A RAMSEY THEOREM FOR MEASURABLE SETS

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Abstract. We prove that if \( X \) is a perfect Polish space and \( [X]^2 = P_0 \cup \ldots \cup P_{k-1} \) is a partition with universally measurable pieces, then there is Cantor set \( C \subset X \) with \( [C]^2 \subset P_i \) for some \( i \).

By a theorem of F. Galvin, if \( X \) is a perfect Polish space and \( [X]^2 = P_0 \cup \ldots \cup P_{k-1} \) is a partition with pieces having the Baire property, then there is Cantor set \( C \subset X \) with \( [C]^2 \subset P_i \) for some \( i \). (See [4], [5], and [6, 19.7, p. 130].) Our aim is to prove the analogous statement for universally measurable sets.

Theorem 1. Let \( X \) be a nonempty perfect Polish space and \( [X]^2 = P_0 \cup \ldots \cup P_{k-1} \) is a partition, where each \( P_i \) is universally measurable in the sense that \( P_i^* = \{(x, y) \in X^2 : \{x, y\} \in P_i\} \) is universally measurable. Then there is a Cantor set \( C \subset X \) with \( [C]^2 \subset P_i \) for some \( i \).

Note that the statement is not true for infinitely many pieces or for \( [X]^3 \) instead of \( [X]^2 \) (see [6, 19.9 and 19.10]). We shall prove Theorem 1 by presenting a sufficient condition for the existence of “squares” (sets of the form \( P \times P \)) contained in a given subset of \( \mathbb{R}^2 \). By a theorem of M. L. Brodskii [1], every subset of \( \mathbb{R}^2 \) of positive measure contains the product of two perfect sets (see also [3, p. 114]). Of course, a set of positive measure need not contain squares. As the trivial example \( \{(x, y) : |y - x| > \varepsilon\} \) suggests, in order to find squares in a set \( H \subset \mathbb{R}^2 \) we need an extra condition saying that \( H \) is large in the vicinity of the diagonal \( \Delta = \{(x, x) : x \in \mathbb{R}\} \). We remark that no reasonable condition implies the existence of a subset \( P \times P \) where \( P \) is a perfect set of positive measure (take \( H = \{(x, y) : x - y \text{ is irrational}\} \)). Moreover, as Z. Buczolich showed [2] there is a set \( H \) of full measure such that whenever \( P \) and \( Q \) are closed sets such that \( P \times Q \subset H \) and \( P \) is of positive measure, then the Hausdorff dimension of \( Q \) is zero.

We shall use the following notation. If \( H \subset \mathbb{R}^2 \), then we shall denote \( H^+ = \{(x, y) : (y, x) \in H\} \). The set \( H \) will be called symmetric if \( H^+ = H \). The sections of \( H \) are denoted by \( H_x = \{y : (x, y) \in H\} \) and \( H^y = \{x : (x, y) \in H\} \). The Lebesgue outer measure in \( \mathbb{R}^k \) will be denoted by \( \lambda_k \). The symmetric upper density of the set \( A \subset \mathbb{R} \) at the point \( x \in \mathbb{R} \) is defined by

\[
\overline{d}(A, x) = \lim_{h \to 0^+} \frac{\lambda_1(A \cap (x - h, x + h))}{2h}.
\]
The symmetric lower density \( \underline{d}(A, x) \) is defined by taking the lim inf of the same quotient.

**Theorem 2.** Let \( H \subset \mathbb{R}^2 \) be a symmetric Lebesgue measurable set satisfying
\[
\lambda_1(\{x \in \mathbb{R} : \underline{d}(H, x) > 0\}) > 0.
\]
Then there is a nonempty perfect set \( P \subset \mathbb{R} \) such that \( P \times P \subset H \cup \Delta \).

**Proof of Theorem 2.** First we note that if \( H \subset \mathbb{R}^2 \) is measurable, then, by Fubini’s theorem, the function \( f_h \) defined by \( f_h(x) = \lambda_1(H_{x} \cap [x - h, x + h]) \) (\( x \in \mathbb{R} \)) is also measurable for every fixed \( h \). Therefore the functions \( f(x) = \underline{d}(H_x, x) \) and \( f(x) = \overline{d}(H_x, x) \) are measurable as well, since \( \bar{f} = \limsup_{n \to \infty} f_{1/n} \cdot n/2 \) and \( \underline{f} = \liminf_{n \to \infty} f_{1/n} \cdot n/2 \).

Now we show that if \( H \subset \mathbb{R}^2 \) is closed, symmetric and satisfies (1), then there is a nonempty perfect set \( P \) such that \( P \times P \subset H \).

If \( n \) is a positive integer and \( n = 2^k m \), where \( m \) is odd, then we shall denote \( r(n) = k \). Then \( 0 \leq r(n) < n \) for every \( n > 0 \). Also, for every nonnegative integer \( k \) there are infinitely many \( n \)'s with \( r(n) = k \).

It is well-known that for every measurable \( H \subset \mathbb{R}^2 \) there is a set \( A \subset H \) such that \( \lambda_2(H \setminus A) = 0 \), and for every \( (x, y) \in A \), \( x \) is a density point of \( A_y \) and \( y \) is a density point of \( A_x \) (see [2] pp. 130-131). If \( H \) is symmetric, then \( A \) can also be chosen symmetric, since otherwise we take \( A \cap A^c \) instead of \( A \). Suppose that \( H \) is a symmetric closed set satisfying (1), and let \( A \) be a symmetric subset of \( H \) with the property described above. Since \( \lambda_2(H \setminus A) = 0 \), it follows from Fubini’s theorem that \( \lambda_1((H \setminus A)_x) = 0 \) for a.e. \( x \), and hence \( \overline{d}(H_x, x) = \underline{d}(A_x, x) \) for a.e. \( x \). Then the set \( D = \{x : \underline{d}(A_x, x) > 0\} \) is measurable and has positive measure. Let \( D^0 \) denote the set of those elements of \( D \) which are density points of \( D \). We shall define a sequence \( x_n \in D^0 \) as follows. Let \( x_0 \in D^0 \) be arbitrary. Let \( n > 0 \) and suppose that \( x_i \in D^0 \) has been selected for every \( i < n \) such that \( (x_i, x_j) \in A \) for every \( i, j < n \), \( i \neq j \). Then \( x_{r(n)} \in A_{x_i} \) for every \( i < n \), \( i \neq r(n) \) and, consequently, \( x_{r(n)} \) is a density point of the set \( E = \bigcap \{A_{x_i} : i < n, i \neq r(n)\} \). Since \( x_{r(n)} \in D^0 \), \( x_{r(n)} \) is also a density point of \( D^0 \). Finally, as \( x_{r(n)} \in D \), it follows from the definition of \( D \) that the outer density of \( A_{x_{r(n)}} \) at \( x_{r(n)} \) is positive. This implies that the outer density of the set
\[
M = E \cap A_{x_{r(n)}} \cap D^0 = \bigcap \{A_{x_i} : i < n\} \cap D^0
\]
at \( x_{r(n)} \) is positive. In particular, \( x_{r(n)} \) is a point of accumulation of \( M \). Let \( x_n \in M \) be such that \( 0 < |x_n - x_{r(n)}| < 1/n \). Then \( x_n \in D^0 \) and \( (x_i, x_n) \in A \) for every \( i < n \).

In this way we have defined \( x_n \) for every \( n \). The set \( S = \{x_n : n = 0, 1, \ldots\} \) is dense in itself, since for every \( k \) there are infinitely many \( n \)'s with \( r(n) = k \) and for these \( n \)'s we have \( 0 < |x_n - x_k| < 1/n \). Let \( P \) be the closure of \( S \), then \( P \) is perfect. Since \( (S \times S) \setminus \Delta \subset A \subset H \) and \( H \) is closed, it follows that \( P \times P \subset H \).

Finally, we prove that if \( H \) is measurable, symmetric and satisfies (1), then \( H \cup \Delta \) contains a symmetric closed subset which also satisfies (1); this will finish the proof of the theorem.

Put \( H_n = \{(x, y) \in H : x - \frac{1}{n} \leq y \leq x + \frac{1}{n}\} \), and let \( F_n \) be a closed subset of \( H_n \) such that \( \lambda_2(H_n \setminus F_n) < n^{-4} \) (\( n = 1, 2, \ldots \)). We define \( F_0 = \bigcup_{n=1}^{\infty} F_n \) and \( \Delta = \bigcup_{n=1}^{\infty} F_n^* \cup \Delta \). Then \( F \) is closed, symmetric, and is contained in \( H \cup \Delta \). We
shall prove that \( \bar{d}(F_x, x) = \bar{d}(H_x, x) \) for a.e. \( x \). Since \( \lambda_1(\{x : \bar{d}(H_x, x) > 0\}) > 0 \) by assumption, this will prove that \( F \) also satisfies (1). Let

\[
G_n = \{x \in \mathbb{R} : \lambda_1((H \setminus F)_x \cap [x - \frac{1}{n}, x + \frac{1}{n}]) > n^{-2}\}.
\]

By \( \lambda_2(H_n \setminus F_n) < n^{-4} \) we have \( \lambda_1(G_n) < n^{-2} \). Thus \( \sum_{n=1}^{\infty} \lambda_1(G_n) < \infty \) and hence the set \( G = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} G_n \) is of measure zero. If \( x \notin G \), then, for \( n \) large enough,

\[
\lambda_1((H \setminus F)_x \cap [x - \frac{1}{n}, x + \frac{1}{n}]) \leq n^{-2}.
\]

Therefore \( \bar{d}((H \setminus F)_x, x) = 0 \) and \( \bar{d}(F_x, x) = \bar{d}(H_x, x) \), which completes the proof.

\( \square \)

**Proof of Theorem 1.** We may assume that \( X \) is a Cantor set. Applying a suitable homeomorphism, we may also suppose that \( X \) is a closed subset of \( \mathbb{R} \) having positive measure. The sets \( P^*_t \) are symmetric and, being universally measurable, they are also measurable with respect to the Lebesgue measure. If \( x \) is a density point of \( X \), then

\[
1 = \bar{d}(X, x) \leq \sum_{i=0}^{k-1} \bar{d}(P^*_t, x),
\]

and thus \( \max_{0 \leq i < k} \bar{d}(P^*_t, x) > 0 \). Therefore \( \lambda_1(\{x \in \mathbb{R} : \bar{d}(P^*_t, x) > 0\}) > 0 \) holds for at least one of \( i = 0 \ldots k-1 \). For such an \( i \) the set \( P^*_t \) satisfies the condition of Theorem 2. Therefore \( P \times P \subset P^*_t \cup \Delta \) for a suitable nonempty perfect \( P \), and thus \( |P|^2 \subset P \).

It is obvious that the condition of symmetry cannot be omitted from Theorem 2 (consider the set \( \{(x, y) : y > x\} \)). For nonsymmetric sets we can prove the following.

**Theorem 3.** If \( H \subset \mathbb{R}^2 \) is measurable and

\[
\lambda_1(\{x \in \mathbb{R} : \bar{d}(H_x, x) > 1/2\}) > 0,
\]

then there is a nonempty perfect set \( P \) such that \( P \times P \subset H \cup \Delta \).

**Proof.** First we show that if (2) holds, then the set \( E = H \cap H^* \) satisfies the following condition: \( E \) is symmetric and there is an interval \( I \) such that

\[
\lim_{h \to 0^+} \lambda_2(\{x : y - h \leq x \leq y + h\} \cap E)/h > 0.
\]

Indeed, (2) implies that there is an \( \varepsilon > 0 \) and there is an interval \( I \) such that the set \( B = \{x \in I : \bar{d}(H_x, x) > \frac{1}{4} + \varepsilon\} \) is of positive measure. As we saw in the proof of Theorem 2, \( B \) is measurable. We put \( T(h) = \{(x, y) : x \in B, \ x - h \leq y \leq x + h\} \) for every \( h > 0 \). Then \( T(h)^* = \{(x, y) : y \in B, \ y - h \leq x \leq y + h\} \). We prove that

\[
\lim_{h \to 0^+} \lambda_2(T(h)^* \setminus T(h))/h = 0.
\]

If \( A \subset \mathbb{R}^2 \) is measurable, then \( \lambda_2(A) = \int_{-\infty}^{\infty} \lambda_1(\{x : (x, x + t) \in A\}) \, dt \) by Fubini's theorem. Since

\[
\{x : (x, x + t) \in T(h)^* \setminus T(h)\} = (B - t) \setminus B
\]
for every $|t| \leq h$, we have
\[
\lambda_2(T(h)^* \cap T(h)) = \int_{-h}^{h} \lambda_1((B - t) \setminus B) \, dt.
\]
Now $\lambda_1((B - t) \setminus B) \to 0$ as $t \to 0$, and (4) follows. It is easy to see that
\[
\liminf_{h \to 0^+} \lambda_2(T(h) \cap H)/(2h) \geq \left( \frac{1}{2} + \epsilon \right) \lambda_1(B),
\]
and thus
\[
\liminf_{h \to 0^+} \lambda_2(T(h)^* \cap H^*)/(2h) \geq \left( \frac{1}{2} + \epsilon \right) \lambda_1(B).
\]
Since
\[
\lambda_2(H \cap H^* \cap T(h) \cap T(h)^*) = \lambda_2(H \cap T(h)) + \lambda_2(H^* \cap T(h)^*) - \lambda_2((H \cap T(h)) \cup (H^* \cap T(h)^*)) \geq \lambda_2(H \cap T(h)) + \lambda_2(H^* \cap T(h)^*) - \lambda_2(T(h)) - \lambda_2(T(h)^* \setminus T(h)),
\]
it follows from (4), (5), and (6) that
\[
\liminf_{h \to 0^+} \lambda_2(H \cap H^* \cap T(h) \cap T(h)^*)/(2h) \geq (1 + 2\epsilon)\lambda_1(B) - \lambda_1(B) - 0 > 0.
\]
Therefore, $E = H \cap H^*$ satisfies (3), even if the lim sup is replaced by lim inf.

Now we prove that $\lambda_1(\{x : d(E_x, x) > 0\}) > 0$. By Theorem 2, this implies that $E \cup \Delta$ (and thus $H \cup \Delta$ as well) contains a set of the form $P \times P$, where $P$ is nonempty perfect. By (3), there is a $\delta > 0$ such that
\[
\lambda_2(\{(x, y) : x \in I, \ y - h_n \leq x \leq y + h_n \} \cap E) > \delta h_n \quad (n = 1, 2, \ldots)
\]
for a suitable sequence of positive numbers $h_n$ converging to zero. Let
\[
\eta = \delta/(2 \max(|I|, 1))
\]
and
\[
C_n = \{x \in I : \lambda_1(E_x \cap [x - h_n, x + h_n]) > \eta h_n\}.
\]
Then $C_n$ is measurable for every $n$. Also, we have $\lambda_1(C_n) > \delta/4$ since otherwise $\lambda_1(C_n) \leq \delta/4$ would imply, by Fubini’s theorem,
\[
\lambda_2(\{(x, y) : x \in I, \ y - h_n \leq x \leq y + h_n \} \cap E) \leq \frac{\delta}{4} \cdot 2h_n + \frac{\delta h_n}{2|I|} \cdot |I| = \delta h_n,
\]
which is impossible. Let $C = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_n$. Then $\lambda_1(C) \geq \delta/4 > 0$, and for every $x \in C$ there are infinitely many $n$’s with $x \in C_n$. Therefore $d(E_x, x) \geq \eta/2 > 0$ for every $x \in C$, completing the proof.

References


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