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A RAMSEY THEOREM FOR MEASURABLE SETS

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Abstract. We prove that if X is a perfect Polish space and $[X]^2 = P_0 \cup \ldots \cup P_{k-1}$ is a partition with universally measurable pieces, then there is Cantor set $C \subset X$ with $[C]^2 \subset P_i$ for some i.

By a theorem of F. Galvin, if X is a perfect Polish space and $[X]^2 = P_0 \cup \ldots \cup P_{k-1}$ is a partition with pieces having the Baire property, then there is Cantor set $C \subset X$ with $[C]^2 \subset P_i$ for some i. (See [4], [5], and [6, 19.7, p. 130].) Our aim is to prove the analogous statement for universally measurable sets.

Theorem 1. *Let X be a nonempty perfect Polish space and $[X]^2 = P_0 \cup \ldots \cup P_{k-1}$ is a partition, where each P_i is universally measurable in the sense that $P_i^* = \{(x,y) \in X^2 : \{x,y\} \in P_i\}$ is universally measurable. Then there is a Cantor set $C \subset X$ with $[C]^2 \subset P_i$ for some i.*

Note that the statement is not true for infinitely many pieces or for $[X]^3$ instead of $[X]^2$ (see [6, 19.9 and 19.10]). We shall prove Theorem 1 by presenting a sufficient condition for the existence of "squares" (sets of the form $P \times P$) contained in a given subset of $\mathbf{R}^2$. By a theorem of M. L. Brodskii [1], every subset of $\mathbf{R}^2$ of positive measure contains the product of two perfect sets (see also [3, p. 114]). Of course, a set of positive measure need not contain squares. As the trivial example $\{(x,y) : |y-x| > \varepsilon\}$ suggests, in order to find squares in a set $H \subset \mathbf{R}^2$ we need an extra condition saying that H is large in the vicinity of the diagonal $\Delta = \{(x,x) : x \in \mathbf{R}\}$. We remark that no reasonable condition implies the existence of a subset $P \times P$ where P is a perfect set of positive measure (take $H = \{(x,y) : x-y$ is irrational$\}$). Moreover, as Z. Buczolich showed [2] there is a set H of full measure such that whenever P and Q are closed sets such that $P \times Q \subset H$ and P is of positive measure, then the Hausdorff dimension of Q is zero.

We shall use the following notation. If $H \subset \mathbf{R}^2$, then we shall denote $H^* = \{(x,y) : (y,x) \in H\}$. The set H will be called symmetric if $H^* = H$. The sections of H are denoted by $H_x = \{y : (x,y) \in H\}$ and $H^y = \{x : (x,y) \in H\}$. The Lebesgue outer measure in $\mathbf{R}^k$ will be denoted by λ_k. The symmetric upper density of the set $A \subset \mathbf{R}$ at the point $x \in \mathbf{R}$ is defined by

$$\overline{d}(A,x) = \limsup_{h \to 0+} \frac{\lambda_1(A \cap (x-h, x+h))}{2h}.$$

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The symmetric lower density $\underline{d}(A, x)$ is defined by taking the lim inf of the same quotient.

Theorem 2. *Let $H \subset \mathbf{R}^2$ be a symmetric Lebesgue measurable set satisfying*

$$\lambda_1(\{x \in \mathbf{R} : \bar{d}(H_x, x) > 0\}) > 0. \tag{1}$$

Then there is a nonempty perfect set $P \subset \mathbf{R}$ such that $P \times P \subset H \cup \Delta$.

Proof of Theorem 2. First we note that if $H \subset \mathbf{R}^2$ is measurable, then, by Fubini's theorem, the function f_h defined by $f_h(x) = \lambda_1(H_x \cap [x-h, x+h])$ $(x \in \mathbf{R})$ is also measurable for every fixed h. Therefore the functions $\bar{f}(x) = \bar{d}(H_x, x)$ and $\underline{f}(x) = \underline{d}(H_x, x)$ are measurable as well, since $\bar{f} = \limsup_{n\to\infty} f_{1/n} \cdot n/2$ and $\underline{f} = \liminf_{n\to\infty} f_{1/n} \cdot n/2$.

Now we show that if $H \subset \mathbf{R}^2$ is closed, symmetric and satisfies (1), then there is a nonempty perfect set P such that $P \times P \subset H$.

If n is a positive integer and $n = 2^k m$, where m is odd, then we shall denote $r(n) = k$. Then $0 \le r(n) < n$ for every $n > 0$. Also, for every nonnegative integer k there are infinitely many n's with $r(n) = k$.

It is well-known that for every measurable $H \subset \mathbf{R}^2$ there is a set $A \subset H$ such that $\lambda_2(H \setminus A) = 0$, and for every $(x, y) \in A$, x is a density point of A^y and y is a density point of A_x (see [7, pp. 130-131]). If H is symmetric, then A can also be chosen symmetric, since otherwise we take $A \cap A^*$ instead of A. Suppose that H is a symmetric closed set satisfying (1), and let A be a symmetric subset of H with the property described above. Since $\lambda_2(H \setminus A) = 0$, it follows from Fubini's theorem that $\lambda_1((H \setminus A)_x) = 0$ for a.e. x, and hence $\bar{d}(H_x, x) = \bar{d}(A_x, x)$ for a.e. x. Then the set $D = \{x : \bar{d}(A_x, x) > 0\}$ is measurable and has positive measure. Let D^0 denote the set of those elements of D which are density points of D. We shall define a sequence $x_n \in D^0$ as follows. Let $x_0 \in D^0$ be arbitrary. Let $n > 0$ and suppose that $x_i \in D^0$ has been selected for every $i < n$ such that $(x_i, x_j) \in A$ for every $i, j < n,\ i \ne j$. Then $x_{r(n)} \in A_{x_i}$ for every $i < n,\ i \ne r(n)$ and, consequently, $x_{r(n)}$ is a density point of the set $E = \bigcap\{A_{x_i} : i < n,\ i \ne r(n)\}$. Since $x_{r(n)} \in D^0$, $x_{r(n)}$ is also a density point of D^0. Finally, as $x_{r(n)} \in D$, it follows from the definition of D that the outer density of $A_{x_{r(n)}}$ at $x_{r(n)}$ is positive. This implies that the outer density of the set

$$M = E \cap A_{x_{r(n)}} \cap D^0 = \bigcap\{A_{x_i} : i < n\} \cap D^0$$

at $x_{r(n)}$ is positive. In particular, $x_{r(n)}$ is a point of accumulation of M. Let $x_n \in M$ be such that $0 < |x_n - x_{r(n)}| < 1/n$. Then $x_n \in D^0$ and $(x_i, x_n) \in A$ for every $i < n$.

In this way we have defined x_n for every n. The set $S = \{x_n : n = 0, 1, \dots\}$ is dense in itself, since for every k there are infinitely many n's with $r(n) = k$ and for these n's we have $0 < |x_n - x_k| < 1/n$. Let P be the closure of S, then P is perfect. Since $(S \times S) \setminus \Delta \subset A \subset H$ and H is closed, it follows that $P \times P \subset H$.

Finally, we prove that if H is measurable, symmetric and satisfies (1), then $H \cup \Delta$ contains a symmetric closed subset which also satisfies (1); this will finish the proof of the theorem.

Put $H_n = \{(x, y) \in H : x - \frac{1}{n} \le y \le x + \frac{1}{n}\}$, and let F_n be a closed subset of H_n such that $\lambda_2(H_n \setminus F_n) < n^{-4}$ $(n = 1, 2, \dots)$. We define $F_0 = \bigcup_{n=1}^{\infty} F_n$ and $F = F_0 \cup F_0^* \cup \Delta$. Then F is closed, symmetric, and is contained in $H \cup \Delta$. We shall prove that $\bar{d}(F_x, x) = \bar{d}(H_x, x)$ for a.e. x. Since $\lambda_1(\{x : \bar{d}(H_x, x) > 0\}) > 0$ by assumption, this will prove that F also satisfies (1). Let

$$G_n = \big\{x \in \mathbf{R} : \lambda_1\big((H \setminus F)_x \cap \big[x - \frac{1}{n}, x + \frac{1}{n}\big]\big) > n^{-2}\big\}.$$

By $\lambda_2(H_n \setminus F_n) < n^{-4}$ we have $\lambda_1(G_n) < n^{-2}$. Thus $\sum_{n=1}^{\infty} \lambda_1(G_n) < \infty$ and hence the set $G = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} G_n$ is of measure zero. If $x \notin G$, then, for n large enough,

$$\lambda_1\bigg((H \setminus F)_x \cap \big[x - \frac{1}{n}, x + \frac{1}{n}\big]\bigg) \leq n^{-2}.$$

Therefore $\overline{d}((H \setminus F)_x, x) = 0$ and $\bar{d}(F_x, x) = \bar{d}(H_x, x)$, which completes the proof. $\square$

Proof of Theorem 1. We may assume that X is a Cantor set. Applying a suitable homeomorphism, we may also suppose that X is a closed subset of $\mathbf{R}$ having positive measure. The sets P_i^* are symmetric, and, being universally measurable, they are also measurable with respect to the Lebesgue measure. If x is a density point of X, then

$$1 = \overline{d}(X, x) \leq \sum_{i=0}^{k-1} \overline{d}((P_i^*)_x, x),$$

and thus $\max_{0 \leq i < k} \overline{d}((P_i^*)_x, x) > 0$. Therefore $\lambda_1(\{x \in \mathbf{R} : \bar{d}((P_i^*)_x, x) > 0\}) > 0$ holds for at least one of $i = 0 \ldots k-1$. For such an i the set P_i^* satisfies the condition of Theorem 2. Therefore $P \times P \subset P_i^* \cup \Delta$ for a suitable nonempty perfect P, and thus $[P]^2 \subset P_i$. $\square$

It is obvious that the condition of symmetry cannot be omitted from Theorem 2 (consider the set $\{(x, y) : y > x\}$). For nonsymmetric sets we can prove the following.

Theorem 3. *If* $H \subset \mathbf{R}^2$ *is measurable and*

$$\lambda_1(\{x \in \mathbf{R} : \underline{d}(H_x, x) > 1/2\}) > 0, \tag{2}$$

then there is a nonempty perfect set P *such that* $P \times P \subset H \cup \Delta$.

Proof. First we show that if (2) holds, then the set $E = H \cap H^*$ satisfies the following condition: E is symmetric and there is an interval I such that

$$\limsup_{h \to 0+} \lambda_2(\{(x, y) : x \in I,\ y - h \leq x \leq y + h\} \cap E)/h > 0. \tag{3}$$

Indeed, (2) implies that there is an $\varepsilon > 0$ and there is an interval I such that the set $B = \{x \in I : \underline{d}(H_x, x) > \frac{1}{2} + \varepsilon\}$ is of positive measure. As we saw in the proof of Theorem 2, B is measurable. We put $T(h) = \{(x, y) : x \in B,\ \ x - h \leq y \leq x + h\}$ for every $h > 0$. Then $T(h)^* = \{(x, y) : y \in B,\ \ y - h \leq x \leq y + h\}$. We prove that

$$\lim_{h \to 0+} \lambda_2(T(h)^* \setminus T(h))/h = 0. \tag{4}$$

If $A \subset \mathbf{R}^2$ is measurable, then $\lambda_2(A) = \int_{-\infty}^{\infty} \lambda_1(\{x : (x, x + t) \in A\})\, dt$ by Fubini's theorem. Since

$$\{x : (x, x + t) \in T(h)^* \setminus T(h)\} = (B - t) \setminus B$$

for every $|t| \le h$, we have

$$\lambda_2(T(h)^* \setminus T(h)) = \int_{-h}^{h} \lambda_1((B-t) \setminus B)\, dt.$$

Now $\lambda_1((B-t) \setminus B) \to 0$ as $t \to 0$, and (4) follows. It is easy to see that

$$\liminf_{h \to 0+} \lambda_2\left(T(h) \cap H\right)/(2h) \ge \left(\frac{1}{2} + \varepsilon\right) \lambda_1(B), \tag{5}$$

and thus

$$\liminf_{h \to 0+} \lambda_2\left(T(h)^* \cap H^*\right)/(2h) \ge \left(\frac{1}{2} + \varepsilon\right) \lambda_1(B). \tag{6}$$

Since

$$\begin{aligned} &\lambda_2\big(H \cap H^* \cap T(h) \cap T(h)^*\big) \\ &\qquad = \lambda_2\big(H \cap T(h)\big) + \lambda_2\big(H^* \cap T(h)^*\big) - \lambda_2\big(\big(H \cap T(h)\big) \cup \big(H^* \cap T(h)^*\big)\big) \\ &\qquad \ge \lambda_2\big(H \cap T(h)\big) + \lambda_2\big(H^* \cap T(h)^*\big) - \lambda_2(T(h)) - \lambda_2(T(h)^* \setminus T(h)), \end{aligned}$$

it follows from (4), (5), and (6) that

$$\liminf_{h \to 0+} \lambda_2\big(H \cap H^* \cap T(h) \cap T(h)^*\big)/(2h) \ge (1+2\varepsilon)\lambda_1(B) - \lambda_1(B) - 0 > 0.$$

Therefore, $E = H \cap H^*$ satisfies (3), even if the lim sup is replaced by lim inf.

Now we prove that $\lambda_1(\{x : \overline{d}(E_x, x) > 0\}) > 0$. By Theorem 2, this implies that $E \cup \Delta$ (and thus $H \cup \Delta$ as well) contains a set of the form $P \times P$, where P is nonempty perfect. By (3), there is a $\delta > 0$ such that

$$\lambda_2(\{(x,y) : x \in I,\ y - h_n \le x \le y + h_n\} \cap E) > \delta h_n \quad (n = 1, 2, \dots)$$

for a suitable sequence of positive numbers h_n converging to zero. Let

$$\eta = \delta/(2\max(|I|, 1))$$

and

$$C_n = \{x \in I : \lambda_1(E_x \cap [x - h_n, x + h_n]) > \eta h_n\}.$$

Then C_n is measurable for every n. Also, we have $\lambda_1(C_n) > \delta/4$ since otherwise $\lambda_1(C_n) \le \delta/4$ would imply, by Fubini's theorem,

$$\lambda_2(\{(x,y) : x \in I,\ y - h_n \le x \le y + h_n\} \cap E) \le \frac{\delta}{4} \cdot 2h_n + \frac{\delta h_n}{2|I|} \cdot |I| = \delta h_n,$$

which is impossible. Let $C = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_n$. Then $\lambda_1(C) \ge \delta/4 > 0$, and for every $x \in C$ there are infinitely many n's with $x \in C_n$. Therefore $\overline{d}(E_x, x) \ge \eta/2 > 0$ for every $x \in C$, completing the proof. □

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