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A PROOF OF WEINBERG'S CONJECTURE ON LATTICE-ORDERED MATRIX ALGEBRAS

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Dedicated to Professor Melvin Henriksen on his 75th birthday

ABSTRACT. Let $\mathbf{F}$ be a subfield of the field of real numbers and let $\mathbf{F}_n$ $(n \geq 2)$ be the $n \times n$ matrix algebra over $\mathbf{F}$. It is shown that if $\mathbf{F}_n$ is a lattice-ordered algebra over $\mathbf{F}$ in which the identity matrix 1 is positive, then $\mathbf{F}_n$ is isomorphic to the lattice-ordered algebra $\mathbf{F}_n$ with the usual lattice order. In particular, Weinberg's conjecture is true.

Let L be a totally ordered field, and let L_n $(n \geq 2)$ be the $n \times n$ matrix algebra over L. Then L_n may be lattice-ordered by requiring that a matrix in L_n is positive exactly when each of its entries is positive, that is, the positive cone is $(L^+)_n$. This lattice order is called the usual lattice order of L_n.

Let $\mathbf{Q}$ be the field of rational numbers. In 1966, Weinberg conjectured that $(\mathbf{Q}^+)_n$ is the only lattice order of $\mathbf{Q}_n$ (up to an isomorphism) such that $\mathbf{Q}_n$ is a *lattice-ordered algebra* (ℓ*-algebra*) over $\mathbf{Q}$ in which 1 is positive, and he proved his conjecture for $n = 2$ [8]. Recently some conditions have been obtained to ensure an ℓ-algebra L_n, in which 1 is positive, is isomorphic to the ℓ-algebra L_n with the usual lattice order [5], [7].

In this paper, we show that Weinberg's conjecture is true for a matrix ℓ-algebra over any subfield of real numbers. More precisely, suppose that $\mathbf{F}$ is a subfield of the field of real numbers; it is shown that an ℓ-algebra $\mathbf{F}_n$ over $\mathbf{F}$ in which 1 is positive is isomorphic to the ℓ-algebra $\mathbf{F}_n$ with the usual lattice order.

We begin by collecting some definitions and results we will use later. The reader is referred to Birkhoff & Pierce [2] and Fuchs [4] for the general theory of *lattice-ordered rings* (ℓ*-rings*). A *partially ordered ring* (*po-ring*) R is an (associative) ring which is partially ordered, and in which i) $a \geq b$ implies $a + c \geq b + c$, for any $c \in R$, and ii) $a \geq 0$ and $b \geq 0$ imply $ab \geq 0$. Let R be a po-ring. The set $R^+ = \{a \in R : a \geq 0\}$ is called the *positive cone* of R. Clearly R^+ is closed under addition and multiplication, and $R^+ \cap -R^+ = \{0\}$. Conversely, if P is a subset of a ring R which is closed under addition and multiplication, and satisfies $P \cap -P = \{0\}$, then the partial order $\geq$ defined by $a \geq b$ if and only if $a - b \in P$ makes R into a po-ring with the positive cone equal to P. We will refer to such a

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positive cone as a partial order of a po-ring R, and denote a po-ring R with positive cone P by (R, P). If R is a po-ring and an algebra over a totally ordered field L, then R is called a *partially ordered algebra* (*po-algebra*) over L if $L^+R^+ \subseteq R^+$. A po-ring (po-algebra) is called an ℓ-ring (ℓ-algebra) if the partial order is a lattice order. Similarly, *partially ordered groups* (*po-groups*) and *lattice-ordered groups* (*ℓ-groups*) and *partially ordered vector spaces* (*po-vector spaces*) and *vector lattices* over a totally ordered field can be defined. We skip these standard definitions, and refer the reader to [2] and [4].

An element $a > 0$ in a po-group G is called *basic* if the interval $[0, a] = \{b \in G : 0 \leq b \leq a\}$ is a *chain*. A subset S of an ℓ-group G is called *disjoint* if $s > 0$ for each $s \in S$ and $s \wedge t = 0$ for any $s, t \in S$, $s \neq t$. If G is an ℓ-group and $a \in G$, then the *absolute value* of a is $|a| = a \vee (-a)$, the *positive part* of a is $a^+ = a \vee 0$, and the *negative part* of a is $a^- = (-a) \vee 0$.

For the rest of the paper, we always assume that $\mathbf{F}$ is a subfield of the totally ordered field $\mathbf{R}$ of real numbers; $\mathbf{F}_n$ is the $n \times n$ matrix algebra over $\mathbf{F}$, and $\mathbf{F}^n$ is the vector space $\mathbf{F} \oplus \ldots \oplus \mathbf{F}$ (n times) over $\mathbf{F}$. We also consider $\mathbf{F}^n$ as a topological subspace of the Euclidean space $\mathbf{R}^n$. If $(\mathbf{F}_n, P)$ is a po-algebra over $\mathbf{F}$ with the positive cone P, then we simply say that P is a partial order of $\mathbf{F}_n$. The positive cone of a po-vector space $\mathbf{F}^n$ over $\mathbf{F}$ is simply called a *cone* in $\mathbf{F}^n$.

Let P be a partial order of $\mathbf{F}_n$. A nonempty subset S of $\mathbf{R}^n$ is said to be a *P-invariant set* if for every $f \in P$, $fS \subseteq S$. A cone O in $\mathbf{F}^n$ is said to be a *P-invariant cone* in $\mathbf{F}^n$ if O is also a P-invariant set. We remark that in these definitions, a P-invariant set is not necessarily a subset of $\mathbf{F}^n$, and although P-invariant cones can be defined more generally, we will only consider P-invariant cones in $\mathbf{F}^n$ in this paper. Obviously, $\{0\}$ is a P-invariant cone for every P. We will refer to this cone as the *trivial* P-invariant cone. Other cones will be called *nontrivial*.

Given a subset K of $\mathbf{R}^n$, let

$$\mathrm{cone}_{\mathbf{F}}(K) = \{\sum \alpha_i v_i : \alpha_i \in \mathbf{F}^+, v_i \in K\}.$$

It is clear that $\mathrm{cone}_{\mathbf{F}}(K) \subseteq \mathbf{F}^n$ exactly when $K \subseteq \mathbf{F}^n$. Also, it is easily verified that $\mathrm{cone}_{\mathbf{F}}(K) + \mathrm{cone}_{\mathbf{F}}(K) \subseteq \mathrm{cone}_{\mathbf{F}}(K)$ and $\mathbf{F}^+\mathrm{cone}_{\mathbf{F}}(K) \subseteq \mathrm{cone}_{\mathbf{F}}(K)$. But $\mathrm{cone}_{\mathbf{F}}(K) \cap -\mathrm{cone}_{\mathbf{F}}(K) \neq \{0\}$ in general. In case $\mathbf{F} = \mathbf{R}$, many interesting properties of cones and matrices that leave a cone invariant are discussed in Berman & Plemmons [1] and some papers referenced there. Note that if K is finite, then $\mathrm{cone}_{\mathbf{R}}(K)$ is closed in the Euclidean space $\mathbf{R}^n$ [1, Theorem 2.5] and if K consists of n linearly independent vectors in $\mathbf{R}^n$, then $\mathrm{cone}_{\mathbf{R}}(K)$ has a nonempty interior. It actually satisfies definition 2.10 of a *proper cone* in [1].

Theorem 1. *Every nontrivial partial order P of $\mathbf{F}_n$ has a nontrivial P-invariant cone in $\mathbf{F}^n$.*

Proof. Consider the collection $\boldsymbol{M}$ of all the null spaces in $\mathbf{F}^n$ of the nonzero matrices from P. Let N denote an element of $\boldsymbol{M}$ with maximum dimension, and let $u \notin N$. Define $O = \{fu : f \in P \text{ and } fN = 0\}$. The set O is closed under addition and $\mathbf{F}^+O \subseteq O$. Also, if some $fu, gu \in O$ and $fu + gu = 0$, then $(f + g)u = 0$ and $(f + g)N = 0$. Thus $(f + g)(\mathbf{F}u + N) = 0$. But since $f + g \in P$ and $N \subset \mathbf{F}u + N$, it follows from the maximality of the dimension of N that $f + g = 0$. Therefore, $f = g = 0$ and $fu = gu = 0$. This proves that O is a cone in $\mathbf{F}^n$. It is obvious that O is nontrivial and P-invariant. □

Lemma 2. *If P is a directed partial order of $\mathbf{F}_n$ and $S \neq \{0\}$ is a P-invariant set, then S contains n linearly independent vectors over $\mathbf{R}$.*

Proof. Let M be the subspace of $\mathbf{R}^n$ generated by S. Then for every $f \in P$, $fM \subseteq M$. But then, since P is directed, every matrix $g \in \mathbf{F}_n$ satisfies $g = f_1 - f_2$ with $f_1, f_2 \in P$, so $gM = (f_1 - f_2)M \subseteq M$, and hence M is a nontrivial invariant subspace for every matrix in $\mathbf{F}_n$. Thus $M = \mathbf{R}^n$, so S contains n linearly independent vectors over $\mathbf{R}$. □

Let S be a subset of $\mathbf{R}^n$. The *topological closure* of S in the Euclidean space $\mathbf{R}^n$ will be denoted by $\overline{S}$.

Lemma 3. *If P is a directed partial order of $\mathbf{F}_n$ and O is a P-invariant cone in $\mathbf{F}^n$, then $\overline{O} \cap \overline{-O} = \{0\}$.*

Proof. The set $\overline{O} \cap \overline{-O}$ is a P-invariant set. Moreover, since both $\overline{O}$ and $\overline{-O}$ are closed under addition and scalar multiplication by nonnegative real numbers, so is $\overline{O}\cap\overline{-O}$. If we suppose that $\overline{O}\cap\overline{-O} \neq \{0\}$, then by Lemma 2, $\overline{O}\cap\overline{-O}$ contains a set K of n linearly independent vectors of $\mathbf{R}^n$ over $\mathbf{R}$. Therefore, $\text{cone}_{\mathbf{R}}(K) \subseteq \overline{O}\cap\overline{-O}$. Let U be the interior of $\text{cone}_{\mathbf{R}}(K)$. Note that $0 \notin U$. Since $U \cap -O \neq \emptyset$, $\overline{O} \cap -O \neq \{0\}$, and hence by Lemma 2, it contains a set K_1 of n linearly independent vectors of $\mathbf{R}^n$ over $\mathbf{R}$. Let U_1 be the interior of $\text{cone}_{\mathbf{R}}(K_1)$. Then $U_1 \cap O \neq \emptyset$, since $U_1 \subseteq \text{cone}_{\mathbf{R}}(K_1) \subseteq \overline{O}$. Since $K_1 \subseteq -O \subseteq \mathbf{F}^n$, $U_1 \cap O \subseteq U_1 \cap \mathbf{F}^n \subseteq \text{cone}_{\mathbf{R}}(K_1) \cap \mathbf{F}^n$ $= \text{cone}_{\mathbf{F}}(K_1) \subseteq -O$, and hence $U_1 \cap O \subseteq O \cap -O$; so $O \cap -O \neq \{0\}$, which is a contradiction. □

Note that we have just proven that if P is a directed partial order of $\mathbf{F}_n$ and O is a P-invariant cone in $\mathbf{F}^n$, then $(\mathbf{R}^n, \overline{O})$ is a po-vector space over $\mathbf{R}$ with the positive cone $\overline{O}$, and $\overline{O}$ is P-invariant.

Lemma 4. *Let T be a subspace of $\mathbf{F}^m$ $(m > 1)$ over $\mathbf{F}$ which is totally ordered. If $\dim_{\mathbf{F}}(T) > 1$, then $\overline{T^+} \cap \overline{-T^+} \neq \{0\}$.*

Proof. First notice that $\overline{T}$ is a vector subspace of $\mathbf{R}^m$ over $\mathbf{R}$. Suppose that $\overline{T^+} \cap \overline{-T^+} = \{0\}$. Since $T = T^+ \cup -T^+$, we have that $\overline{T} = \overline{T^+} \cup \overline{-T^+}$. If we write $X = \overline{T} \setminus \{0\}$, $A = \overline{T^+} \setminus \{0\} = \overline{T} \setminus \overline{-T^+}$ and $B = \overline{-T^+} \setminus \{0\} = \overline{T} \setminus \overline{T^+}$, then we have that $X = A \cup B$ and $A \cap B = \emptyset$, so X is a union of two open, disjoint, nonempty subsets of $\overline{T}$, proving that X is disconnected as a subset of $\overline{T}$. Since $\overline{T}$ is a Euclidean space, this is true only if $\dim_{\mathbf{R}}(\overline{T}) = 1$. Finally, since $T \subseteq \mathbf{F}^m$, we have $\dim_{\mathbf{F}}(T) = 1$. □

Corollary 5. *Let $\mathbf{F}^m$ $(m > 1)$ be a po-vector space over $\mathbf{F}$ such that $(\mathbf{F}^m)^+ \subseteq (\mathbf{F}^+)^m$, and let $T \neq 0$ be a totally ordered subspace of $\mathbf{F}^m$. Then T is one-dimensional.*

Proof. Since $T^+ \subseteq (\mathbf{F}^+)^m$, $\overline{T^+} \subseteq \overline{(\mathbf{F}^+)^m} = (\mathbf{R}^+)^m$. Similarly, $\overline{-T^+} \subseteq (-\mathbf{R}^+)^m$. Since $(\mathbf{R}^+)^m \cap (-\mathbf{R}^+)^m = \{0\}$, $\overline{T^+} \cap \overline{-T^+} = \{0\}$, and hence T is one-dimensional by Lemma 4. □

A vector lattice over a totally ordered field L is called *Archimedean* over L if it has no nonzero bounded subspaces.

Corollary 6. *Let $(\mathbf{F}_n, P)$ be an ℓ-algebra over $\mathbf{F}$ with the positive cone P. If P is contained in the usual lattice order $(\mathbf{F}^+)_n$, then $(\mathbf{F}_n, P)$ contains a set of exactly n^2 disjoint basic elements.*

Proof. By [2, Corollary 1, p.51], $\mathbf{F}_n$ is Archimedean over $\mathbf{F}$. So $\mathbf{F}_n$ ($= \mathbf{F}^{n^2}$ as a vector lattice over $\mathbf{F}$) is a vector lattice direct sum of finitely many totally ordered subspaces of $\mathbf{F}_n$ (see [3, p. 3.27] or [6, Theorem 2.12]). Each of these subspaces is one-dimensional by Corollary 5. Therefore, there exist n^2 disjoint basic elements. □

Theorem 7. *Let $(\mathbf{F}_n, P)$ be an ℓ-algebra over $\mathbf{F}$. If for some vector $v \in \mathbf{F}^n$, $O = Pv$ is a P-invariant cone in $\mathbf{F}^n$, then there exists a finite subset $K \subseteq \mathbf{F}^n$ such that $O = cone_{\mathbf{F}}(K)$.*

Proof. We know that $\mathbf{F}_n$ is a vector lattice direct sum of finitely many totally ordered subspaces, say, $\mathbf{F}_n = \bigoplus_{i=1}^m T_i$. For every i, $T_i v$ is a totally ordered subspace of $(\mathbf{F}^n, O)$ with the positive cone $(T_i)^+ v$. Since $\overline{(T_i v)^+} \subseteq \overline{O}$ and $\overline{-(T_i v)^+} \subseteq \overline{-O}$, $\overline{(T_i v)^+} \cap \overline{-(T_i v)^+} \subseteq \overline{O} \cap \overline{-O} = \{0\}$ by Lemma 3. So by Lemma 4, $\dim_{\mathbf{F}}(T_i v) \leq 1$. This means that for every $i = 1, \ldots, m$, there exists $f_i \in T_i^+$ such that $T_i v = \mathbf{F} f_i v$. Therefore $O = Pv = (\sum_{i=1}^m T_i^+)v = \sum_{i=1}^m (T_i^+)v = \sum_{i=1}^m (T_i v)^+ = \sum_{i=1}^m \mathbf{F}^+ f_i v = \text{cone}_{\mathbf{F}}(K)$, where $K = \{f_i v : i = 1, \ldots, m\}$. □

Let O be a nontrivial cone in $\mathbf{F}^n$. If $O = \text{cone}_{\mathbf{F}}(K)$ for some finite subset K of $\mathbf{F}^n$, then O will be called *a polyhedral cone.* If there exists such a K which has precisely n elements, and they are linearly independent, then O will be called *a simplicial cone*. In the case $\mathbf{F} = \mathbf{R}$, these notions coincide with definitions 2.4 and 2.13 from [1]. If K is a minimal finite set with the property that $O = \text{cone}_{\mathbf{F}}(K)$ for a polyhedral cone O, then the vectors from K will be called the *edges* of O. Let P be a directed partial order of $\mathbf{F}_n$, and O a nontrivial P-invariant cone in $\mathbf{F}^n$. Then, by Lemma 2, O contains n linearly independent vectors; therefore, if $O = \text{cone}_{\mathbf{F}}(K)$ for some $K \subseteq \mathbf{F}^n$, then K contains at least n elements.

Theorem 8. *Let P be a lattice order of $\mathbf{F}_n$. Then every nontrivial P-invariant cone in $\mathbf{F}^n$ contains a minimal nontrivial P-invariant cone in $\mathbf{F}^n$. In particular, there exists a minimal nontrivial P-invariant cone.*

Proof. First, we consider a chain $\{Pv_\alpha\}$ of nontrivial P-invariant cones Pv_α and show that $\bigcap Pv_\alpha$ is a nontrivial P-invariant cone. Let S denote the unit sphere in $\mathbf{R}^n$. The family $\{\overline{Pv_\alpha} \cap S\}$ is a chain of closed nonempty sets in the space S, and therefore their intersection $\bigcap_\alpha (\overline{Pv_\alpha} \cap S) \neq \emptyset$ since S is compact. Therefore, $\bigcap_\alpha \overline{Pv_\alpha} \neq \{0\}$. Since this set is P-invariant, it contains a subset K of n linearly independent vectors from $\mathbf{R}^n$ over $\mathbf{R}$ by Lemma 2. By Theorem 7, for each α, $Pv_\alpha = \text{cone}_{\mathbf{F}}(K_\alpha)$ for some finite subset $K_\alpha \subseteq \mathbf{F}^n$. It is clear that $\overline{Pv_\alpha} = \overline{\text{cone}_{\mathbf{F}}(K_\alpha)} = \text{cone}_{\mathbf{R}}(K_\alpha)$ and $\overline{Pv_\alpha} \cap \mathbf{F}^n = \text{cone}_{\mathbf{R}}(K_\alpha) \cap \mathbf{F}^n = \text{cone}_{\mathbf{F}}(K_\alpha)$, where $\text{cone}_{\mathbf{R}}(K_\alpha) \cap \mathbf{F}^n = \text{cone}_{\mathbf{F}}(K_\alpha)$ since a polyhedral cone containing n linearly independent vectors is a union of simplicial cones and the equality is clearly true for a simplicial cone. Thus $\text{cone}_{\mathbf{R}}(K) \cap \mathbf{F}^n \subseteq \overline{Pv_\alpha} \cap \mathbf{F}^n = \text{cone}_{\mathbf{F}}(K_\alpha) = Pv_\alpha$, for every Pv_α, and hence $\text{cone}_{\mathbf{R}}(K) \cap \mathbf{F}^n \subseteq \cap Pv_\alpha$, so $\cap Pv_\alpha \neq \{0\}$.

Now let O be a nontrivial P-invariant cone in $\mathbf{F}^n$. Let $0 \neq v \in O$. Then Pv is contained in O, so Pv is a nontrivial P-invariant cone. By the above argument and Zorn's Lemma, Pv contains a minimal nontrivial P-invariant cone, so O contains a minimal nontrivial P-invariant cone. Finally, because of Theorem 1, there exists a minimal nontrivial P-invariant cone in $\mathbf{F}^n$. □

Lemma 9. *Let O be a polyhedral cone in $\mathbf{F}^n$, and let k be an edge of O. If $0 < u < k$ for some $u \in \mathbf{F}^n$, then $u = \alpha k$ for some $\alpha \in \mathbf{F}^+$. In particular, the edges of O are basic elements of the partial order O in $\mathbf{F}^n$.*

Proof. Let $K = \{k, k_1, \dots, k_m\}$ be minimal with the property $O = \mathrm{cone}_{\mathbf{F}}(K)$. Then, $u = \alpha k + \sum_{j=1}^m \alpha_j k_j$ and $k - u = \beta k + \sum_{j=1}^m \beta_j k_j$, where $\alpha, \beta, \alpha_j, \beta_j \in \mathbf{F}^+$ for $j = 1, \dots, m$. Therefore, $k = (\alpha + \beta)k + \sum_{j=1}^m (\alpha_j + \beta_j)k_j$. Since K is minimal, we must have $\alpha + \beta = 1$, and $\alpha_j = \beta_j = 0$ for $j = 1, \dots, m$. So $u = \alpha k$. □

Theorem 10. *Let $(\mathbf{F}_n, P)$ be an ℓ-algebra over $\mathbf{F}$ and let O be a minimal nontrivial P-invariant cone in $\mathbf{F}^n$. Then*:

(1) *O is a lattice order in $\mathbf{F}^n$.*
(2) *O is a simplicial cone in $\mathbf{F}^n$, so O has exactly n disjoint edges.*

Proof. (1) By Lemma 2, O contains n linearly independent vectors v_i, $i = 1, \dots, n$, over $\mathbf{F}$. Let $v = \sum_i v_i$. Then $Pv = O$ from the minimality of O, and hence O is a polyhedral cone by Theorem 7. Let k be an edge of O. There exists a matrix $f_k \in P$ such that $k = f_k v = \sum f_k v_i$. Since k is an edge of O, by Lemma 9, we have that for all $i = 1, \dots, n$, $f_k v_i = \alpha_i k$ for some scalars $\alpha_i \geq 0$. Consequently, $\mathrm{rank}(f_k) = 1$. Thus the null space N of f_k has dimension $n - 1$. By the proof of Theorem 1, $O' = \{fv : f \in P \text{ and } fN = 0\}$ is a nontrivial P-invariant cone. Obviously, $O' \subseteq Pv$ and we conclude that $O = O' = Pv$.

Since O contains n linearly independent vectors, it is a directed partial order in $\mathbf{F}^n$. Let $u \in \mathbf{F}^n$. There exist $u_1, u_2 \in O$ such that $u = u_1 - u_2$. Let $u_1 = f_1 v$ and $u_2 = f_2 v$ for some $f_1, f_2 \in P$ and $f_1 N = f_2 N = 0$. Then $u = (f_1 - f_2)v$. Let $f = f_1 - f_2$. Then $f^+ v \in O$ since $f^+ \in P$. Similarly $(f^+ - f)v = f^- v \in O$. Thus $f^+ v$ is an upper bound for both 0 and u. Below we show that $f^+ v$ is the least upper bound of 0 and u.

Let $0, u \leq w$. Then $w \in O$, so $w = gv$ for some $g \in P$ with $gN = 0$. Similarly $w - u \in O$ implies $w - u = hv$ for some $h \in P$ with $hN = 0$. Thus $w - u = gv - fv = hv$, so $(g - f)v = hv$ and also $(g - f)N = hN = 0$. Therefore, $g - f = h$ since $v \notin N$ and N has dimension $n - 1$. Since $h \in P$, we have $g = h + f \geq f$, and since $g \in P$, we obtain $g \geq f^+$. Thus $g - f^+ \in P$, and hence $(g - f^+)v \in O$; so $gv - f^+ v \geq 0$; i.e., $w \geq f^+ v$.

(2) By (1) O is a lattice order of $\mathbf{F}^n$. By Lemma 9, the edges of O are basic elements in this lattice order, and therefore they are linearly independent over $\mathbf{F}$. Therefore there are at most n disjoint edges, and thus O has exactly n disjoint edges. □

Theorem 11. *Every ℓ-algebra $(\mathbf{F}_n, P_1)$ is isomorphic to an ℓ-algebra $(\mathbf{F}_n, P_2)$ with $P_2 \subseteq (\mathbf{F}^+)_n$, and $(\mathbf{F}^+)^n$ is a minimal P_2-invariant cone.*

Proof. Let O_1 be a minimal nontrivial P_1-invariant cone in $\mathbf{F}^n$. By Theorem 10, O_1 is simplicial. Let $k_1, \dots, k_n$ be the edges of O_1. Let ψ be the $n \times n$ matrix with the columns $k_1, \dots, k_n$. Since the edges form a linearly independent set, ψ is nonsingular and yields both an algebra isomorphism $\mathbf{F}_n \to \psi^{-1}\mathbf{F}_n\psi$ and a vector space isomorphism $\mathbf{F}^n \to \psi^{-1}\mathbf{F}^n$. Let $P_2 = \psi^{-1}P_1\psi$. Then $(\mathbf{F}_n, P_2)$ becomes an ℓ-algebra which is isomorphic to $(\mathbf{F}_n, P_1)$. If we let $O_2 = \psi^{-1}O_1$, then $O_2 = \psi^{-1}(\mathbf{F}^+ k_1 + \ldots + \mathbf{F}^+ k_n) = \psi^{-1}\psi(\mathbf{F}^+)^n = (\mathbf{F}^+)^n$, and it is clear that O_2 is a minimal P_2-invariant cone. Finally, $P_2 \subseteq (\mathbf{F}^+)_n$ since $(\mathbf{F}^+)^n$ is P_2-invariant. □

Given $f \in \mathbf{F}_n$, the *transpose* of f is denoted by f^T. It is well-known that if P is a partial order (lattice order) of F_n, then so is $P^T = \{f^T : f \in P\}$.

Theorem 12. *Let* $(\mathbf{F}_n, P)$ *be an* ℓ*-algebra over* $\mathbf{F}$ *such that* $1 \in P$*. Then* $(\mathbf{F}_n, P)$ *is isomorphic to the* ℓ*-algebra* $(\mathbf{F}_n, (\mathbf{F}^+)_n)$*.*

Proof. By Theorem 11 we may assume that $P \subseteq (\mathbf{F}^+)_n$ and $O = (\mathbf{F}^+)^n$ is a minimal nontrivial P-invariant cone. We show that $P = (\mathbf{F}^+)_n$. Let $v \in O$ be any vector with all components strictly positive. Then $Pv \neq \{0\}$, and since O is minimal, $Pv = O$. Hence there are n matrices $f_i \in P$ such that $f_i v = e_i$, $i = 1, \dots, n$, where e_i denotes the vector in which the i^{th} component is 1 and the other components are zero. Since every f_i, $i = 1, \dots, n$, has all its entries nonnegative, all rows of f_i except the i^{th} one are zero rows. It follows that $f_i^T e_j = 0$ for $i \neq j$, and $f_i^T v \neq 0$, $i, j = 1, \dots, n$. Let N_i denote the subspace of $\mathbf{F}^n$ spanned by $\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n\}$. Then for every $i = 1, \dots, n$, $f_i^T N_i = 0$. Now $(\mathbf{F}_n, P^T)$ is an ℓ-algebra over $\mathbf{F}$. It follows from the proof of Theorem 1 that the sets $O_i = \{f^T v : f^T N_i = 0, f \in P\}$, $i = 1, \dots, n$, are nontrivial P^T-invariant cones. By Lemma 2 each O_i contains n linearly independent vectors. For each $1 \leq i \leq n$, suppose $f_{ij}^T v$, $j = 1, \dots, n$, are linearly independent vectors in O_i, where $f_{ij} \in P$ and $f_{ij}^T N_i = 0$. Then it is clear that for each i, f_{ij}, $j = 1, \dots, n$, are linearly independent over $\mathbf{F}$, and all rows of f_{ij} except the i^{th} one are zero rows. Then it follows that the whole set $\{f_{ij} : 1 \leq i, j \leq n\}$ is linearly independent over $\mathbf{F}$. Since $P \subseteq (\mathbf{F}^+)_n$, by Corollary 6, $(\mathbf{F}_n, P)$ contains a set G of n^2 disjoint basic elements. Then $P = \sum_{g \in G} \mathbf{F}^+ g$. Since $\{f_{ij} : 1 \leq i, j \leq n\} \subseteq P$, each f_{ij} is a nonnegative linear combination of elements in G. Since $\{f_{ij} : 1 \leq i, j \leq n\}$ is linearly independent, each $g \in G$ must appear in at least one such linear combination with a nonzero coefficient from $\mathbf{F}^+$. Keeping in mind that all entries of the matrices $g \in G$ are nonnegative numbers, we conclude that each $g \in G$ has all but one row zero.

Now let $G = \{g_{ij} : 1 \leq i, j \leq n\}$. Then for every pair of indices $i, j = 1, \dots, n$, there exists a matrix from G, say, g_{ij}, whose ij^{th} entry is not 0. Moreover, since $1 \in P$, for each $i = 1, \dots, n$ there is a matrix from G which we will, without loss of generality, call g_{ii}, such that it has only a nonzero ii^{th} entry. Let e_{ij}, $i, j = 1, \dots, n$, denote the usual matrix units in $\mathbf{F}_n$. Then $g_{ii} = \lambda_{ii} e_{ii}$ for some $0 < \lambda_{ii} \in \mathbf{F}$, for every $i = 1, \dots, n$. Therefore, for every $i = 1, \dots, n$, the matrices $e_{ii} \in P$. Now for every $i, j = 1, \dots, n$, $g_{ij} e_{jj} = \lambda_{ij} e_{ij}$ for some $0 < \lambda_{ij} \in \mathbf{F}$. Thus $\{e_{ij} : i, j = 1, \dots, n\} \subseteq P$, and hence $P = (\mathbf{F}^+)_n$. □

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