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ON THE COMPLEXITY OF THE DESCRIPTION OF $*$-ALGEBRA REPRESENTATIONS BY UNBOUNDED OPERATORS

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ABSTRACT. We study the complexity of the problem to describe, up to unitary equivalence, representations of $*$-algebras by unbounded operators on a Hilbert space. A number of examples are developed in detail.

Representations of algebras and $*$-algebras by bounded and unbounded operators on a Hilbert space have many applications in various branches of analysis and mathematical physics and is a powerful tool for the study of algebras itself. Often it is important to estimate how complicated the problem of describing representations (up to an equivalence) is. In the theory of representations of algebras the representation problem is considered to be extremely difficult (wild) if it "contains" the classical unsolved problem of describing up to a similarity a pair of matrices without relations (see [DF]). For $*$-algebras the complexity of the structure of their representations by bounded operators on a Hilbert space was studied in [KS] (see also [OS, S]). The authors introduced a quasiorder $\succ$ (majorization) of $*$-algebras with respect to how difficult their representations are and proved that the enveloping C^*-algebra, $C^*(\mathcal{F}_2)$, of the free group with two generators, majorizes any finitely generated $*$-algebra showing that the problem of describing $*$-representations (up to unitary equivalence) of a $*$-algebra $\mathfrak{A}$ such that $\mathfrak{A} \succ C^*(\mathcal{F}_2)$ is difficult (such an algebra $\mathfrak{A}$ is called $*$-wild). Note that for a unital C^*-algebra $\mathfrak{A}$, $\mathfrak{A} \succ C^*(\mathcal{F}_2)$ is equivalent to $\mathfrak{A}/J \simeq M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$ where J is a C^*-ideal of $\mathfrak{A}$ and $n \in \mathbb{N}$ ([OS]). Boundedness of $*$-representations is essential for their consideration.

In the present paper we study the complexity of $*$-algebras representations by unbounded operators on a Hilbert space. Unbounded representations occur, for example, in the representation theory of non-compact Lie algebras, non-compact quantum groups and quantum algebras, etc. (see [JSW, OS, Wor1, Wor2] and references therein). For instance, the $*$-algebra with two selfadjoint generators p, q and the canonical (Heisenberg) commutation relation $pq - qp = i$ of the quantum mechanics does not have any representations by bounded operators but it has well known unbounded ones: $p = i\frac{d}{dx}$, $q = x$ on $L_2(\mathbb{R}, dx)$. That the structure of unbounded representations can be very complicated for $*$-algebras is already evident from [S, Chapter 9.4] and [ST1]: the problem of describing unbounded

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$*$-representations for the $*$-algebra $\mathbb{C}[x_1, x_2]$ of all polynomials in commuting self-adjoint elements x_1, x_2 "contains" as a subproblem the problem of classification up to unitary equivalence all representations of $C^*(\mathcal{F}_2)$.

In Section 3, developing [KS, OS], we give a definition of complicated ($*$-wild) classes of $*$-representations by unbounded operators. An invaluable tool for our study is the notion of unbounded elements which generate a C^*-algebra introduced and studied by Woronowicz [Wor2]. We prove that the class of bounded representations of a unital $*$-algebra $\mathfrak{A}$ is complicated ($*$-wild) iff the algebra $\mathfrak{A}$ is $*$-wild in the sense of [KS] (Theorem 4). Similar to [KS], we show that the class of representations which "arises from" representations of a C^*-algebra $\mathfrak{A}$ is complicated iff $\mathfrak{A}/J \simeq CB(H) \otimes C^*(\mathcal{F}_2)$, where $CB(H)$ are now compact operators on a Hilbert space H (Theorem 5). From this we derive that integrable representations of finite-dimensional Lie algebras, i.e. infinitesimal representations dU of unitary representations U of the corresponding Lie group, are not $*$-wild (Corollary 1).

In Section 4, we study, from the point of view developed in Section 3, the complexity of representations by unbounded operators of some $*$-algebras. The first example is devoted to non-integrable representations of the polynomial $*$-algebra $\mathbb{C}[x_1, x_2]$. The second example deals with the $*$-algebra $\mathfrak{A} = \mathbb{C}\langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 0, x_i^* = x_i, i = 1, 2\rangle$. Note that operators satisfying the double commutation relations were studied in [Dal] in the framework of abstract hyperbolic equations. In the third example we discuss a class of unbounded representations generated by idempotents with zero sum.

We attempt to give sufficient background references for all concepts involved. For the basic definitions and notions of the theory of representations of $*$-algebras and C^*-algebras we refer the reader to [D, Ped, S]. Throughout the paper H is a separable Hilbert space, $CB(H)$ and $B(H)$ denote the algebra of compact and respectively bounded operators on H. For $*$-algebras A and B the algebraic tensor product of A and B is denoted by $A \odot B$. We write $\mathrm{Rep}(A)$ for the category of $*$-representations of A, with bounded non-degenerate representations as objects and intertwining operators as morphisms. Recall that a $*$-representation π of a $*$-algebra A on a Hilbert space H is non-degenerate if $\pi(A)H$ is dense in H.

1. $*$-wild algebras

We follow [KS] in introducing the notion of $*$-wild algebras (see also [OS]). Within this section we assume that all $*$-algebras are unital and representations of $*$-algebras are unital $*$-homomorphisms into $B(H)$.

Let $\mathfrak{A}$ be a $*$-algebra. If $\psi : \mathfrak{A} \to M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$, $n \in \mathbb{N}$, is a unital $*$-homomorphism of the $*$-algebra $\mathfrak{A}$ and the C^*-algebra $M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$ ($= M_n(C^*(\mathcal{F}_2))$), then there is a natural way to construct the functor $F_\psi\colon \mathrm{Rep}(C^*(\mathcal{F}_2)) \to \mathrm{Rep}(\mathfrak{A})$:

- $F_\psi(\pi) = (id \otimes \pi) \circ \psi$, for any $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$,
- $F_\psi(\alpha) = I \otimes \alpha$ for any operator α intertwining π_1, $\pi_2 \in \mathrm{Rep}(C^*(\mathcal{F}_2))$.

(I is the identity operator in $B(\mathbb{C}^n)$.)

Definition 1. A $*$-algebra $\mathfrak{A}$ is called $*$-wild if there exist $n \in \mathbb{N}$ and a $*$-homomorphism $\psi\colon \mathfrak{A} \to M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$ such that the functor $F_\psi\colon \mathrm{Rep}(C^*(\mathcal{F}_2)) \to \mathrm{Rep}(\mathfrak{A})$ is full.

In order to verify that the functor F_ψ is full one has to show that for representations π_1, $\pi_2 \in \mathrm{Rep}(C^*(\mathcal{F}_2))$ in H_1 and H_2 respectively, an operator A intertwines the representations $F_\psi(\pi_1)$ and $F_\psi(\pi_2)$ iff $A = I \otimes \alpha$, where I is the identity operator in $B(\mathbb{C}^n)$ and α intertwines the representations π_1, π_2. It follows from the definition that two representations π_1, π_2 of $C^*(\mathcal{F}_2)$ are unitarily equivalent iff so are the representations $F_\psi(\pi_1)$, $F_\psi(\pi_2)$ of $\mathfrak{A}$, a representation π of $C^*(\mathcal{F}_2)$ is irreducible iff so is $F_\psi(\pi)$. Thus the unitary classification problem of all representations of $\mathfrak{A}$ contains, as a subproblem, the problem of unitary classification of all representations of $C^*(\mathcal{F}_2)$. As it was noticed in the introduction, $C^*(\mathcal{F}_2)$ majorizes any finitely-generated $*$-algebra and the unitary classification of all representations of $C^*(\mathcal{F}_2)$ contains, as a subproblem, the problem of unitary classification of any affine $*$-algebra. An example of a $*$-wild algebra is $\mathfrak{S}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*\rangle$. For C^*-algebras we have the following:

Theorem 1 ([OS])**.** *A C^*-algebra $\mathfrak{A}$ is $*$-wild if and only if there exist a C^*-ideal $J \subset \mathfrak{A}$ and $n \in \mathbb{N}$ such that $\mathfrak{A}/J \simeq M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$.*

Remark 1. It follows from the above theorem that a unital C^*-algebra $\mathfrak{A}$ is $*$-wild iff there exists a $*$-epimorphism $\psi\colon \mathfrak{A} \to M_n(C^*(\mathcal{F}_2))$ and if, in particular, $\mathfrak{A}$ is generated by some elements $T_1, \dots, T_n$, i.e. $\mathfrak{A}$ is the closure of algebraic combinations of $1, T_1, \dots, T_n, T_1^*, \dots, T_n^*$, then $\psi(T_1), \dots \psi(T_n)$ generate $M_n(C^*(\mathcal{F}_2))$.

For other results and examples of $*$-wild algebras we refer the reader to [OS].

2. C^*-algebras generated by a finite number of affiliated elements

In order to generalise the notion of $*$-wild algebra to include representations by unbounded operators on a Hilbert space, we need unbounded elements which are "related" to a C^*-algebra. The notion of a C^*-algebra generated by unbounded affiliated elements was introduced and investigated by S. L. Woronowicz ([Wor2], see also [Wor1]). We recall here some definitions and facts from [Wor2] which will be used in the sequence.

Let H be a Hilbert space, $C^*(H)$ the set of separable non-degenerate C^*-subalgebras of $B(H)$, and $A \in C^*(H)$. The set of all *multipliers*, $M(A)$, of A is defined by

$$M(A) = \{a \in B(H) \mid ab, ba \in A, \text{ for any } b \in A\}.$$

Let T be a closed operator acting on a Hilbert space H. We say that T is *affiliated* with $A \in C^*(H)$ if the *z-transform* $z_T = T(I + T^*T)^{-1/2}$ of T belongs to $M(A)$, $z_T^* z_T \leq I$ and $(I - z_T^* z_T)A$ is dense in A. We write $T \eta A$. The set of all elements affiliated with A will be denoted by A^η. T is a linear mapping acting on A with the domain $D(T) = (I - z_T^* z_T)^{1/2} A$. $D \subseteq D(T)$ is a core of T if T coincides with the closure of $T|_D$. One should distinguish between $D(T)$ and the domain of T as an operator acting on a Hilbert space which will be denoted by $\mathcal{D}(T)$.

Let A be a C^*-algebra, $B \in C^*(H)$. The *set of morphisms*, $\mathrm{Mor}(A, B)$, consists of all $\pi \in \mathrm{Rep}(A, H)$ such that $\pi(A)B$ is dense in B, where $\mathrm{Rep}(A, H)$ is the set of all non-degenerate representations of A on H. In particular, $\mathrm{Mor}(A, CB(H)) = \mathrm{Rep}(A, H)$. If $\varphi \in \mathrm{Mor}(A, B)$, then φ can be uniquely extended to a mapping from A^η to B^η.

The notions of multiplier algebra, affiliated elements are independent of the choice of embedding of C^*-algebras into $B(H)$ (see [Wor2]).

Let A be a C^*-algebra and $T_1, \dots, T_n$ be elements affiliated with A. We say that A *is generated* by $T_1, \dots, T_n$ if for any Hilbert space H, $B \in C^*(H)$ and any $\pi \in \mathrm{Rep}(A, H)$ the condition $\pi(T_i)\eta B$, $i = 1, \dots, n$, implies $\pi \in \mathrm{Mor}(A, B)$. We will use the following sufficient condition for a C^*-algebra A to be generated by elements affiliated with A.

Theorem 2. *Let A be a C^*-algebra and $T_1, \dots, T_n$ be elements affiliated with A. The subset of $M(A)$ composed of all elements of the form $(I + T_i^* T_i)^{-1}$ and $(I + T_i T_i^*)^{-1}$, $i = 1, \dots, n$, will be denoted by Γ. Assume that:*

1. *$T_1, \dots, T_n$ separate representations of A: if φ_1, φ_2 are different elements of Rep(A, H), then $\varphi_1(T_i) \neq \varphi_2(T_i)$ for some $i = 1, \dots, n$.*
2. *There exist elements $r_1, \dots, r_k \in \Gamma$ such that the product $r_1 \dots r_k \in A$.*

Then A is generated by $T_1, \dots, T_n$.

Remark 2. A C^*-algebra A is generated by $T_1, \dots, T_n \eta A$ with $||T_i|| < \infty$ for all $i = 1, \dots, n$, if and only if A is unital and A coincides with the norm closure of all algebraic combinations of I, $T_1, \dots, T_n$, $T_1^*, \dots, T_n^*$.

See [Wor2] for other discussions on C^*-algebras generated by affiliated elements.

3. The complexity of unbounded representations of $*$-algebras

In this section we shall extend the notion of a $*$-wild problem to unbounded representations. We restrict our attention to finitely presented unital $*$-algebras, i.e. $*$-algebras introduced in terms of a finite number of generators and relations (algebraic equalities imposed on the generators).

Let $\mathfrak{A}$ be a unital $*$-algebra generated by elements $t_1, \dots, t_n$, $t_1^*, \dots, t_n^*$ and relations

$$w_j(t_1, \dots, t_n, t_1^*, \dots, t_n^*) = 0, \qquad j = 1, \dots, m, \tag{1}$$

where w_j are polynomials over $\mathbb{C}$ in the non-commuting variables $t_1, \dots, t_n$, $t_1^*, \dots, t_n^*$ and 1.

A family of closed operators $T_1, \dots, T_n, T_1^*, \dots, T_n^*$ on a Hilbert space H is called a *representation, π, of* $\mathfrak{A}$ if there exists a dense domain, $\mathcal{D}$, such that $\mathcal{D}$ is invariant with respect to all operators of the family, $\mathcal{D}$ is a core for T_i, T_i^*, $i = 1, \dots, n$, and the relations

$$w_j(T_1, \dots, T_n, T_1^*, \dots, T_n^*)\varphi = 0, \qquad j = 1, \dots, m,$$

hold for any $\varphi \in \mathcal{D}$ (with the identity operator instead of 1). We write $\pi(t_i) = T_i$. We denote by $\mathrm{Rep}_{unb}(\mathfrak{A})$ the category of unbounded representations of $\mathfrak{A}$. The objects of $\mathrm{Rep}_{unb}(\mathfrak{A})$ are representations defined above and the morphisms of $\mathrm{Rep}_{unb}(\mathfrak{A})$ are bounded operators $C \in B(H, \tilde{H})$ intertwining representations π and $\tilde{\pi}$ which act on Hilbert spaces H and $\tilde{H}$ respectively, i.e. $CT_i \subseteq \tilde{T}_i C$, $CT_i^* \subseteq \tilde{T}_i^* C$, $i = 1, \dots, n$ (we write $C \in I(\pi, \tilde{\pi})$). We say that two representations, π_1 and π_2, are unitarily equivalent if there exists a unitary operator of $H(\pi_1)$ onto $H(\pi_2)$ such that $U \in I(\pi_1, \pi_2)$ and $U^{-1} \in I(\pi_2, \pi_1)$. In this case we write $\pi_1 \simeq \pi_2$.

Proving that the problem of unitary classification of bounded representations of a $*$-algebra A is $*$-wild we construct a unital $*$-homomorphism $\psi\colon A \to M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$, $n \in \mathbb{N}$, which generates a full functor $F_\psi\colon \mathrm{Rep}(C^*(\mathcal{F}_2)) \to \mathrm{Rep}(\mathfrak{A})$. In order to include unbounded representations we shall first replace the above $*$-homomorphism by a "$*$-homomorphism" into the set of affiliated elements

$(CB(H)\otimes C^*(\mathcal{F}_2))^\eta$, where $CB(H)$ is the C^*-algebra of a compact operator on a Hilbert space (not necessarily finite-dimensional) and $CB(H)\otimes C^*(\mathcal{F}_2)$ is the C^*-algebra obtained by the completion of the algebraic tensor product $CB(H)\odot C^*(\mathcal{F}_2)$ in a C^*-norm (it is known that it does not depend on norm).

Namely, let ψ be a unital mapping from $(1, t_1, \dots, t_n)$ to $(CB(H)\otimes C^*(\mathcal{F}_2))^\eta$ such that there exists a dense linear subset D of $CB(H)\otimes C^*(\mathcal{F}_2)$ satisfying the following conditions:

- $D\subseteq D(\psi(t_i))$, $D\subseteq D(\psi(t_i)^*)$, $\psi(t_i)D\subseteq D$, $\psi(t_i)^*D\subseteq D$,
- $w_j(\psi(t_1),\dots,\psi(t_n),\psi(t_1)^*,\dots,\psi(t_n)^*)a=0, \qquad j=1,\dots,m, \quad a\in D$,
- D is a core for $\psi(t_1),\dots,\psi(t_n)$, $\psi(t_1)^*,\dots,\psi(t_n)^*$.

In the sequel, whenever we write a $*$-homomorphism ψ from $\mathfrak{A}$ to $(CB(H)\otimes C^*(\mathcal{F}_2))^\eta$, we mean a unital mapping $\psi\colon (1,t_1,\dots,t_n)\to (CB(H)\otimes C^*(\mathcal{F}_2))^\eta$ satisfying the above conditions.

As before, the mapping ψ generates a functor $F_\psi\colon \mathrm{Rep}(C^*(\mathcal{F}_2))\to \mathrm{Rep}_{unb}(\mathfrak{A})$:

- $F_\psi(\pi)(t_i)=(id\otimes\pi)(\psi(t_i))$ for any representation $\pi\in\mathrm{Rep}(C^*(\mathcal{F}_2))$, where $id\otimes\pi$ is the unique extension to the affiliated elements,
- $F_\psi(\alpha)=I\otimes\alpha$ for any α intertwining π_1, $\pi_2\in\mathrm{Rep}(C^*(\mathcal{F}_2))$.

If π is a representation of $C^*(\mathcal{F}_2)$ on a Hilbert space $H(\pi)$, then $F_\psi(\pi)(t_i)$, $i=1,\dots,n$, define a representation of $\mathfrak{A}$ with $\mathcal{D}=\{(id\otimes\pi)(D)\varphi, \varphi\in H\otimes H(\pi)\}$, $\overline{\mathcal{D}}=H\otimes H(\pi)$.

Consider the following two properties of the mapping $\psi\colon\mathfrak{A}\to(CB(H)\otimes C^*(\mathcal{F}_2))^\eta$: (**P.1.**) the corresponding functor F_ψ is full, and (**P.2.**) $\psi(t_1),\dots,\psi(t_n)$ generate $CB(H)\otimes C^*(\mathcal{F}_2)$ as affiliated elements. We will see below that in the case of unital C^*-algebra, $\mathcal{A}$, the second condition (not the first one) implies $*$-wildness of $\mathcal{A}$ and the result of Theorem 1 partially explains this (see also Remark 1). Theorem 3 shows that (**P.2**) implies (**P.1**) but the inverse is false (see Remark 5).

Theorem 3. *Let $\psi\colon\mathfrak{A}\to(CB(H)\otimes C^*(\mathcal{F}_2))^\eta$ be a $*$-homomorphism in the sense defined above. Assume that $\psi(t_1),\dots,\psi(t_n)$ generate $CB(H)\otimes C^*(\mathcal{F}_2)$. Then the corresponding functor $F_\psi\colon$ Rep$(C^*(\mathcal{F}_2))\to\mathrm{Rep}_{unb}(\mathfrak{A})$ is full.*

Proof. To prove the statement it is enough to show that given an operator $C\in B(H)$, a representation $\pi\in\mathrm{Rep}(C^*(\mathcal{F}_2))$ such that $CF_\psi(\pi)(t_i)\subseteq F_\psi(\pi)(t_i)C$, $CF_\psi(\pi)(t_i)^*\subseteq F_\psi(\pi)(t_i)^*C$, we have $C=I\otimes\alpha$, where $\alpha\in I(\pi,\pi)$.

We first show that C commutes with $(id\otimes\pi)(a)$ for any $a\in CB(H)\otimes C^*(\mathcal{F}_2)$. Define the following set:

$$\mathfrak{B}_C=\left\{a\in M(\mathfrak{B})\colon\ \begin{array}{l} C(id\otimes\pi)(a)=(id\otimes\pi)(a)C,\\ C(id\otimes\pi)(a^*)=(id\otimes\pi)(a^*)C.\end{array}\right\}$$

Here $\mathfrak{B}=CB(H)\otimes C^*(\mathcal{F}_2)$. $\mathfrak{B}_C$ is a C^*-subalgebra of $M(\mathfrak{B})$. Let us now show that $\mathfrak{B}_C=M(\mathfrak{B})$. We will use [Wor2, Proposition 2.2], an analogue of the Stone-Weierstrass Theorem.

(1) $\mathfrak{B}_C$ is non-empty, since the z-transforms $z_{\psi(t_i)}$, $z^*_{\psi(t_i)}$, $i=1,\dots,n$, belong to $\mathfrak{B}_C$.

(2) $\mathfrak{B}_C$ separates representations of $\mathfrak{B}$. Indeed, let π_1, π_2 be two different representations of $\mathfrak{B}$. Assuming that $\pi_1(q)=\pi_2(q)$ for any $q\in\mathfrak{B}_C$ we get $\pi_1(z_{\psi(t_i)})=\pi_2(z_{\psi(t_i)})$ which implies $\pi_1(\psi(t_i))=\pi_2(\psi(t_i))$ for any $i=1,\dots,n$. Since $\psi(t_1),\dots,\psi(t_n)$ generate $\mathfrak{B}$, we get $\pi_1=\pi_2$. A contradiction.

(3) $\mathfrak{B}_C$ is strictly closed. In fact, let $\{x_\lambda\} \in \mathfrak{B}_C$ strictly converge to x, i.e. $||a(x_\lambda - x)|| \to 0$ and $||(x_\lambda - x)a|| \to 0$ for any $a \in \mathfrak{B}$. If $\pi \in \mathrm{Rep}(\mathfrak{B}, H)$, we get $||\pi(x_\lambda a) - \pi(xa)|| \to 0$ and

$$(\pi(x_\lambda) - \pi(x))\pi(a)\varphi \to 0, \quad a \in \mathfrak{B},\ \varphi \in H.$$

Since π is a non-degenerate representation of $\mathfrak{B}$, the set $\{\pi(a)\varphi\colon a \in \mathfrak{B}, \varphi \in H\}$ is dense in H and any strictly convergent sequence is bounded, we deduce that $(\pi(x_\lambda)-\pi(x))\varphi \to 0$, $\varphi \in H$. Therefore, if $C\pi(x_\lambda) = \pi(x_\lambda)C$ we get $C\pi(x) = \pi(x)C$ and $x \in \mathfrak{B}_C$.

According to [Wor2, Proposition 2.2], (1), (2), (3) imply $\mathfrak{B}_C = M(\mathfrak{B})$ and hence $[C, (id \otimes \pi)(a)] = 0$ for any $a \in CB(H) \otimes C^*(\mathcal{F}_2)$. This gives $C = I \otimes \alpha$, where $[\alpha, \pi(a)] = 0$ for any $a \in C^*(\mathcal{F}_2)$, and hence the functor F_ψ is full. □

In the sequel we shall consider classes, R, of representations from $\mathrm{Rep}_{unb}(\mathfrak{A})$ which are closed with respect to the direct sum and taking subrepresentations, i.e. R satisfies the following conditions: a) if $\pi_1 \in R$ and $\pi_1 \simeq \pi_2$, then $\pi_2 \in R$; b) if $(\pi_\lambda)_{\lambda\in\Lambda}$ is a family of representations from R, where Λ is a countable set of indexes, then $\bigoplus_{\lambda\in\Lambda} \pi_\lambda \in R$; c) if $\pi_1 \oplus \pi_2 \in R$, then $\pi_i \in R$, $i = 1, 2$.

Definition 2. Let $\mathfrak{A}$ be a unital $*$-algebra generated by $t_1, \dots, t_n, t_1^*, \dots, t_n^*$ and relations (1). We say that a class $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ is $*$-wild if there exists a $*$-homomorphism $\psi\colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that $(id \otimes \pi)(\psi)$ belongs to R for any $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$ and $\psi(t_1), \dots, \psi(t_n)$ generate the C^*-algebra $CB(H) \otimes C^*(\mathcal{F}_2)$.

Theorem 4. *The class of bounded representations of $\mathfrak{A}$ is $*$-wild if and only if $\mathfrak{A}$ is $*$-wild.*

Proof. Assume that there exists a $*$-homomorphism $\psi\colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that the operators $(id\otimes\pi)(\psi(t_i))$ are bounded for any $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$, and $\psi(t_1), \dots, \psi(t_n)$ generate $\mathfrak{B} = CB(H) \otimes C^*(\mathcal{F}_2)$. Clearly, $\psi(t_i)$, $i = 1, \dots, n$, are bounded. It follows from [Wor2, Example 1] that $\mathfrak{B}$ is unital and coincides with the norm closure of all algebraic combinations of I, $\psi(t_1), \dots, \psi(t_n)$, $\psi(t_1)^*, \dots, \psi(t_n)^*$. This implies that H is finite-dimensional and ψ is a $*$-homomorphism from $\mathfrak{A}$ to $CB(H) \otimes C^*(\mathcal{F}_2)$. Moreover, by Theorem 3, the functor generated by ψ is full which means that the $*$-algebra $\mathfrak{A}$ is $*$-wild.

Conversely, suppose that $\mathfrak{A}$ is $*$-wild. Then there exists a $*$-homomorphism $\psi\colon \mathfrak{A} \to CB(H) \otimes C^*(\mathcal{F}_2)$ with $\dim H < \infty$. Let $\tilde{\mathfrak{B}}$ be the norm closure of the set of all algebraic combinations of I, $\psi(t_1), \dots, \psi(t_n)$, $\psi(t_1)^*, \dots, \psi(t_n)^*$. Then $\tilde{\mathfrak{B}}$ is a C^*-subalgebra of $CB(H) \otimes C^*(\mathcal{F}_2)$. Since $\mathfrak{A}$ is $*$-wild, it is not difficult to see that $\pi(\psi(\mathfrak{A}))' = \pi(CB(H) \otimes C^*(\mathcal{F}_2))'$ for any representation π of $CB(H) \otimes C^*(\mathcal{F}_2)$ (see, for example, the proof of [OS, Theorem 50]). This implies $\pi(\tilde{\mathfrak{B}})' = \pi(CB(H)\otimes C^*(\mathcal{F}_2))'$. By [OS, Lemma 14], the inclusion $i\colon \tilde{\mathfrak{B}} \to CB(H)\otimes C^*(\mathcal{F}_2)$ is a surjection, and hence $\tilde{\mathfrak{B}} = CB(H) \otimes C^*(\mathcal{F}_2)$. Since the C^*-algebra $CB(H) \otimes C^*(\mathcal{F}_2)$ is unital, by Remark 2, we see that $\psi(t_1), \dots, \psi(t_n)$ generate $CB(H) \otimes C^*(\mathcal{F}_2)$ in the sense of affiliated elements. □

Let $\mathfrak{A}$ be a $*$-algebra generated by $t_1, \dots, t_n$, $t_1^*, \dots, t_n^*$ and relations (1). We say that a class $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ is *manageable* (see [Wor2]) if there exist a separable C^*-algebra $\mathfrak{B}$ (unital or non-unital) and $T_1, \dots, T_n$ η $\mathfrak{B}$ such that $\mathfrak{B}$ is generated by $T_1, \dots, T_n$ and R is equal to the set

$$\{\pi \in \mathrm{Rep}_{unb}(\mathfrak{A}) : \pi(t_i) = \tilde{\pi}(T_i) \text{ for some } \tilde{\pi} \in \mathrm{Rep}(\mathfrak{B})\}. \tag{2}$$

Example 1 ([Wor2]). Let G be a connected, simply connected Lie group, c^i_{kl}, $k, l = 1, \dots, n$, be the structure constants of the Lie algebra A of G and the class R consists of integrable representations of A, or equivalently families $(T_1, \dots, T_n)$ such that $T_1, \dots, T_n$ are skew-adjoint operators acting on a Hilbert space H such that on a dense invariant domain $[T_k, T_l] = \sum_{i=1}^n c^i_{kl} T_i$ and $\sum_{i=1}^n T_i^2$ is essentially self-adjoint on the same domain. Then the class R is manageable and $\mathfrak{B} = C^*(G)$.

Theorem 5. *Let R be a manageable class of $*$-representations of $\mathfrak{A}$. Let $\mathfrak{B}$ be the C^*-algebra defined above. Then R is $*$-wild if and only if there exist a C^*-ideal J and a Hilbert space H such that*

$$\mathfrak{B}/J \simeq CB(H) \otimes C^*(\mathcal{F}_2). \tag{3}$$

Proof. Assume that R is $*$-wild. Then there exists $\psi \colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that $\psi(t_1), \dots, \psi(t_n)$ generate $CB(H) \otimes C^*(\mathcal{F}_2)$ and $(id \otimes \pi)(\psi) \in R$ for any $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$. We can assume that $C^*(\mathcal{F}_2)$ is embedded into $B(H_0)$. Let $\tilde{\pi} \in \mathrm{Rep}(C^*(\mathfrak{B}))$ such that $\psi(t_i) = \tilde{\pi}(T_i)$. Since $T_1, \dots, T_n$ generate $\mathfrak{B}$, we get $\tilde{\pi} \in \mathrm{Mor}(\mathfrak{B}, CB(H) \otimes C^*(\mathcal{F}_2))$. Applying [Wor2, Proposition 3.2] we conclude that $\tilde{\pi}(\mathfrak{B}) = CB(H) \otimes C^*(\mathcal{F}_2)$ and hence there exists an ideal $J = \ker \tilde{\pi}$ such that (3) holds.

Assume now that (3) holds with an isomofism $\varphi : \mathfrak{B}/J \to CB(H) \otimes C^*(\mathcal{F}_2)$. Let π denote the quotient map $\pi : \mathfrak{B} \to \mathfrak{B}/J$. Since $T_1, \dots, T_n$ generate $\mathfrak{B}$ it follows from the definition that $(\varphi \circ \pi)(T_1), \dots, (\varphi \circ \pi)(T_n)$ generate $CB(H) \otimes C^*(\mathcal{F}_2)$. Setting $\psi(t_i) = (\varphi \circ \pi)(T_i)$ we get a $*$-homomorphism from $\mathfrak{A}$ to $(CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ satisfying the conditions of Definition 2. Therefore the class R is $*$-wild. □

Corollary 1. *Integrable representations of a finite-dimensional Lie algebra are not $*$-wild.*

Proof. Assuming the class of an integrable representation of a finite-dimensional Lie algebra to be $*$-wild we get by Theorem 5 and Example 1 that the C^* algebra $C^*(G)$ of the corresponding connected, simply connected Lie group G contains a C^*-ideal J such that $C^*(G)/J \simeq CB(H) \otimes C^*(\mathcal{F}_2)$. However $C^*(G)$ is nuclear (see [Pat]), and therefore has only hyperfinite factor representations while $CB(H) \otimes C^*(\mathcal{F}_2)$ has non-hyperfinite representations. A contradiction. □

Remark 3. If we have a $*$-homomorphism $\psi \colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ which generates a full functor $F_\psi : \mathrm{Rep}(C^*(\mathcal{F}_2)) \to R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$, then the problem of unitary classification of $*$-representations R of $\mathfrak{A}$ is difficult and contains as a subproblem the problem of unitary classification of all representations of $C^*(\mathcal{F}_2)$. Namely, the representation $((id \otimes \pi)(\psi)(t_1), \dots, (id \otimes \pi)(\psi)(t_n))$, $\pi \in C^*(\mathcal{F}_2)$, is irreducible iff so is π, two such representations are unitarily equivalent iff so are the corresponding representations of $C^*(\mathcal{F}_2)$.

In Section 4 we will use the following proposition.

Proposition 1. *Let $\mathfrak{B}$ be a C^*-algebra such that* (3) *holds, and let $\psi \colon \mathfrak{A} \to (CB(H_0) \otimes \mathfrak{B})^\eta$ be a unital $*$-homomorphism such that $\psi(t_1), \dots, \psi(t_n)$ generate $CB(H_0) \otimes \mathfrak{B}$. Assume that the representation $((id \otimes \pi)(\psi(t_1)), \dots, (id \otimes \pi)(\psi(t_n)))$ belongs to a class $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ for any $\pi \in \mathrm{Rep}(\mathfrak{B})$. Then R is $*$-wild.*

Proof. By assumption, there exists a $*$-homomorphism $\varphi\colon \mathfrak{B} \to CB(H_1) \otimes C^*(\mathcal{F}_2)$ such that

$$\varphi(\mathfrak{B}) = CB(H_1) \otimes C^*(\mathcal{F}_2). \tag{4}$$

If we prove that $(id_{H_0} \otimes \varphi)(\psi(t_i))$, $i = 1, \dots, n$, generate $\mathcal{A} = CB(H_0) \otimes CB(H_1) \otimes C^*(\mathcal{F}_2)$, the assertion follows. Here id_H is the identity representation of $CB(H)$. Let $\mathcal{H}$ be a separable Hilbert space and $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$. Then there exist a unitary operator $V \in B(\mathcal{H})$ and a representation $\rho \in \mathrm{Rep}(C^*(\mathcal{F}_2))$ such that

$$\pi = V(id_{H_0} \otimes id_{H_1} \otimes \rho)V^{-1}.$$

Let $\mathcal{C} \in C^*(\mathcal{H})$ and $\pi((id_{H_0} \otimes \varphi)(\psi(t_i)))\eta\ \mathcal{C}$. Then

$$(id_{H_0} \otimes id_{H_1} \otimes \rho)((id_{H_0} \otimes \varphi)(\psi(t_i)))\ \eta\ V^{-1}\mathcal{C}V =: \tilde{\mathcal{C}}.$$

Since $\tilde{\rho} := (id_{H_1} \otimes \rho)(\varphi)$ is a representation of $\mathfrak{B}$, $id_{H_0} \otimes \tilde{\rho} \in \mathrm{Mor}(CB(H_0) \otimes \mathfrak{B}, \tilde{\mathcal{C}})$. From this and (4) we conclude that $id_{H_0} \otimes id_{H_1} \otimes \rho \in \mathrm{Mor}(\mathcal{A}, \tilde{\mathcal{C}})$ and $\pi \in \mathrm{Mor}(\mathcal{A}, \mathcal{C})$. The proof is finished. ☐

Remark 4. Let R be a class of $\mathrm{Rep}_{unb}(\mathfrak{A})$, and let H and H_0 be separable Hilbert spaces, $\dim H = \infty$. We denote by $R(\mathcal{H})$ the set of all representations $\pi \in R$ acting on $\mathcal{H} = H \otimes H_0$. Assume that there exists a $*$-homomorphism $\psi : \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$. We shall write $R_\psi(\mathcal{H})$ for the subset of those representations $\pi \in R(\mathcal{H})$ which are generated by ψ, i.e., $R_\psi(\mathcal{H}) = \{(\pi)(\psi(t_1)), \dots, (\pi)(\psi(t_n)),\ \pi \in \mathrm{Rep}(CB(H) \otimes C^*(\mathcal{F}_2))\}$. Then $R(\mathcal{H})$ and $R_\psi(\mathcal{H})$ are complete subsets of $(\mathbb{C}^N \otimes CB(\mathcal{H}))^\eta$. Denote by $C_\infty(R_\psi(\mathcal{H}))$ and $C_\infty(R(\mathcal{H}))$ the C^*-algebras of all "vanishing at infinity" continuous operator functions defined on $R_\psi(\mathcal{H})$ and $R(\mathcal{H})$ respectively (see [Wor2]). Then one can show that if R is $*$-wild, then $CB(H) \otimes C^*(\mathcal{F}_2) \simeq C_\infty(R_\psi(\mathcal{H}))$ and if $R(\mathcal{H})$ is "locally compact", then $*$-wildness of R is equivalent to $C_\infty(R(\mathcal{H}))/J \simeq CB(H) \otimes C^*(\mathcal{F}_2)$, where J is a C^*-ideal of $C_\infty(R(\mathcal{H}))$.

4. Examples

4.1. Unbounded representations of $\mathbb{C}[x_1, x_2]$. Let $\mathfrak{A} = \mathbb{C}[x_1, x_2]$ be a unital $*$-algebra of all polynomials in two commuting hermitian generators x_1, x_2. It is known that any irreducible integrable representation π of the algebra is one-dimensional. Recall that for $\mathbb{C}[x_1, x_2]$ a representation $\pi = (X_1 = X_1^*, X_2 = X_2^*)$ is integrable if the spectral projections of X_1 and X_2 commute. It was shown by Schmüdgen [S] that for any properly infinite von Neumann algebra $\mathcal{N}$ on a separable Hilbert space there exists a non-integrable $*$-representation $\rho = (X_1 = X_1^*, X_2 = X_2^*)$ of $\mathfrak{A}$ such that the spectral projections of these operators generate the von Neumann algebra $\mathcal{N}$. The result which was given without proof in [ST1] is that the classification of such representations "contains as a subproblem" the problem of unitary classification of representations of the $*$-algebra $\mathfrak{S}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*\rangle$. We repeat relevant material from [ST1].

Let $\alpha, \beta, \varepsilon_1, \varepsilon_2 > 0$. Consider the set $\mathcal{R}$ of all representations π of $\mathfrak{S}_2$ such that $||\pi(a)|| \le \alpha$, $||\pi(b)|| \le \beta$ and $\pi(a) \ge \varepsilon_1$, $\pi(b) \ge \varepsilon_2$. Denote by $\mathfrak{B}_{\alpha,\beta}^{\varepsilon_1,\varepsilon_2}$ the completion of $\mathfrak{S}_2/\{z : |||z||| = 0\}$ under $|||z||| = sup\{||\rho(z)||; \rho \in \mathcal{R}\}$. Consider the following construction, analogous to the one in [S]. Consider $p, q \in M_3(\mathfrak{B}_{\alpha,\beta}^{\varepsilon_1,\varepsilon_2})$ given by

$$p = \begin{pmatrix} \lambda e_1 & \mu a & 0 \\ \mu a & \lambda e_1 & \mu b \\ 0 & \mu b & \lambda e_1 \end{pmatrix}, \qquad q = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where λ, $\mu \in \mathbb{R}$ are such that $1/2 < p < 3/4$, e_n is the unit element in $M_n(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$. Let w_1, $w_2 \in M_2(M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})) \simeq M_6(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$ be defined by

$$w_1 = \begin{pmatrix} i(e_3 - 2q) & 0 \\ 0 & e_3 \end{pmatrix}, \quad w_2 = \begin{pmatrix} e_3 - 2p & -2(p-p^2)^{1/2} \\ -2(p-p^2)^{1/2} & 2p - e_3 \end{pmatrix}.$$

Since $1/2 < p < 3/4$, the element w_2 is well-defined.

Let H be an infinite-dimensional separable Hilbert space and let $\{f_k, k \in \mathbb{Z}\}$ be an orthonormal basis in H. Let P_k be the projection onto $\mathbb{C}\langle f_k\rangle$ and v be the shift operator, i.e., $vf_k = f_{k+1}$, $k \in \mathbb{Z}$. We now define v_1, v_2, $v_3 \in B(H) \odot M_6(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$ to be

$$v_1 = v \otimes e_6, \quad v_2 = v(I - P_1) \otimes e_6 + vP_1 \otimes w_1,$$
$$v_3 = v(I - P_1 - P_2) \otimes e_6 + vP_1 \otimes w_1 + vP_2 \otimes w_2.$$

Finally, we define operators U_1 and $U_2 \in B(H) \odot B(H) \odot M_6(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \subset B(\mathcal{H}) \odot \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$, $\mathcal{H} = \bigoplus_{i=1}^{6} H \otimes H$ by

$$U_1 = v \otimes E, \quad U_2 = (\sum_{i=-\infty}^{1} P_i) \otimes v_1 + P_2 \otimes v_2 + (\sum_{i=3}^{+\infty} P_i) \otimes v_3,$$

where E is the unity in $B(H) \odot M_6(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$.

Proposition 2. *There exist self-adjoint elements X_1, X_2 η $CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ such that U_1, U_2 are the Cayley transforms of X_1 and X_2 respectively.*

Proof. We denote by A the C^*-algebra $CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$. U_1, U_2 are unitary elements of $M(CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$. According to [WN, Proposition 5.1, Theorem 5.2], it suffices to show that $(I - U_i^*)A$ is dense in A, $i = 1, 2$. Suppose for the moment that the statement is false. Then $(I - U_i^*)A$ is a proper right ideal in A and there exists a pure state on A such that $f((I - U_i^*)a) = 0$, for any $a \in A$ ([D, Theorem 2.9.5]). Using the GNS procedure we can construct a representation $\pi \in \mathrm{Rep}(A, H)$ and a cyclic vector $\varphi \in H$ such that $f(a) = (\varphi \mid \pi(a)\varphi)$ for all $a \in A$. Thus

$$0 = f((I - U_i^*)a) = (\varphi \mid \pi(I - U_i^*)\pi(a)\varphi) = ((I - \pi(U_i))\varphi \mid \pi(a)\varphi).$$

Since φ is a cyclic vector, $\pi(A)\varphi$ is dense in H. This implies $(I - \pi(U_i))\varphi = 0$ and hence $\varphi \in \ker(I - \pi(U_i))$. Any non-degenerate representation π of $CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ is of the form $V id \otimes \pi_0 V^*$, where π_0 is a non-degenerate representation of $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$. It is easy to check that $\ker((id \otimes \pi_0)(U_i) - I) = \{0\}$, for any π_0. A contradiction. By [WN, Proposition 5.1], U_1, U_2 are the Cayley transforms of self-adjoint elements $X_1 \eta A$ and $X_2 \eta A$ respectively. Moreover, $D(X_i) = (I - U_i^*)(CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$, $i = 1, 2$. □

Proposition 3. *ψ: $\mathbb{C}[x_1, x_2] \ni x_i \to X_i \in (CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})^\eta$ is a $*$-homomorphism.*

Proof. Let us consider the following set $A_{1,1} = [U_1, U_2]A$, where $A = CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$. Using a simple computation one can show that

$$[U_1, U_2] = P_3 v \otimes P_3 v \otimes (e_6 - w_2) + P_2 v \otimes P_2 v \otimes (e_6 - w_1)$$

and $Q_{1,1} = P_3 \otimes P_3 \otimes ((e_6 - w_2)/2) + P_2 \otimes P_2 \otimes \begin{pmatrix} e_3 & 0 \\ 0 & 0 \end{pmatrix} \in A$ is the projection of A onto $A_{1,1}$. Now define

$$D = (U_1^* - I)(U_2^* - I)(I - Q_{1,1})A.$$

The proposition follows from the following lemma.

Lemma 1. *D is dense in A, D is a core for X_1, X_2 and $X_1X_2a = X_2X_1a$, for any $a \in D$.*

Proof. The proof is similar to that of [S, Lemmas 9.3.2, 9.3.3, 9.3.4]. We begin by showing that $X_1X_2a = X_2X_1a$ for any $a \in D$. By definition of the projection $Q_{1,1}$, we have $(I - Q_{1,1})[U_1, U_2] = (I - Q_{1,1})[U_1 - I, U_2 - I] = 0$ which implies

$$[U_1^* - I, U_2^* - I](I - Q_{1,1}) = 0 \text{ and } (U_1^* - I)(U_2^* - I)b = (U_2^* - I)(U_1^* - I)b$$

for any $b \in (I - Q_{1,1})A$. Now let $a \in D$. Then

$$a = (U_1^* - I)(U_2^* - I)b = (U_2^* - I)(U_1^* - I)b$$

with $b \in (I - Q_{1,1})A$. Remembering that each element of $D(X_i)$ is of the form $(I - U_i^*)a$, $a \in A$, $i = 1, 2$, we see that $a \in D(X_1X_2) \cap D(X_2X_1)$ and

$$b = ((X_2 - i)/2i)((X_1 - i)/2i)a = ((X_1 - i)/2i)((X_2 - i)/2i)a$$

which implies $X_2X_1a = X_1X_2a$ for any $a \in D$.

To prove that D is a core for X_1 and X_2 we have to show that $D = (I - z_i^*z_i)^{1/2}D_i$, where D_i is a dense subset in A and z_i is the z-transform of X_i, $i = 1, 2$. Let $U_i - I = V_i|U_i - I|$ be the polar decomposition of $U_i - I$ (see [WN, Proposition 0.2]). Then $V_i \in M(A)$ and V_i is unitary because $(U_i - I)A$ and $(U_i^* - I)A$ are dense in A. One can check that $(I - z_i^*z_i)^{1/2} = i|U_i - I|/2 = i(U_i^* - I)V_i/2$. Since $D = (U_1^* - I)(U_2^* - I)(I - Q_{1,1})A = (U_2^* - I)(U_1^* - I)(I - Q_{1,1})A$, it is sufficient to now show that $(U_i^* - I)(I - Q_{1,1})A$ is dense in A for $i = 1, 2$. This follows by the same method as in the proof of Proposition 2 and completes the proof of the lemma and the proposition. □

Theorem 6. *The functor F_ψ is full.*

Proof. Let H_1, H_2 be two separable Hilbert spaces and $\pi_i \in \mathrm{Rep}(\mathfrak{B}_{\alpha,\beta}, H_i)$, $i = 1, 2$. Then $\tilde{\pi}_i = id_{\mathcal{H}} \otimes \pi_i$ is a representation of $CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}^{\varepsilon_1,\varepsilon_2}$, $i = 1, 2$. Let C be a bounded operator intertwining representations $(\tilde{\pi}_1(X_1), \tilde{\pi}_1(X_2))$ and $(\tilde{\pi}_2(X_1), \tilde{\pi}_2(X_2))$, i.e.

$$C\tilde{\pi}_1(X_1) \subseteq \tilde{\pi}_2(X_1)C, \quad C\tilde{\pi}_1(X_2) \subseteq \tilde{\pi}_2(X_2)C \tag{5}$$

where $\tilde{\pi}_i$, $i = 1, 2$, are the extensions to affiliated elements. (5) now implies

$$C\tilde{\pi}_1(U_i) = \tilde{\pi}_2(U_i)C, \quad C\tilde{\pi}_1(U_i)^* = \tilde{\pi}_2(U_i)^*C, \ i = 1, 2.$$

We have to show that $C = I \otimes A$ where $A \in I(\pi_1, \pi_2)$. This follows by direct calculation using simple arguments similar to that of [S, Theorem 9.4.1]. We leave it to the reader. □

Corollary 2. *There exists a $*$-homomorphism $\phi\colon \mathbb{C}[x_1, x_2] \to (CB(\mathcal{H}) \otimes C^*(\mathcal{F}_2))^\eta$ such that the corresponding functor F_ϕ is full.*

Proof. One can easily show that the C^*-algebra $\mathfrak{B}_{\alpha,\beta}^{\varepsilon_1,\varepsilon_2}$ is $*$-wild and there exists a $*$-homomorphism $\varphi\colon \mathfrak{B}_{\alpha,\beta}^{\varepsilon_1,\varepsilon_2} \to M_n(C^*(\mathcal{F}_2))$, $n > 0$, which generates the full functor F_φ from the category $\mathrm{Rep}(C^*(\mathcal{F}_2))$ into the category $\mathrm{Rep}(\mathfrak{B}_{\alpha,\beta}^{\varepsilon_1,\varepsilon_2})$. Setting $\phi = \varphi \circ \psi$, we obtain that ϕ is a $*$-homomorphism from $\mathbb{C}[x_1, x_2]$ to $(CB(\mathcal{H}) \otimes C^*(\mathcal{F}_2))^\eta$ and the corresponding functor is full. □

Remark 5. $\psi\colon C^*(\mathcal{F}_2) \ni u_i \to U_i \in M(CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \subset (CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})^\eta$, $i = 1, 2$, is a $*$-homomorphism and the corresponding functor F_ψ is full. However, U_1, U_2 do not generate $CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$, because $CB(\mathcal{H}) \otimes \mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ is a non-unital C^*-algebra.

4.2. Representations of the double commutator relations. Consider the $*$-algebra $\mathfrak{A} = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 0, x_i^* = x_i, i = 1, 2\rangle$. Clearly, any representation of the commutative algebra $\mathbb{C}[x_1, x_2]$ is a representation of the $*$-algebra $\mathfrak{A}$. It follows from the preceding example that non-integrable representations can be complicated, i.e. the problem of unitary classification of such $*$-representations contains as a subproblem the problem of unitary classification of representations of $C^*(\mathcal{F}_2)$. In this example we show that the class of representations π defined on a domain formed by analytic vectors for $\pi(x_1)$ and $\pi(x_2)$ is $*$-wild.

Let, as before, $\alpha, \beta, \varepsilon_1, \varepsilon_2 > 0$ and let $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ be the C^*-algebra which is defined in the first example. On the Hilbert space $H = L_2(\mathbb{R}, dx)$ we consider the multiplication operator q by x and the operator of differentiation $p = i\frac{d}{dx}$. Let a_1, a_2 denote the following elements in $M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$:

$$a_1 = \begin{pmatrix} l_1 e_1 & 0 & 0 \\ 0 & l_2 e_1 & 0 \\ 0 & 0 & l_3 e_1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$$

where $l_i \in \mathbb{R}$, with $l_i \neq l_j$, $i \neq j$, e_n is the unity of $M_n(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta})$. Since any closed operator on H is affiliated with $CB(H)$, we have that q, $p\ \eta\ CB(H)$ and there exist uniquely defined self-adjoint elements $X_1 = e_3 \otimes q$ and $X_2 = a_1 \otimes p + a_2 \otimes I_H$ such that $X_1, X_2\ \eta\ M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$. Here I_H is the identity operator on H.

Proposition 4. *$\psi\colon \mathfrak{A} \ni x_i \to X_i \in (M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H))^\eta$ is a $*$-homomorphism.*

Proof. Let G be the Heisenberg group, i.e., the group of matrices of the form

$$g = g(t, s, r) = \begin{pmatrix} 1 & t & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \ t, s, r \in \mathbb{R}.$$

Then $u\colon G \to B(H)$ defined by $u(g(t, 0, 0)) = e^{itq}$, $u(g(0, s, 0)) = e^{isp}$, $u(g(0, 0, r)) = e^{ir} I$ is a unitary representation of G in $CB(H)$. By [WN], given a unitary representation u of a real Lie group in a C^*-algebra A, there always exists a dense in A domain Φ which is invariant with respect to operators of infinitesimal representation of the Lie algebra and is their essential domain. Let $D = M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \odot \Phi$. Clearly D satisfies all the required conditions: D is dense in $M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$, D is a core for X_1, X_2 and $[X_1, [X_1, X_2]]a = 0$ for any $a \in D$. ◻

We now denote by R the set of all representations π of $\mathfrak{A}$ on a Hilbert space H_π defined on a dense invariant domain consisting of analytic vectors for $\pi(x_1)$, $\pi(x_2)$.

Theorem 7. *R is a $*$-wild class of representations.*

Proof. Let id be the identity representation of $M_3(CB(H))$ on $H \oplus H \oplus H$. Given a representation π of $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$, $((\pi \otimes id)(X_1), (\pi \otimes id)(X_2))$ defines a representation which belongs to the class R. Here $\pi \otimes id$ is the unique extension to affiliated elements of $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta} \otimes M_3(CB(H)) = M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$. To prove that R is $*$-wild it is sufficient to show that X_1, X_2 generate $M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$. By Theorem 2, the statement will be proved once we prove that X_1, X_2 separate representations of $M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$ and that $(I+X_2^2)^{-1}(I+X_1^2)^{-1}(I+X_2^2)^{-1} \in M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$.

We realize $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ as an algebra of operators in a Hilbert space $\mathcal{H}$. Let F be the Fourier transform operator in $H = L_2(\mathbb{R}, dx)$. Then $\mathcal{F} = I^3_{\mathcal{H}} \otimes F$ is a bounded operator acting on the space $(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}) \otimes H$ and such that $\mathcal{F}X_1\mathcal{F}^{-1} = I^3_{\mathcal{H}} \otimes p$, $\mathcal{F}X_1\mathcal{F}^{-1} = a_1 \otimes q + a_2 \otimes I_H$ (here $I^3_{\mathcal{H}}$ is the identity operator on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$). The operator $(1+q^2)^{-1}(1+p^2)^{-1}(1+q^2)^{-1}$ is integral with the kernel $K(x,y) = \frac{1}{2}(1+x^2)e^{-|x-y|}(1+y^2)^{-1}$ as it is easy to check. Moreover, this operator is positive with finite trace which implies that it is compact. Therefore,

$$r = I^3_{\mathcal{H}} \otimes (I_H + q^2)^{-1}(I_H + p^2)^{-1}(I_H + q^2)^{-1} \in M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H).$$

Let

$$s = (I + (a_1 \otimes q + a_2 \otimes I)^2)^{-1}(I + I \otimes q^2).$$

Clearly, s is bounded. Moreover, s is affiliated with $M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$. This is due to the following lemma.

Lemma 2. *Let A be a C^*-algebra, $S \eta A$, $v \in M(A)$. Assume that $[v, z_S] = [v, z_S^*] = 0$. Then there exists $T \eta A$ such that $Ta = vSa$, for any $a \in D(S)$.*

Proof. We shall use [Wor1, Theorem 2.3]. Let

$$a = (I - z_S z_S^*)^{1/2},\ b = vz_S,\ c = vz_S,\ d = (I - z_S^* z_S)^{1/2}.$$

One can easily see that a, b, c, $d \in M(A)$, $ab = cd$, and the sets $a^*A = aA$, $dA = d^*A$ are dense in A.

For

$$Q = \begin{pmatrix} d & -c^* \\ b & a \end{pmatrix}$$

we have

$$Q^*Q = \begin{pmatrix} I - z_S^* z_S + v^* v z_S^* z_S & 0 \\ 0 & I - z_S z_S^* + v v^* z_S z_S^* \end{pmatrix}.$$

Let π be an irreducible representation of A on a Hilbert space H_π, id the canonical representation of $M_2(\mathbb{C})$ on $\mathbb{C}^2$. Since $(I - z_S^* z_S)^{1/2}A$, $(I - z_S z_S^*)^{1/2}A$ are dense in A and $v^* v z_S^* z_S$, $v v^* z_S z_S^* \geq 0$, one can easily deduce that the range of $(id \otimes \pi)(Q^*Q)$ is dense in H_π which implies that $(id \otimes \pi)(Q)$ is dense in H_π. Using [Wor1, Proposition 2.5] we see that $Q(A \oplus A)$ is dense in $A \oplus A$. By [Wor1, Theorem 2.3] there exists an element $T\ \eta\ A$ such that $dA = (I - z_S^* z_S)^{1/2}A$ is a core of T and $T(I - z_S^* z_S)^{1/2}x = vz_S x$ for any $x \in A$. Since $(I - z_S^* z_S)^{1/2}A = D(S)$, $Ta = vSa$ for any $a \in D(S)$. ∎

It is known that multipliers are the only bounded elements affiliated with a C^*-algebra. Therefore $s \in M(M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H))$ and $srs \in M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$. On the other hand,

$$srs = \mathcal{F}(I + X_2^2)^{-1}(I + X_1^2)^{-1}(I + X_2^2)^{-1}\mathcal{F}^{-1}$$

which yields $(I+X_2^2)^{-1}(I+X_1^2)^{-1}(I+X_2^2)^{-1} \in M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$. Since $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ is a $*$-wild C^*-algebra, we conclude that R is a $*$-wild class of representations due to Proposition 1. The fact that X_1, X_2 separate representations follows from [NT, Theorem 3] and the fact that any representation of $M_3(\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}) \otimes CB(H)$ is of the form $V^{-1}(\pi \otimes id)V$, where V is a unitary operator, π is a representation of $\mathfrak{B}^{\varepsilon_1,\varepsilon_2}_{\alpha,\beta}$ and id is the identity representation of $M_3(CB(H))$. The proof is finished. $\square$

4.3. On unbounded idempotents. Let B be an algebra and let p_1, p_2, p_3, p_4 be idempotents in B such that $p_1 + p_2 + p_3 + p_4 = 0$. Idempotents with this property were studied in [BES]. They arise, in particular, in the study of logarithmic residues in Banach algebras. In [BES] it is shown that non-trivial zero sums of four idempotents do not exist in Banach algebras, however, there are unbounded idempotents in a Hilbert space having this property. Unbounded representations of a $*$-algebra generated by idempotents p_1, p_2, p_3, p_4 satisfying $p_1+p_2+p_3+p_4 = 0$ were discussed in [ST2]. In this example we shall see that the class of representations defined in [ST2] is $*$-wild. Let $p = (p_1 + p_2)/2$, $q = (p_3 + p_4)/2$, $r = (p_1 - p_2)/2$, $s = (p_3 - p_4)/2$ (we have $p_1 = p + r$, $p_2 = p - r$, $p_3 = q + s$, $p_4 = q - s$). Direct computation shows that they satisfy the following relations:

$$pr = r(1 - p), \quad ps = s(-1 - p), r^2 = p(1 - p), \quad s^2 = -p(p + 1). \tag{6}$$

We assume additionally that

$$pr^* = rp,\ ps^* = sp,\ p = p^*, \tag{7}$$

and denote by $\mathfrak{A}$ the $*$-algebra generated by p, q, r, s and relations (6)–(7). We define $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ as follows: a family of closed operators (P, Q, R, S) on a Hilbert space H belongs to R iff there exists a linear dense subset $\Phi \subseteq H$ such that $\Phi \subseteq \mathcal{H}_w(P, Q, R^*R, S^*S)$, Φ is a core for the operators R, R^*, S and S^* and relations (6)–(7) hold on Φ. Here $H_w(P, Q, R, S)$ denotes the set of analytical vectors for P, Q, R, S. It was proved in [ST1] that a subclass $\tilde{R}$ of R defined by the condition $\ker P \neq \{0\}$ is manageable, i.e. there exists a C^*-algebra A and elements p, q, r, s η A generating A and such that $\tilde{R} = \{(\pi(p), \pi(q), \pi(r), \pi(s)) \mid \pi \in \mathrm{Rep}(A)\}$. Moreover, all such representations were classified up to a unitary equivalence.

Let $\alpha, \beta > 0$. Consider the set $\mathcal{R}$ of all representations π of $\mathfrak{S}_2$ such that $||\pi(a)|| \le \alpha$, $||\pi(b)|| \le \beta$. Denote by $\mathcal{A}_{\alpha,\beta}$ the completion of $\mathfrak{S}_2/\{z : |||z||| = 0\}$ under $|||z||| = sup\{||\rho(z)||; \rho \in \mathcal{R}\}$. Let H be a separable infinite-dimensional Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$, and let P_k be the orthoprojection onto $\mathbb{C}\langle e_k\rangle$, $k \in \mathbb{Z}$. We consider operators v, w defined by $ve_k = e_{k+1}$, $ve_{k+1} = e_k$ if k is even and $we_k = e_{k+1}$, $we_{k+1} = e_k$ if k is odd. Clearly, $(P_{2k} + P_{2k+1})H$ $((P_{2k+1} + P_{2k+2})H)$ is invariant with respect to v (respectively w). Now let

$$\tilde{p} = \sum_{k\neq 0}(-1)^{k+1}kP_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},\ \tilde{q} = \sum_{k\neq 0}(-1)^{k}kP_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

$$\tilde{r} = (\sum_{k\neq 0}(2k+1)vP_{2k} - 2kvP_{2k+1}) \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + vP_0 \otimes \begin{pmatrix} e & 0 & 0 \\ 0 & 2e & 0 \end{pmatrix},$$

$$\tilde{s} = (\sum_{k\neq 0}(2k+1)wP_{2k+2} - (2k+2)P_{2k+1}) \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$
$$+ wP_0 \otimes \begin{pmatrix} e & e & a+ib \\ 0 & e & e \end{pmatrix}.$$

Here e is the identity element in $\mathcal{A}_{\alpha,\beta}$. We write $\mathcal{H}$ for the Hilbert space $P_0H \oplus P_0H \oplus P_0H \oplus ((I-P_0)H \oplus (I-P_0)H)$. Direct verification shows that $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ are affiliated with $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ and separate representations of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$; moreover, since

$$(I+\tilde{p}^2)^{-1} = \sum_{k\neq 0}(1+k^2)^{-1}P_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, (I+\tilde{p}^2)^{-1} \in CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}.$$

Therefore, by Theorem 2, $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ generate the C^*-algebra $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$.

Let $D = l.s.\ \{a \otimes b \mid a \in \mathcal{F}, b \in \mathcal{A}_{\alpha,\beta}\}$, where $\mathcal{F}$ is the space of finite-dimensional operators in $\mathcal{H}$. Then D is dense in $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ and invariant with respect to $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$, D is a core for the elements $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ and $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ satisfy relations (6)–(7) on D. Moreover, any representation $(\pi(\psi)(p))$, $\pi(\psi(q))$, $\pi(\psi(r)$, $\pi(\psi(s)))$ belongs to R, where $\psi(x) = \tilde{x}$, π is a representation of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$. It implies that the class R is $*$-wild.

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