ERGODICITY OF THE ACTION
OF THE POSITIVE RATIONALS
ON THE GROUP OF FINITE ADELES AND
THE BOST-CONNES PHASE TRANSITION THEOREM

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Abstract. We study relatively invariant measures with the multiplicators $\mathbb{Q}^*_+ \ni q \mapsto q^{-\beta}$ on the space $A_f$ of finite adeles. We prove that for $\beta \in (0,1]$ such measures are ergodic, and then deduce from this the uniqueness of KMS$_\beta$-states for the Bost-Connes system. Combining this with a result of Blackadar and Boca-Zaharescu, we also obtain ergodicity of the action of $\mathbb{Q}^*$ on the full adeles.

Bost and Connes [BC] constructed a remarkable C$^*$-dynamical system which has a phase transition with spontaneous symmetry breaking involving an action of the Galois group $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, and whose partition function is the Riemann $\zeta$ function. In their original definition the underlying algebra arises as the Hecke algebra associated with an inclusion of certain $ax+b$ groups. Recently Laca and Raeburn [LR, L2] have realized the Bost-Connes algebra as a full corner of the crossed product algebra $C_0(A_f) \rtimes \mathbb{Q}^*_+$. This new look at the system has allowed to simplify significantly the proof of the existence of KMS-states for all temperatures, and the classification of KMS$_\beta$-states for $\beta > 1$ [L1]. On the other hand, for $\beta \leq 1$ the uniqueness of KMS$_\beta$-states implies ergodicity of the action of $\mathbb{Q}^*_+$ on $A_f$ for certain measures (in particular, for the Haar measure). The aim of this note is to give a direct proof of the ergodicity, and then to show that the uniqueness of KMS$_\beta$-states easily follows from it.

So let $\mathcal{P}$ be the set of prime numbers, $A_f$ the restricted product of the fields $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, $\mathbb{P} = \prod_p \mathbb{Z}_p$ its maximal compact subring, $W = \mathbb{R}^* = \prod_p \mathbb{Z}_p^*$. The group $\mathbb{Q}^*_+$ of positive rationals is embedded diagonally into $A_f$, and so acts by multiplication on the additive group of finite adeles. Then the Bost-Connes algebra $C_\mathbb{Q}$ is the full corner of $C_0(A_f) \rtimes \mathbb{Q}^*_+$ determined by the characteristic function of $\mathbb{R}$ [L2]. The dynamics $\sigma_t$ is defined as follows [L1]: it is trivial on $C_0(A_f)$, and $\sigma_t(u_q) = q^{it}u_q$, where $u_q$ is the multiplier of $C_0(A_f) \rtimes \mathbb{Q}^*_+$ corresponding to $q \in \mathbb{Q}^*_+$. Then (L1) there is a one-to-one correspondence between...
(β, σ,)-KMS-states on CQ and measures μ on A_f such that

\[ \mu(R) = 1 \] and \( q_\ast \mu = q^\beta \mu \) for all \( q \in \mathbb{Q}_+^* \) (i.e., \( \mu(q^{-1}X) = q^\beta \mu(X) \)).

Namely, the KMS-state corresponding to μ is the restriction of the dual weight on \( C_0(A_f) \approx \mathbb{Q}_+^* \) to CQ.

Note that if \( \beta > 1 \) and μ is a measure with property (1β), then

\[ \mu(W) = \prod_{p \in P} (1 - p^{-\beta}) = \frac{1}{\zeta(\beta)} > 0, \]

since \( W = R \setminus \bigcup_p pR \). Moreover, the sets \( qW, q \in \mathbb{Q}_+^* \), are disjoint, and their union is a set of full measure (since \( \sum_{n \in N} \mu(nW) = \frac{1}{\zeta(1)} \sum_{n \in N} n^{-\beta} = 1 \)). Thus there exists a one-to-one correspondence between probability measures on W and measures on A_f satisfying (1β). On the other hand, if \( \beta \leq 1 \), then \( \mu(W) = 0 \).

**Proposition.** For β ∈ (0, 1] and any measure μ satisfying (1β), the action of \( \mathbb{Q}_+^* \) on \((A_f, \mu)\) is ergodic.

**Proof.** Consider the space \( L^2(R, d\mu) \) and the subspace \( H \) of it consisting of the functions that are constant on N-orbits. In other words,

\[ H = \{ f \in L^2(R, d\mu) | V_n f = f, \ n \in N \}, \]

where \( (V_n f)(x) = f(nx) \). Since any \( \mathbb{Q}_+^* \)-invariant subset of A_f is completely determined by its intersection with R, it suffices to prove that H consists of constant functions. For this we will compute the action of the projection \( P : L^2(R, d\mu) \to H \) on functions spanning a dense subspace of \( L^2(R, d\mu) \).

Let B be a finite subset of \( P \). Consider the projection \( \pi_B : R \to \prod_{p \in B} \mathbb{Z}_p \), and set \( \mu_B = (\pi_B)_\ast \mu \). Then \( L^2(\prod_{p \in B} \mathbb{Z}_p, d\mu_B) \) can be considered as a subspace of \( L^2(R, d\mu) \), and the union of these subspaces over all finite B is dense in \( L^2(R, d\mu) \). The characters of \( \prod_{p \in B} \mathbb{Z}_p \) span a dense subspace of \( L^2(\prod_{p \in B} \mathbb{Z}_p, d\mu_B) \). Let \( N_B \) be the unitary multiplicative subsemigroup of N generated by \( p \in B \). Note that the sets \( n \prod_{p \in B} \mathbb{Z}_p, n \in N_B, \) are disjoint, their union is a subset of \( \prod_{p \in B} \mathbb{Z}_p \) of full measure (condition (1β) implies that the set \( \{ x \in R | x_p = 0 \} \) has zero measure), and the operator \( n^{-\beta/2} V_n^\ast \) maps isometrically \( L^2(\prod_{p \in B} \mathbb{Z}_p, d\mu_B) \) onto \( L^2(n \prod_{p \in B} \mathbb{Z}_p, d\mu_B) \) for any \( n \in N_B \). Hence the functions \( V_n^\ast \chi, \ n \in N_B, \chi \in (\prod_{p \in B} \mathbb{Z}_p) \), span a dense subspace of \( L^2(\prod_{p \in B} \mathbb{Z}_p, d\mu_B) \). So we have to compute \( PV_n^\ast \chi \). But if \( g \in H \), then \( (V_n^\ast \chi)(g) = (\chi, g) \), whence \( PV_n^\ast \chi = P \chi \). Thus we have only to compute \( P \chi \).

For a finite subset A of \( P \), let \( H_A \) be the subspace consisting of the functions that are constant on \( N_A \)-orbits, \( P_A \) the projection onto \( H_A \). Then \( P_A \setminus P \) as \( A \setminus P \). Set

\[ W_A = \prod_{p \in A} \mathbb{Z}_p \times \prod_{q \in P \setminus A} \mathbb{Z}_q \subset R. \]

Note, as above, that \( \bigcup_{n \in N_A} nW_A \) is a subset of R of full measure. We assert that

\[ P_A f|_{N_A x} = \frac{1}{\zeta_A(\beta)} \sum_{n \in N_A} n^{-\beta} f(nx) \text{ for } x \in W_A, \]
Hence
\[ (f_A, g) = \sum_{n \in \mathbb{N}_A} \int_{\mathbb{N}_A} f_A(x) \overline{g(x)} \, d\mu(x) = \sum_{n \in \mathbb{N}_A} n^{-\beta} \int_{\mathbb{N}_A} f_A(x) \overline{g(x)} \, d\mu(x) \]
\[ = \zeta_A(\beta) \int_{\mathbb{N}_A} f_A(x) \overline{g(x)} \, d\mu(x) = \sum_{n \in \mathbb{N}_A} n^{-\beta} \int_{\mathbb{N}_A} f(nx) \overline{g(x)} \, d\mu(x) \]
\[ = \sum_{n \in \mathbb{N}_A} \int_{\mathbb{N}_A} f(x) \overline{g(x)} \, d\mu(x) = (f, g). \]

Returning to the computation of \( P_\chi \), we see that
\[ P_{A \chi|\mathbb{N}_A} = \frac{\chi(x)}{\zeta_A(\beta)} \sum_{n \in \mathbb{N}_A} n^{-\beta} \chi(n) = \chi(x) \prod_{p \in A} \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \text{ for } x \in \mathbb{N}_A. \]

Hence
\[ \|P_\chi\|_2 = \lim_{A \rightarrow D} \prod_{\beta \in A} \left| \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \right|. \]

If \( \chi \) is trivial, then using (3) we see that \( P_{A \chi} = \prod_{\beta \in B}(1 - p^{-\beta}) \) for all \( A \supset B \), hence \( P_\chi \) is a constant. Suppose \( \chi \) is non-trivial. The limit in (4) is an increasing function in \( \beta \) on \((0, +\infty)\) (because each factor is increasing), which is equal to \( |L(\beta, \chi)|\zeta(\beta)^{-1} \) for \( \beta > 1 \), where \( L(\beta, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-\beta} \) is the Dirichlet \( L \)-function corresponding to the number character \( \chi \) [6]. By elementary properties of Dirichlet series, \( |L(\beta, \chi)| \) tends to a finite value as \( \beta \rightarrow 1 + 0 \), while \( \zeta(\beta) \rightarrow \infty \). Thus \( P_\chi = 0 \).

Since the set of measures satisfying (1.3) is convex and consists of ergodic measures, there exists at most one measure satisfying (1.3). Such a measure does exist. In fact, for each \( \beta \in (0, +\infty) \) there is a unique \( W \)-invariant measure \( \mu_\beta \) satisfying (1.3) [6, 13]. Explicitly, \( \mu_\beta = \prod_{p} \mu_{\beta, p} \), where \( \mu_{\beta, p} \) is the measure on \( \mathbb{Q}_p \) such that \( \mu_{1, p} \) is the Haar measure \( (\mu_{1, p}(\mathbb{Z}_p) = 1) \), and
\[ \frac{d\mu_{\beta, p}}{d\mu_{1, p}}(a) = \frac{1 - p^{-\beta}}{1 - p^{-1}} |a|_p^{\beta - 1} \text{ for } a \in \mathbb{Q}_p. \]

Let \( \phi_\beta \) be the \((\beta, \sigma_1)\) KMS state on \( C_\mathbb{Q} \) corresponding to \( \mu_\beta \). Then \( \pi_{\phi_\beta}(C_\mathbb{Q})'\) is a factor, which is a reduction of the factor \( L^\infty(\mathcal{A}_f, \mu_\beta) \times \mathbb{Q}_+^\ast \). It is easy to describe its flow of weights \([\mathbb{Q}_+^\ast] \). Consider a standard measure space \( X_\beta \) with the measure algebra consisting of \( \mathbb{Q}_+^\ast \)-invariant \((\lambda \times \mu_\beta)\)-measurable subsets of \( \mathbb{R}_+ \times \mathcal{A}_f \), where \( \lambda \) is the Lebesgue measure, and the flow \( F_t^\beta \) on it coming from the action \( t(x, a) = (e^{-t/\beta}x, a) \) of \( \mathbb{R}_+ \times \mathcal{A}_f \). Then \( F_t^\beta \) is ergodic by Proposition, and it is the flow of weights of the factors \( L^\infty(\mathcal{A}_f, \mu_\beta) \times \mathbb{Q}_+^\ast \) and \( \pi_{\phi_\beta}(C_\mathbb{Q})'\).

**Theorem** ([6, 13]). For \( \beta \in (0, 1] \), \( \mu_\beta \) is a unique measure satisfying (1.3). The action of \( \mathbb{Q}_+^\ast \) on \( (\mathcal{A}_f, \mu_\beta) \) is ergodic, moreover, the action of \( \mathbb{Q}_+^\ast \) on \( (\mathcal{A}, \nu_\beta) \), where \( \mathcal{A} = \mathbb{R} \times \mathcal{A}_f \) is the space of full adeles and \( \nu_\beta = \lambda \times \mu_\beta \), is ergodic. Equivalently, \( \phi_\beta \) is a unique \((\beta, \sigma_1)\) KMS state on \( C_\mathbb{Q} \), and \( \pi_{\phi_\beta}(C_\mathbb{Q})'\) is the hyperfinite factor of type \( III_1 \).
Proof. In view of the above description of the flow of weights, the factor $\pi_\beta(CQ)$ is of type III$_1$ if and only if the action of $Q^*_+$ on $(\mathbb{R}_+ \times A_f, \lambda \times \mu_\beta)$ is ergodic, or equivalently, the action of $Q^*$ on $(A, \nu_\beta)$ is ergodic.

To prove the ergodicity, first note that the action of $W$ on $X_\beta$ is ergodic. Indeed, the induced flow on $X_\beta/W$ is the flow of weights of the factor $L^\infty(A_f/W, \mu_\beta) \rtimes Q^*_+ = (L^\infty(A_f, \mu_\beta) \rtimes Q^*_+)^W$.

It is easy to see ([BC]) that this factor is ITPFI with eigenvalue list
\[
\{p^{-n\beta}(1 - p^{-\beta}) | n \geq 0\}_{p \in \mathbb{P}}.
\]
Hence it is of type III$_1$ by [B] (see also [BZ]). Thus its flow of weights is trivial, i.e. the action of $W$ on $X_\beta$ is ergodic. Since $W$ is compact, the action is transitive. So we may identify $X_\beta$ with $W/W_\beta$ for some closed subgroup $W_\beta$ of $W$. Then the flow $F^\beta_t$ is given by a continuous one-parametric subgroup of $W/W_\beta$. Since $W/W_\beta$ is totally disconnected, this one-parametric subgroup is trivial, and since the flow is ergodic, $W_\beta = W$. Thus $X_\beta$ is singlepoint, and the action of $Q^*_+$ on $(\mathbb{R}_+ \times A_f, \lambda \times \mu_\beta)$ is ergodic.

Remarks. (i) In order to prove that $\phi_\beta$ is a unique KMS$_\beta$-state, it is enough to know that $\mu_\beta$ is ergodic. Indeed, this means that $\phi_\beta$ is an extremal KMS$_\beta$-state. Since $\phi_\beta$ is a unique $W$-invariant KMS$_\beta$-state, we can argue as in the proof of [BC, Theorem 25]: if $\psi$ is an extremal KMS$_\beta$-state, then
\[
\int_W w_* \psi \, dw = \phi_\beta.
\]
Since KMS$_\beta$-states form a simplex, we conclude that $\psi = \phi_\beta$.

Thus, the uniqueness and the assertion about the type for KMS$_\beta$-states follow easily from ergodicity of the action of $Q^*$ on $(A, \nu_\beta)$.

(ii) A slight modification of the argument in the proof of the Theorem gives the following general result, apparently well-known to specialists: if $M$ is a factor and $G$ a compact totally disconnected group acting on $M$ such that $M^G$ is a factor of type III$_1$, then $M$ is also of type III$_1$.

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References


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