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LAWLESSNESS AND RANK RESTRICTIONS IN CERTAIN FINITARY GROUPS

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ABSTRACT. We give two applications of the recent classification of locally finite simple finitary skew linear groups. We show that certain irreducible finitary skew linear groups of infinite dimension generate the variety of all groups and have infinite Prüfer rank.

1. INTRODUCTION

In this note, we give an example of how the recent classification of locally finite simple finitary skew linear groups (see [1] and [12]) can be used. We shall only be interested in finitary skew linear groups over division rings which are locally finite-dimensional over some subfield. These groups have a local Zariski topology, which can be applied to pull up results from the theory of linear groups.

Throughout, D will be a division ring which is locally finite-dimensional over some subfield F, unless stated otherwise. Let V be a left vector space over D. A *finitary skew linear group* on V is a subgroup of

$$\mathrm{FGL}(V) = \{g \in \mathrm{GL}(V) : \dim_D V(g-1) < \infty\}.$$

A permutation group G on the set Ω is called *finitary* if every permutation in G fixes all but finitely many elements of Ω.

A group G is said to be *lawless* if it generates the variety of all groups; that is, if G satisfies no non-trivial law.

The group G has *finite Prüfer rank* r if every finitely generated subgroup of G can be generated by r elements, and r is the least integer with this property. Of course, if no such r exists, we say that G has *infinite Prüfer rank*.

P. M. Neumann ([3], Theorem 1) has proved that a transitive finitary permutation group of infinite degree is lawless. For a linear group $G \leq \mathrm{GL}(n, F)$, V. Platonov has proved that

1. G satisfies a non-trivial law if and only if G is soluble-by-finite ([10], Theorem 10.15).
2. If G has finite Prüfer rank r, then it is soluble-by-finite. If further, the characteristic of F is $p > 0$, then G has an abelian normal subgroup of index bounded in terms of n, p and r ([10], Theorem 10.9).

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We shall extend these results to finitary skew linear versions. We prove the following:

Theorem 1. *Let G be a finitary skew linear group on V.*

1. *The group G is either lawless or is locally-soluble by locally-finite.*
2. *If G is irreducible and $\dim_D V$ is infinite, then G is lawless.*

Theorem 2. *Let G be a group of finite Prüfer rank.*

1. *Suppose that G is a finitary permutation group on Ω. Then G has finite orbits on Ω.*
2. *Suppose that G is a finitary skew linear group on V.*
   (a) *The group G is locally-soluble by locally-finite. If further,* $\mathrm{char} D > 0$, *then G is abelian by locally-finite.*
   (b) *If G is irreducible, then $\dim_D V$ is finite.*

Remarks. 1. The unitriangular group $G = \mathrm{Tr}_1(3, \mathbb{Z})$ is polycyclic and thus has finite Prüfer rank. Now G is not abelian by locally-finite, so the hypothesis of positive characteristic in Theorem 2, Part 2(a) is required.

2. By Theorem 1, Part 2, one can remove the hypothesis "$\eta(G)$ is non-trivial" from [5], Corollary, Part 2, when considering division rings which are locally finite-dimensional over a subfield.

2. The Zariski topology

This section extends some of the results of Puglisi [6], Section 2, to our situation.

Let $G \leq \mathrm{FGL}(V)$ and let X be a finitely generated subgroup of G. Now X is skew linear over D, say $X = \langle x_1, \ldots, x_s \rangle \leq \mathrm{GL}(n, D)$ and further, $X \leq GL(n, E)$ where E is the subring of D generated by F together with the entries of the matrices $x_1, \ldots, x_s$. Then E is finite-dimensional over F, say of dimension m, and so $X \leq \mathrm{GL}(mn, F)$. Therefore X carries the usual Zariski topology of linear groups (see [10], Chapter 5) and has a connected component X° containing the identity. Using the proof of [9], Proposition 2.2, we see that if Y is a finitely generated subgroup of G containing X, then the topology induced on X from that on Y coincides with the Zariski topology on X. Thus X° is well-defined and $X^\circ \leq Y^\circ$.

Let G^- denote the union of the subgroups X° where X ranges through all the finitely generated subgroups of G. The following result is well-known.

2.1. Lemma. *Let $G \leq \mathrm{FGL}(V)$.*

1. *$G^- \lhd G$ and G/G^- is locally finite.*
2. *If $\mathfrak{X}$ is a class of groups and if for each finitely generated subgroup X of G we have $X^\circ \in \mathfrak{X}$, then G^- is locally-$\mathfrak{X}$.*

Proof. Now $X^\circ \lhd X$ and the X° form a local system of G^-, where X is a finitely generated subgroup of G. Thus $G^- \lhd G$ and 2 follows. A finitely generated subgroup of G/G^- has the form G^-X/G^- where X is some finitely generated subgroup of G. Now X° has finite index in X by [10], Lemma 5.2, and $X^\circ \leq G^-$. Thus G^-X/G^- is finite. □

2.2. Proposition. *Let $G \leq \mathrm{FGL}(V)$.*

1. *Suppose that $\mathfrak{X}$ is a subgroup closed class of groups such that if $P \leq L$, where L is linear over a field and $P \in \mathfrak{X}$, then the Zariski closure of P in L lies in $\mathfrak{X}$. If G is locally $\mathfrak{X}$-by-finite, then G is locally-$\mathfrak{X}$ by locally-finite.*

2. *Suppose that $\mathfrak{X}$ is a variety. If G is locally $\mathfrak{X}$-by-finite, then G is $\mathfrak{X}$ by locally-finite.*

Part 2 of Proposition 2.2 for D a field is [6], Theorem 2.3.

Proof. Let H be a finitely generated subgroup of G. Then there is a normal subgroup N of H such that $N \in \mathfrak{X}$ and H/N is finite.

In the case of 1, the closure $\overline{N}$ of N in H lies in $\mathfrak{X}$. Now $N \leq \overline{N}$ so that $(H : \overline{N})$ is finite. Thus $H^\circ \leq \overline{N}$ by [10], Lemma 5.3. Since $\mathfrak{X}$ is subgroup closed, $H^\circ \in \mathfrak{X}$.

In the case of 2, H has a closed normal $\mathfrak{X}$-subgroup M of finite index in H by [10], Lemma 10.7 (for example, in the notation there $H \cap \mathcal{A}_F(N)$). By [10], Lemma 5.3 again, $H^\circ \leq M$. A variety is always subgroup closed and thus $H^\circ \in \mathfrak{X}$.

The result now follows from Lemma 2.1, using the fact that a variety is locally closed. □

The proofs of Theorem 1, Part 1 and Theorem 2, Part 2(a). Let $G \leq \mathrm{FGL}(V)$. Suppose that G has finite Prüfer rank. Let X be a finitely generated subgroup of G. Now X is linear and so by Platonov's Theorem, X has a soluble normal subgroup N of finite index in X. By [10], Theorem 5.11(i), the closure $\overline{N}$ of N is X is soluble. By Proposition 2.2, Part 1, G is locally-soluble by locally-finite.

Suppose that $\mathrm{char} D = \mathrm{char} F > 0$. In this case, X is abelian-by-finite by Platonov's Theorem. The abelian groups form a variety. Thus by Proposition 2.2, Part 2, G is abelian by locally-finite.

We have now proved Theorem 2, Part 2(a). A similar argument gives Theorem 1, Part 1. □

Theorem 2.1 of [6] is a finitary linear version of the Tits' alternative. Using the finite-dimensional Tits' alternative ([10], Theorem 10.16) and Proposition 2.2 we obtain:

2.3. Proposition. *Let $G \leq \mathrm{FGL}(V)$. Then G is either locally-soluble by locally-finite or contains a non-cyclic free group.*

A. Lichtman (see [8], Theorem 1.4.9) has constructed a finitely generated skew linear group G which satisfies a non-trivial law (so, in particular, contains no non-cyclic free subgroup) and is not soluble-by-finite. Because G is finitely generated, it cannot be locally-soluble by locally-finite. Thus Theorem 1, Part 1 and Proposition 2.3 are not true when D is an arbitrary division ring.

3. Techniques for primitive groups

For this section only, D is any division ring. Recently much progress has been made in determining the structure of locally finite primitive irreducible finitary skew linear groups. The following results rely on the Classification of Finite Simple Groups.

3.1. Theorem. *Let G be a locally finite primitive irreducible subgroup of* $\mathrm{FGL}(V)$ *where* $\dim_D V$ *is infinite. Then G contains a simple normal subgroup $S \lhd G$.*

This was proved by Phillips ([4], Theorem A) for D a field and by Wehrfritz ([12], Theorem) in the general case. In fact S turns out to be G', a result proved by Leinen and Puglisi ([2], Theorem B) and independently by Redford [7] for D a field, and by Wehrfritz [12] for all division rings.

We also know all possibilities for S in Theorem 3.1.

3.2. Theorem. *Let G be a non-linear simple locally finite subgroup of* $\mathrm{FGL}(V)$*. Then G is either an infinite alternating group, or is one of the following types over a locally finite field: finitary symplectic, finitary special unitary, finitary orthogonal or special transvection.*

This classification was obtained by J. I. Hall ([1], Theorem 1.3) for D a field and by Wehrfritz ([12], Corollary 1) for any division ring D. For a description of the groups in Theorem 3.2, see [1].

3.3. Proposition. *Let G be a non-linear locally finite simple subgroup of* $\mathrm{FGL}(V)$*. Then G contains a copy of every finite group.*

Proof. This is left as an exercise using Theorem 3.2. For example, if G is a special transvection group over the field K, it contains $\mathrm{SL}(n, K)$ for all n (see [1] or [2]). Thus G contains a copy of every finite group. □

4. The proofs

We have already established Theorem 1, Part 1 and Theorem 2, Part 2(a).

4.1. Proposition. *Let G be a transitive finitary permutation group on Ω and suppose that G has finite Prüfer rank. Then Ω is finite.*

Theorem 2, Part 1 clearly follows from Proposition 4.1. We note two key examples. Let n be any positive integer and p be any prime. Then the direct product $C_p^{(n)}$ of n copies of C_p has Prüfer rank n. In particular, any infinite direct product of copies of C_p has infinite Prüfer rank. Since $C_p^{(n)}$ embeds into $\mathrm{Alt}(p^n + 2)$ for all n, the group $\mathrm{Alt}(\Omega)$ has infinite Prüfer rank when Ω is infinite.

Proof of Proposition 4.1. Suppose that Ω is infinite. Using P. M. Neumann's terminology (see [3], Section 2), G is either primitive, almost primitive or totally imprimitive. By the Jordan-Wielandt Theorem ([13], Satz 9.4), a primitive group contains $\mathrm{Alt}(\Delta)$ for some infinite set Δ. Also an almost primitive group has an image containing some $\mathrm{Alt}(\Delta)$. By the remark preceding this proof, we need only consider the totally imprimitive case.

Since G is locally finite, we can choose an element $g \in G$ of prime order p. Put $\Delta = \mathrm{supp}_\Omega(g)$, the support of g on Ω. Now Δ is a finite non-empty set. By [3], Theorem 2.4(i), we can choose a G-congruence with blocks $(\Omega_i)_{i \in I}$ such that $\Delta \subseteq \Omega_1$, say.

Using the transitivity of G, there is $x_i \in G$ such that $\Omega_1 x_i = \Omega_i$. Now for every $i \in I$, we have $\mathrm{supp}_\Omega(g^{x_i}) = \Delta x_i \subseteq \Omega_i$. Thus the g^{x_i} commute. Furthermore, it is easy to see that $\langle g^{x_i} | i \in I\rangle \cong C_p^{(I)}$. Now I is an infinite set, so $C_p^{(I)}$ has infinite Prüfer rank. Thus G has infinite Prüfer rank. This completes the proof of the proposition. □

Conclusion of the proofs. Let G be an irreducible subgroup of $\mathrm{FGL}(V)$ with $\dim_D(V)$ infinite. Suppose that either G has finite Prüfer rank or satisfies a non-trivial law. If G is imprimitive, then it has an image isomorphic to a transitive finitary permutation group of infinite degree. By [3], Theorem 1, and Proposition 4.1 above, this is not possible. Thus G is primitive.

Now G is locally-soluble by locally-finite. By the main theorem of [11], G is locally finite. Applying Theorem 3.1 and Proposition 3.3, we see that G contains copies of all finite groups. In particular G contains $C_p^{(n)}$ for all n, so G has infinite

Prüfer rank. Also the variety $\mathcal{V}$ generated by G contains the class of all finite groups. Any free group is residually finite and any variety is residually closed. Thus $\mathcal{V}$ contains every free group and hence it contains every group, that is, G generates the variety of all groups. These contradictions give the results. □

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