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FINITE SUMS OF COMMUTATORS

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Abstract. We show that elements of unital C^*-algebras without tracial states are finite sums of commutators. Moreover, the number of commutators involved is bounded, depending only on the given C^*-algebra.

1. Introduction

It was shown in [2] that in finite von Neumann algebras, elements with central trace zero are sums of at most 10 commutators. The C^*-algebra case was considered in [1]. The main result there states that if the unit of a C^*-algebra A is properly infinite (i.e. there exist two orthogonal projections $p, q \in A$ equivalent to 1), then any hermitian element is a sum of at most five self-adjoint commutators. In this paper we consider the more general case of unital C^*-algebras A without tracial states and improve the previous result of T. Fack. Note that if the unit of A is properly infinite, then A has no tracial states. The converse is known to be false, at least when A is non-simple (see [4] for further details). C^*-algebras without tracial states have several nice characterizations, such as [3]. This paper also contains another simple proof of the latter result of [3, Lemma 1].

2. The result

Given $a, b \in A$, their commutator is $[a,b] = ab - ba$. A self-adjoint commutator is just a commutator of the form $[a^*, a] = a^*a - aa^*$.

Theorem 1. *Let A be a unital C^*-algebra. Then the following properties are equivalent:*

1. *A has no tracial states.*
2. *There exist an integer $n \geq 2$ and elements $b_1, b_2, \dots, b_n \in A$ such that*
$$\sum_{i=1}^{n} b_i^* b_i = 1 \quad \text{and} \quad \|\sum_{i=1}^{n} b_i b_i^*\| < 1.$$
3. *There exists an integer $n \geq 2$ such that any element of A can be expressed as a sum of n commutators and any positive element can be expressed as a sum of at most n self-adjoint commutators.*

Remark 2. The equivalence of (1) and (2) is just [3, Lemma 1]. As mentioned, in this paper we give a new simple proof.

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Remark 3. The integer n appearing in (2) above automatically satisfies property (3). If the unit of A is properly infinite, there exist two isometries $v_1, v_2 \in A$ with orthogonal ranges. Let $b_i = (1/\sqrt{2})v_i$ for $i = 1, 2$. Then $b_1^*b_1 + b_2^*b_2 = 1$ and $b_1b_1^* + b_2b_2^* \leq 1/2$; thus the property from (2) is achieved with $n = 2$. Thus in a properly infinite C^*-algebra, every element is the sum of two commutators, every positive element is the sum of two self-adjoint commutators, and every self-adjoint element is the sum of four self-adjoint commutators.

Proof. The implication (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Consider

$$R = \{\sum_{i=1}^{s}(a_i^*a_i - a_ia_i^*)\ ;\ s \geq 1, a_i \in A\}$$

the set of **finite** sums of self-adjoint commutators of A. Note that $R \subset A_{sa}$ is a **real vector subspace** of A_{sa}. Put $\delta = \text{dist}(1, R)$.

We show that $\delta < 1$. Suppose the contrary, i.e. $\delta = 1$. This is equivalent to

$$\|t + x\| \geq |t|, \quad \forall x \in R, \quad \forall t \in \mathbb{R}.$$

It follows that $\varphi(t + x) = t$ is a real bounded functional on $\mathbb{R}1 + R$ of norm 1. By the Hahn–Banach theorem it can be extended to a norm-1 functional on A_{sa} and furthermore to a bounded **complex** functional on A, also denoted by φ. Observe that φ is necessarily a tracial state on A, which contradicts our hypothesis.

Because $\delta < 1$, there exist some elements $a_1, a_2, \dots, a_m \in A$ such that $t_0 = \|1 - \sum_{i=1}^{m}(a_i^*a_i - a_ia_i^*)\| < 1$. In particular we have

$$\sum_{i=1}^{m} a_ia_i^* \leq -1 + t_0 + \sum_{i=1}^{m} a_i^*a_i. \tag{1}$$

Let $k = \|\sum_{i=1}^{m} a_i^*a_i\|$ and $a_{m+1} = (k - \sum_{i=1}^{m} a_i^*a_i)^{1/2}$. Then we have

$$\sum_{i=1}^{m+1} a_i^*a_i = k;$$

but on the other hand, by (1) we have also

$$\sum_{i=1}^{m+1} a_ia_i^* \leq -1 + t_0 + k.$$

The required properties are now fulfilled with $n = m + 1$ and $b_i = (1/\sqrt{k})a_i$.

(2) $\Rightarrow$ (3). Suppose that $b_1, b_2, \dots, b_n$ are as in (2). Define $\Phi(a) = \sum b_iab_i^*$. Then Φ is a bounded positive map on A with norm $\|\Phi\| = \|\sum b_ib_i^*\| < 1$. It follows that $Id_A - \Phi$ is invertible in the Banach algebra $\mathcal{B}(A)$ of bounded maps on A. Let

$$\Psi = (Id_A - \Phi)^{-1}.$$

Note that $\Psi = \sum_{i=0}^{\infty}\Phi^i$, thus Ψ is positive too. By definition of Ψ, for any $a \in A$ we have

$$\begin{aligned} a &= (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i\Psi(a)b_i^* \\ &= \sum_{i=1}^{n} b_i^*b_i\Psi(a) - \sum_{i=1}^{n} b_i\Psi(a)b_i^* = \sum_{i=1}^{n}[b_i^*, b_i\Psi(a)], \end{aligned}$$

so a is a finite sum of at most n commutators. If moreover a is a positive element in A, then

$$\begin{aligned} a &= (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i \Psi(a) b_i^* \\ &= \sum_{i=1}^{n} \Psi(a)^{1/2} b_i^* b_i \Psi(a)^{1/2} - \sum_{i=1}^{n} b_i \Psi(a) b_i^* = \sum_{i=1}^{n} [\Psi(a)^{1/2} b_i^*, b_i \Psi(a)^{1/2}], \end{aligned}$$

so a is a finite sum of at most n self-adjoint commutators. □

3. Questions

For an infinite C^*-algebra A (in the sense that it admits no tracial states) let $\nu(A)$ be the least positive integer such that any element of A is a sum of at most $\nu(A)$ commutators. In all the examples that we know of, we have $\nu(A) = 2$. We believe that it is unlikely to always be the case.

In [3] it was shown that if A is an unital exact C^*-algebra, then there exists an integer m such that $\mathbb{M}_m(A)$ is properly infinite. It follows that $\nu(\mathbb{M}_m(A)) = 2$. Then a simple computation shows that $\nu(A) \leq 2m^2$. It would be interesting to answer the inverse problem, that is: assuming $\nu(A)$ is known, estimate the least positive integer m such that $\nu(\mathbb{M}_m(A)) = 2$.

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