

**ERGODICITY OF THE ACTION  
OF THE POSITIVE RATIONALS  
ON THE GROUP OF FINITE ADELES AND  
THE BOST-CONNES PHASE TRANSITION THEOREM**

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**ABSTRACT.** We study relatively invariant measures with the multipliers  $\mathbb{Q}_+^* \ni q \mapsto q^{-\beta}$  on the space  $\mathcal{A}_f$  of finite adeles. We prove that for  $\beta \in (0, 1]$  such measures are ergodic, and then deduce from this the uniqueness of  $\text{KMS}_\beta$ -states for the Bost-Connes system. Combining this with a result of Blackadar and Boca-Zaharescu, we also obtain ergodicity of the action of  $\mathbb{Q}^*$  on the full adeles.

Bost and Connes [BC] constructed a remarkable  $C^*$ -dynamical system which has a phase transition with spontaneous symmetry breaking involving an action of the Galois group  $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ , and whose partition function is the Riemann  $\zeta$  function. In their original definition the underlying algebra arises as the Hecke algebra associated with an inclusion of certain  $ax + b$  groups. Recently Laca and Raeburn [LR, L2] have realized the Bost-Connes algebra as a full corner of the crossed product algebra  $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$ . This new look at the system has allowed to simplify significantly the proof of the existence of KMS-states for all temperatures, and the classification of  $\text{KMS}_\beta$ -states for  $\beta > 1$  [L1]. On the other hand, for  $\beta \leq 1$  the uniqueness of  $\text{KMS}_\beta$ -states implies ergodicity of the action of  $\mathbb{Q}_+^*$  on  $\mathcal{A}_f$  for certain measures (in particular, for the Haar measure). The aim of this note is to give a direct proof of the ergodicity, and then to show that the uniqueness of  $\text{KMS}_\beta$ -states easily follows from it.

So let  $\mathcal{P}$  be the set of prime numbers,  $\mathcal{A}_f$  the restricted product of the fields  $\mathbb{Q}_p$  with respect to  $\mathbb{Z}_p$ ,  $p \in \mathcal{P}$ ,  $\mathcal{R} = \prod_p \mathbb{Z}_p$  its maximal compact subring,  $W = \mathcal{R}^* = \prod_p \mathbb{Z}_p^*$ . The group  $\mathbb{Q}_+^*$  of positive rationals is embedded diagonally into  $\mathcal{A}_f$ , and so acts by multiplication on the additive group of finite adeles. Then the Bost-Connes algebra  $\mathcal{C}_\mathbb{Q}$  is the full corner of  $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$  determined by the characteristic function of  $\mathcal{R}$  [L2]. The dynamics  $\sigma_t$  is defined as follows [L1]: it is trivial on  $C_0(\mathcal{A}_f)$ , and  $\sigma_t(u_q) = q^{it}u_q$ , where  $u_q$  is the multiplier of  $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$  corresponding to  $q \in \mathbb{Q}_+^*$ . Then ([L1]) there is a one-to-one correspondence between

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$(\beta, \sigma_t)$ -KMS-states on  $\mathcal{C}_{\mathbb{Q}}$  and measures  $\mu$  on  $\mathcal{A}_f$  such that

$$(1\beta) \quad \mu(\mathcal{R}) = 1 \quad \text{and} \quad q_*\mu = q^\beta \mu \quad \text{for all } q \in \mathbb{Q}_+^* \quad (\text{i.e., } \mu(q^{-1}X) = q^\beta \mu(X)).$$

Namely, the KMS-state corresponding to  $\mu$  is the restriction of the dual weight on  $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$  to  $\mathcal{C}_{\mathbb{Q}}$ .

Note ([L1]) that if  $\beta > 1$  and  $\mu$  is a measure with property (1 $\beta$ ), then

$$\mu(W) = \prod_{p \in \mathcal{P}} (1 - p^{-\beta}) = \frac{1}{\zeta(\beta)} > 0,$$

since  $W = \mathcal{R} \setminus \bigcup_p p\mathcal{R}$ . Moreover, the sets  $qW$ ,  $q \in \mathbb{Q}_+^*$ , are disjoint, and their union is a set of full measure (since  $\sum_{n \in \mathbb{N}} \mu(nW) = \frac{1}{\zeta(\beta)} \sum_{n \in \mathbb{N}} n^{-\beta} = 1$ ). Thus there exists a one-to-one correspondence between probability measures on  $W$  and measures on  $\mathcal{A}_f$  satisfying (1 $\beta$ ) [L1]. On the other hand, if  $\beta \leq 1$ , then  $\mu(W) = 0$ .

**Proposition.** *For  $\beta \in (0, 1]$  and any measure  $\mu$  satisfying (1 $\beta$ ), the action of  $\mathbb{Q}_+^*$  on  $(\mathcal{A}_f, \mu)$  is ergodic.*

*Proof.* Consider the space  $L^2(\mathcal{R}, d\mu)$  and the subspace  $H$  of it consisting of the functions that are constant on  $\mathbb{N}$ -orbits. In other words,

$$H = \{f \in L^2(\mathcal{R}, d\mu) \mid V_n f = f, \ n \in \mathbb{N}\},$$

where  $(V_n f)(x) = f(nx)$ . Since any  $\mathbb{Q}_+^*$ -invariant subset of  $\mathcal{A}_f$  is completely determined by its intersection with  $\mathcal{R}$ , it suffices to prove that  $H$  consists of constant functions. For this we will compute the action of the projection  $P: L^2(\mathcal{R}, d\mu) \rightarrow H$  on functions spanning a dense subspace of  $L^2(\mathcal{R}, d\mu)$ .

Let  $B$  be a finite subset of  $\mathcal{P}$ . Consider the projection  $\pi_B: \mathcal{R} \rightarrow \prod_{p \in B} \mathbb{Z}_p$ , and set  $\mu_B = (\pi_B)_*\mu$ . Then  $L^2(\prod_{p \in B} \mathbb{Z}_p, d\mu_B)$  can be considered as a subspace of  $L^2(\mathcal{R}, d\mu)$ , and the union of these subspaces over all finite  $B$  is dense in  $L^2(\mathcal{R}, d\mu)$ . The characters of  $\prod_{p \in B} \mathbb{Z}_p^*$  span a dense subspace of  $L^2(\prod_{p \in B} \mathbb{Z}_p^*, d\mu_B)$ . Let  $\mathbb{N}_B$  be the unital multiplicative subsemigroup of  $\mathbb{N}$  generated by  $p \in B$ . Note that the sets  $n \prod_{p \in B} \mathbb{Z}_p^*$ ,  $n \in \mathbb{N}_B$ , are disjoint, their union is a subset of  $\prod_{p \in B} \mathbb{Z}_p$  of full measure (condition (1 $\beta$ ) implies that the set  $\{x \in \mathcal{R} \mid x_p = 0\}$  has zero measure), and the operator  $n^{-\beta/2} V_n^*$  maps isometrically  $L^2(\prod_{p \in B} \mathbb{Z}_p^*, d\mu_B)$  onto  $L^2(n \prod_{p \in B} \mathbb{Z}_p^*, d\mu_B)$  for any  $n \in \mathbb{N}_B$ . Hence the functions  $V_n^* \chi$ ,  $n \in \mathbb{N}_B$ ,  $\chi \in (\prod_{p \in B} \mathbb{Z}_p^*)$ , span a dense subspace of  $L^2(\prod_{p \in B} \mathbb{Z}_p, d\mu_B)$ . So we have to compute  $PV_n^* \chi$ . But if  $g \in H$ , then  $(V_n^* \chi, g) = (\chi, g)$ , whence  $PV_n^* \chi = P\chi$ . Thus we have only to compute  $P\chi$ .

For a finite subset  $A$  of  $\mathcal{P}$ , let  $H_A$  be the subspace consisting of the functions that are constant on  $\mathbb{N}_A$ -orbits,  $P_A$  the projection onto  $H_A$ . Then  $P_A \searrow P$  as  $A \nearrow \mathcal{P}$ . Set

$$W_A = \prod_{p \in A} \mathbb{Z}_p^* \times \prod_{q \in \mathcal{P} \setminus A} \mathbb{Z}_q \subset \mathcal{R}.$$

Note, as above, that  $\bigcup_{n \in \mathbb{N}_A} nW_A$  is a subset of  $\mathcal{R}$  of full measure. We assert that

$$(2) \quad P_A f|_{\mathbb{N}_A x} \equiv \frac{1}{\zeta_A(\beta)} \sum_{n \in \mathbb{N}_A} n^{-\beta} f(nx) \quad \text{for } x \in W_A,$$

where  $\zeta_A(\beta) = \sum_{n \in \mathbb{N}_A} n^{-\beta} = \prod_{p \in A} (1 - p^{-\beta})^{-1}$ . Indeed, denoting the right hand part of (2) by  $f_A$ , for  $g \in H_A$  we obtain

$$\begin{aligned} (f_A, g) &= \sum_{n \in \mathbb{N}_A} \int_{nW_A} f_A(x) \overline{g(x)} d\mu(x) = \sum_{n \in \mathbb{N}_A} n^{-\beta} \int_{W_A} f_A(x) \overline{g(x)} d\mu(x) \\ &= \zeta_A(\beta) \int_{W_A} f_A(x) \overline{g(x)} d\mu(x) = \sum_{n \in \mathbb{N}_A} n^{-\beta} \int_{W_A} f(nx) \overline{g(x)} d\mu(x) \\ &= \sum_{n \in \mathbb{N}_A} \int_{nW_A} f(x) \overline{g(x)} d\mu(x) = (f, g). \end{aligned}$$

Returning to the computation of  $P\chi$ , we see that

$$(3) \quad P_A \chi|_{\mathbb{N}_A x} \equiv \frac{\chi(x)}{\zeta_A(\beta)} \sum_{n \in \mathbb{N}_A} n^{-\beta} \chi(n) = \chi(x) \prod_{p \in A} \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \quad \text{for } x \in W_A.$$

Hence

$$(4) \quad \|P\chi\|_2 = \lim_{A \nearrow \mathcal{P}} \prod_{p \in A} \left| \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \right|.$$

If  $\chi$  is trivial, then using (3) we see that  $P_A \chi \equiv \prod_{p \in B} (1 - p^{-\beta})$  for all  $A \supset B$ , hence  $P\chi$  is a constant. Suppose  $\chi$  is non-trivial. The limit in (4) is an increasing function in  $\beta$  on  $(0, +\infty)$  (because each factor is increasing), which is equal to  $|L(\beta, \chi)|\zeta(\beta)^{-1}$  for  $\beta > 1$ , where  $L(\beta, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-\beta}$  is the Dirichlet  $L$ -function corresponding to the number character  $\chi$  [S]. By elementary properties of Dirichlet series,  $|L(\beta, \chi)|$  tends to a finite value as  $\beta \rightarrow 1 + 0$ , while  $\zeta(\beta) \rightarrow \infty$ . Thus  $P\chi = 0$ .  $\square$

Since the set of measures satisfying (1 $\beta$ ) is convex and consists of ergodic measures, there exists at most one measure satisfying (1 $\beta$ ). Such a measure does exist. In fact, for each  $\beta \in (0, +\infty)$  there is a unique  $W$ -invariant measure  $\mu_\beta$  satisfying (1 $\beta$ ) [BC, L1]. Explicitly,  $\mu_\beta = \prod_p \mu_{\beta,p}$ , where  $\mu_{\beta,p}$  is the measure on  $\mathbb{Q}_p$  such that  $\mu_{1,p}$  is the Haar measure ( $\mu_{1,p}(\mathbb{Z}_p) = 1$ ), and

$$\frac{d\mu_{\beta,p}}{d\mu_{1,p}}(a) = \frac{1 - p^{-\beta}}{1 - p^{-1}} |a|_p^{\beta-1} \quad \text{for } a \in \mathbb{Q}_p.$$

Let  $\phi_\beta$  be the  $(\beta, \sigma_t)$ -KMS state on  $\mathcal{C}_\mathbb{Q}$  corresponding to  $\mu_\beta$ . Then  $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$  is a factor, which is a reduction of the factor  $L^\infty(\mathcal{A}_f, \mu_\beta) \rtimes \mathbb{Q}_+^*$ . It is easy to describe its flow of weights [CT]. Consider a standard measure space  $X_\beta$  with the measure algebra consisting of  $\mathbb{Q}_+^*$ -invariant  $(\lambda \times \mu_\beta)$ -measurable subsets of  $\mathbb{R}_+ \times \mathcal{A}_f$ , where  $\lambda$  is the Lebesgue measure, and the flow  $F_t^\beta$  on it coming from the action  $t(x, a) = (e^{-t/\beta}x, a)$  of  $\mathbb{R}$  on  $\mathbb{R}_+ \times \mathcal{A}_f$ . Then  $F_t^\beta$  is ergodic by Proposition, and it is the flow of weights of the factors  $L^\infty(\mathcal{A}_f, \mu_\beta) \rtimes \mathbb{Q}_+^*$  and  $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$ .

**Theorem** ([BC]). *For  $\beta \in (0, 1]$ ,  $\mu_\beta$  is a unique measure satisfying (1 $\beta$ ). The action of  $\mathbb{Q}_+^*$  on  $(\mathcal{A}_f, \mu_\beta)$  is ergodic, moreover, the action of  $\mathbb{Q}^*$  on  $(\mathcal{A}, \nu_\beta)$ , where  $\mathcal{A} = \mathbb{R} \times \mathcal{A}_f$  is the space of full adeles and  $\nu_\beta = \lambda \times \mu_\beta$ , is ergodic. Equivalently,  $\phi_\beta$  is a unique  $(\beta, \sigma_t)$ -KMS state on  $\mathcal{C}_\mathbb{Q}$ , and  $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$  is the hyperfinite factor of type III $_1$ .*

*Proof.* In view of the above description of the flow of weights, the factor  $\pi_{\phi_\beta}(C_{\mathbb{Q}})''$  is of type III<sub>1</sub> if and only if the action of  $\mathbb{Q}_+^*$  on  $(\mathbb{R}_+ \times \mathcal{A}_f, \lambda \times \mu_\beta)$  is ergodic, or equivalently, the action of  $\mathbb{Q}^*$  on  $(\mathcal{A}, \nu_\beta)$  is ergodic.

To prove the ergodicity, first note that the action of  $W$  on  $X_\beta$  is ergodic. Indeed, the induced flow on  $X_\beta/W$  is the flow of weights of the factor

$$L^\infty(\mathcal{A}_f/W, \mu_\beta) \rtimes \mathbb{Q}_+^* = (L^\infty(\mathcal{A}_f, \mu_\beta) \rtimes \mathbb{Q}_+^*)^W.$$

It is easy to see ([BC]) that this factor is ITPFI with eigenvalue list

$$\{p^{-n\beta}(1 - p^{-\beta}) \mid n \geq 0\}_{p \in \mathcal{P}}.$$

Hence it is of type III<sub>1</sub> by [B] (see also [BZ]). Thus its flow of weights is trivial, i.e. the action of  $W$  on  $X_\beta$  is ergodic. Since  $W$  is compact, the action is transitive. So we may identify  $X_\beta$  with  $W/W_\beta$  for some closed subgroup  $W_\beta$  of  $W$ . Then the flow  $F_t^\beta$  is given by a continuous one-parametric subgroup of  $W/W_\beta$ . Since  $W/W_\beta$  is totally disconnected, this one-parametric subgroup is trivial, and since the flow is ergodic,  $W_\beta = W$ . Thus  $X_\beta$  is singlepoint, and the action of  $\mathbb{Q}_+^*$  on  $(\mathbb{R}_+ \times \mathcal{A}_f, \lambda \times \mu_\beta)$  is ergodic.  $\square$

*Remarks.* (i) In order to prove that  $\phi_\beta$  is a unique KMS <sub>$\beta$</sub> -state, it is enough to know that  $\mu_\beta$  is ergodic. Indeed, this means that  $\phi_\beta$  is an extremal KMS <sub>$\beta$</sub> -state. Since  $\phi_\beta$  is a unique  $W$ -invariant KMS <sub>$\beta$</sub> -state, we can argue as in the proof of [BC, Theorem 25]: if  $\psi$  is an extremal KMS <sub>$\beta$</sub> -state, then

$$\int_W w_* \psi dw = \phi_\beta.$$

Since KMS <sub>$\beta$</sub> -states form a simplex, we conclude that  $\psi = \phi_\beta$ .

Thus, the uniqueness and the assertion about the type for KMS <sub>$\beta$</sub> -states follow easily from ergodicity of the action of  $\mathbb{Q}^*$  on  $(\mathcal{A}, \nu_\beta)$ .

(ii) A slight modification of the argument in the proof of the Theorem gives the following general result, apparently well-known to specialists: if  $M$  is a factor and  $G$  a compact totally disconnected group acting on  $M$  such that  $M^G$  is a factor of type III<sub>1</sub>, then  $M$  is also of type III<sub>1</sub>.

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