out by the referee the global generation problem is connected to the vanishing of the cohomology groups of line bundles of the type considered here (see [15] for a conjectural picture concerning vanishing of H^1 and [17] for partial results). As the vanishing follows from the positivity for many special values of r one can get better bounds than those presented below using certain results of [6], [13], [14].

Theorem 4. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the blowing up of \mathbb{P}^2 in r general points and let $k \geq 2$ and d be integers such that $d \geq 3k + 1$. If $r \leq \frac{(d+3)^2}{(k+1)^2} - 1$, then the line bundle $L = dH - k \sum_{i=1}^r E_i$ is globally generated.*

Proof. As in the proof of Theorem 1 let $N = L - K_X = (d+3)H - (k+1) \sum_{i=1}^r E_i$. Then Theorem 3 implies that N is ample and satisfies $N^2 > 5$ so that Proposition 1.3 applies. This means that if $L = K_X + N$ fails to be globally generated, then there exists a curve $D \subset X$ such that $N \cdot D = 1$ and $D^2 = 0$. Let p, m_1, \dots, m_r

be integers such that $D \equiv pH - \sum_{i=1}^r m_i E_i$. Computing $N.D$ we get $(d + 3)p = (k + 1) \sum_{i=1}^r m_i + 1$ which implies $\sum_{i=1}^r m_i > 0$, so that in fact it must be

$$(3.10) \quad \sum_{i=1}^r m_i \geq 1.$$

Taking into account $\frac{d+3}{k+1} \geq \sqrt{r+1}$ we have

$$p\sqrt{r+1} \leq \sum_{i=1}^r m_i + \frac{1}{k+1}.$$

Since $D^2 = 0$ we have $p^2 = \sum_{i=1}^r m_i^2$, hence

$$(r+1) \sum_{i=1}^r m_i^2 \leq \left(\sum_{i=1}^r m_i + \frac{1}{k+1} \right)^2.$$

Now, the right-hand side is bounded by $r \sum_{i=1}^r m_i^2 + \frac{2}{k+1} \sum_{i=1}^r m_i + \frac{1}{(k+1)^2}$ so that

$$(k+1)^2 \sum_{i=1}^r m_i^2 \leq 2(k+1) \sum_{i=1}^r m_i + 1.$$

Since $\sum_{i=1}^r m_i^2 \geq \sum_{i=1}^r m_i$ we get $(k^2 - 1) \sum_{i=1}^r m_i \leq 1$ which in view of $k \geq 2$ and (3.10) gives a contradiction. \square

Finally as an easy corollary we obtain

Theorem 2. *Let P_1, \dots, P_r be $r \geq 9$ general points in \mathbb{P}^2 and let k_1, \dots, k_r be fixed non-negative integers. If $C \subset \mathbb{P}^2$ is a curve of degree d such that $\text{mult}_{P_i} C \geq k_i$, then*

$$d \geq \frac{1}{\sqrt{r+1}} \sum_{i=1}^r k_i.$$

Proof. This goes exactly as in Remark 3.1. Using Theorem 3 we have $d_k = \lceil (k+1)\sqrt{r+1} \rceil - 3$ this time. \square

Remark 3.2. After this paper was written we learned that Harbourne [14, Proposition I.2] obtained the same bound in a much broader context of all polarized surfaces of any characteristic.

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