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L^p BOUNDEDNESS OF LOCALIZATION OPERATORS ASSOCIATED TO LEFT REGULAR REPRESENTATIONS

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Abstract. We prove an L^p boundedness result for localization operators associated to left regular representations of locally compact and Hausdorff groups and give an application to wavelet multipliers.

1. Introduction

Let G be a locally compact and Hausdorff group on which the left Haar measure is denoted by μ. Let X be an infinite dimensional, separable and complex Hilbert space in which the inner product and the norm are denoted by $(\ ,\ )$ and $\|\ \|$ respectively. Let $B(X)$ be the C^*-algebra of all bounded linear operators on X. Let $\pi : G \to B(X)$ be an irreducible and unitary representation of G on X such that there exists a nonzero element φ in X for which

$$\int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) < \infty. \tag{1.1}$$

Then we call the representation $\pi : G \to B(X)$ a square-integrable representation of G on X. We call any φ in X, such that $\|\varphi\| = 1$ and (1.1) is valid, an admissible wavelet for the square-integrable representation $\pi : G \to B(X)$ of G on X and we define the constant c_φ by

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g).$$

The formula in the following theorem is known as the resolution of the identity formula.

Theorem 1.1. *Let φ be an admissible wavelet for the square-integrable representation $\pi : G \to B(X)$ of G on X. Then*

$$(x, y) = \frac{1}{c_\varphi} \int_G (x, \pi(g)\varphi)(\pi(g)\varphi, y) d\mu(g)$$

for all x and y in X.

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Theorem 1.1 as stated is an abridged version of Theorem 3.1 in the paper [8] by Grossmann, Morlet and Paul, where the original contributions due to Duflo and Moore in [4] are acknowledged. See also the paper [1] by Carey in this connection. A proof of Theorem 1.1 can also be found in the book [20] by Wong.

Let $F \in L^1(G) \cup L^\infty(G)$. Then it is easy to prove that the linear operator $L_{F,\varphi} : X \to X$ defined by

$$(L_{F,\varphi}x, y) = \frac{1}{c_\varphi} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y)d\mu(g), \quad x, y \in X, \tag{1.2}$$

is a bounded linear operator on X. Using the Riesz-Thorin theorem, which we recall in Section 3, it is proved in the paper [9] by He and Wong, and in the book [20] by Wong that if $F \in L^p(G)$, $1 \le p \le \infty$, then there exists a unique bounded linear operator $L_{F,\varphi} : X \to X$ such that the formula (1.1) is valid for all F in $L^1(G) \cup L^\infty(G)$, and x and y in X. Furthermore,

$$\|L_{F,\varphi}\|_{B(X)} \le {c_\varphi}^{-\frac{1}{p}} \|F\|_{L^p(G)},$$

where $\| \, \|_{B(X)}$ is the norm in $B(X)$.

Remark 1.2. The bounded linear operators $L_{F,\varphi} : X \to X$ are known as localization operators and have been studied in [5] by Du and Wong, in [9] by He and Wong, and in the book [20] by Wong. The origin of these operators can be traced back to the study of a class of bounded linear operators introduced by Daubechies in [2, 3] in signal analysis. The rationale for the terminology lies in the observation that by Theorem 1.1, the bounded linear operator $L_{F,\varphi} : X \to X$ is simply the identity operator I on X if $F(g) = 1$, $g \in G$. Thus, the role of the symbol F is to localize on the group G so as to produce a nontrivial bounded linear operator with various applications in signal analysis and quantization among others. In fact, it is proved in the paper [9] by He and Wong, and the book [20] by Wong that if $F \in L^p(G)$, $1 \le p \le \infty$, then the localization operator $L_{F,\varphi} : X \to X$ is in the Schatten-von Neumann class S_p.

In another direction, guided by the Landau-Pollak-Slepian operator in signal analysis, a theory of wavelet multipliers has been introduced and studied in [6] by Du and Wong, [10] by He and Wong and [20] by Wong. To recall, let $\sigma \in L^\infty(\mathbb{R}^n)$. Then we define the linear operator $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T_\sigma u = \mathcal{F}^{-1}\sigma\mathcal{F}u, \quad u \in L^2(\mathbb{R}^n),$$

where $\mathcal{F}^{-1}$ and $\mathcal{F}$ are the inverse Fourier transform and the Fourier transform respectively. The Fourier transform $\mathcal{F}u$, sometimes denoted by $\hat{u}$, of a function u in $L^2(\mathbb{R}^n)$ is given by

$$\mathcal{F}u = \lim_{R\to\infty} \widehat{\chi_R u},$$

where χ_R is the characteristic function of the ball with center at the origin and radius R,

$$\widehat{\chi_R u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\chi_R(x)u(x)dx, \quad \xi \in \mathbb{R}^n,$$

and the convergence of $\widehat{\chi_R u}$ to $\mathcal{F}u$ as $R \to \infty$ is understood to be in $L^2(\mathbb{R}^n)$. It follows from Plancherel's theorem that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.

Let $B(L^2(\mathbb{R}^n))$ be the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$. Let $\pi : \mathbb{R}^n \to B(L^2(\mathbb{R}^n))$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix\cdot\xi}u(x), \quad x, \xi \in \mathbb{R}^n,$$

for all u in $L^2(\mathbb{R}^n)$. Let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then it is proved in the paper [10] by He and Wong, and in the book [20] by Wong that

$$\langle \varphi u, \varphi v\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \langle u, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, v\rangle d\xi$$

for all u and v in the Schwartz space $\mathcal{S}$, where $\langle\, , \,\rangle$ is the inner product in $L^2(\mathbb{R}^n)$.

Let $\sigma \in L^1(\mathbb{R}^n) \cup L^\infty(\mathbb{R}^n)$. Then it is easy to prove that the linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\langle P_{\sigma,\varphi}u, v\rangle = \int_{\mathbb{R}^n} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, v\rangle d\xi \tag{1.3}$$

for all u and v in $L^2(\mathbb{R}^n)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. Using the Riesz-Thorin theorem again, it is proved in [10] by He and Wong, and in [20] by Wong that if $\sigma \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then there exists a unique bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that the formula (1.3) is valid for all σ in $L^1(\mathbb{R}^n) \cup L^\infty(\mathbb{R}^n)$, and u and v in $L^2(\mathbb{R}^n)$. Moreover,

$$\|P_{\sigma,\varphi}\|_{B(L^2(\mathbb{R}^n))} \le (2\pi)^{-\frac{n}{p}} \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{\frac{2}{p'}} \|\varphi\|_{L^p(\mathbb{R}^n)},$$

where p' is the conjugate index of p and $\|\,\|_{B(L^2(\mathbb{R}^n))}$ is the norm in $B(L^2(\mathbb{R}^n))$.

Remark 1.3. In order to understand the linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ better, let $\sigma \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then it is proved in [10] by He and Wong, and in [20] by Wong that the bounded linear operators $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are equal. Thus, the function φ plays the role of the admissible wavelet in a localization operator. Had the admissible wavelet φ in the linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ been replaced by the function φ_0 on $\mathbb{R}^n$ given by

$$\varphi_0(x) = 1, \quad x \in \mathbb{R}^n,$$

we would have obtained the pseudo-differential operator or Fourier multiplier $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$(T_\sigma u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(\xi)\hat{u}(\xi)d\xi, \quad x \in \mathbb{R}^n.$$

See, for instance, the books [11] and [19] by Kumano-go and Wong for detailed expositions of pseudo-differential operators. Therefore it is reasonable to call the bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a wavelet multiplier. That the Landau-Pollak-Slepian operator in signal analysis (studied in the fundamental papers [12, 13] by Landau and Pollak, [14, 15] by Slepian, and [16] by Slepian and Pollak) is in fact a wavelet multiplier is a result proved in [10] by He and Wong, and in [20] by Wong. As in the case of localization operators, it is proved in [10] by He and Wong, and in [20] by Wong that if $\sigma \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then the wavelet multiplier $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in the Schatten-von Neumann class S_p.

In this paper, we let $B(L^p(G))$, $1 \le p \le \infty$, be the Banach algebra of all bounded linear operators on $L^p(G)$ and look at admissible wavelets associated to left regular representations $\pi : G \to B(L^p(G))$ of unimodular, locally compact and Hausdorff groups G on $L^p(G)$, and define localization operators from $L^p(G)$ into $L^p(G)$ in terms of these admissible wavelets and symbols F in $L^r(G)$, $1 \le r \le \infty$. An L^p boundedness result for all these localization operators is established for $1 \le p \le \infty$. As an application, we show that wavelet multipliers can be looked at as such localization operators when the underlying group is taken to be the additive group $\mathbb{R}^n$ and we obtain the L^p boundedness of wavelet multipliers from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$.

2. L^1 boundedness and L^∞ boundedness

Let G be a unimodular, locally compact and Hausdorff group on which the left Haar measure is denoted by μ. Let $\pi : G \to B(L^p(G))$ be the left regular representation of G on $L^p(G)$, $1 \le p \le \infty$, i.e., $(\pi(g)u)(h) = u(g^{-1}h)$, $g, h \in G$, for all u in $L^p(\mathbb{R}^n)$. Let $\varphi \in \bigcap_{1\le p\le\infty} L^p(G)$ be such that $\|\varphi\|_{L^2(G)} = 1$. By Young's inequality on, say, page 52 of the book [7] by Folland, we get

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) \le \|\varphi\|^2_{L^1(G)} < \infty. \tag{2.1}$$

In view of (2.1), the fact that $\|\varphi\|_{L^2(G)} = 1$ and the notion of admissible wavelets given in Section 1, we call the function φ an admissible wavelet for the left regular representation π of G on $L^p(G)$.

Remark 2.1. In order to obtain an explicit formula for c_φ, we assume that G is a second countable, unimodular, type I, locally compact and Hausdorff group. We let $\hat{G}$ be the set of all equivalence classes of irreducible and unitary representations of G on $L^2(G)$. Let ν be the Plancherel measure on $\hat{G}$. Then by Plancherel's theorem,

$$\begin{aligned} c_\varphi &= \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) \\ &= \int_G |(\varphi * \varphi^*)(g)|^2 d\mu(g) \\ &= \int_{\hat{G}} \mathrm{tr}\{\hat{\psi}(\omega)\hat{\psi}(\omega)^*\} d\nu(\omega), \end{aligned} \tag{2.2}$$

where φ^* is the function on G defined by

$$\varphi^*(g) = \overline{\varphi(g^{-1})}, \quad g \in G,$$

$$\psi(g) = (\varphi * \varphi^*)(g), \quad g \in G, \tag{2.3}$$

$\mathrm{tr}\{\cdots\}$ is the trace of the trace class operator $\{\cdots\}$ and $\{\cdots\}^*$ denotes the adjoint of the bounded linear operator $\{\cdots\}$. Now,

$$\hat{\psi}(\omega) = \hat{\varphi}(\omega)^*\hat{\varphi}(\omega), \quad \omega \in \hat{G}. \tag{2.4}$$

Thus, by (2.2)-(2.4),

$$c_\varphi = \int_{\hat{G}} \mathrm{tr}\{|\hat{\varphi}(\omega)|^4\} d\nu(\omega), \tag{2.5}$$

where

$$|\hat{\varphi}(\omega)| = (\hat{\varphi}(\omega)^*\hat{\varphi}(\omega))^{\frac{1}{2}}. \tag{2.6}$$

Therefore by (2.5) and (2.6), we get

$$c_\varphi = \int_{\hat{G}} \|\hat{\varphi}(\omega)\|_{S_4}^4 d\nu(\omega), \tag{2.7}$$

where $\| \|_{S_4}$ is the norm in the Schatten-von Neumann class S_4. Finally, we can write (2.7) as

$$c_\varphi = \|\hat{\varphi}\|_{L^4(\hat{G},S_4)}^4,$$

where $L^4(\hat{G}, S_4)$ is the Banach space of all S_4-valued functions f on $\hat{G}$ for which

$$\int_{\hat{G}} \|f(\omega)\|_{S_4}^4 d\nu(\omega) < \infty$$

and the norm $\| \|_{L^4(\hat{G},S_4)}$ in it is given by

$$\|f\|_{L^4(\hat{G},S_4)} = \left\{ \int_{\hat{G}} \|f(\omega)\|_{S_4}^4 d\nu(\omega) \right\}^{\frac{1}{4}}$$

for all f in $L^4(\hat{G}, S_4)$.

Let $F \in L^1(G) \cup L^\infty(G)$. Then for $1 \le p \le \infty$, we define the localization operator $L_{F,\varphi} : L^p(G) \to L^p(G)$ associated to the symbol F and the admissible wavelet φ by

$$(L_{F,\varphi}u, v) = \frac{1}{c_\varphi} \int_G (u, \pi(g)\varphi)(\pi(g)\varphi, v) d\mu(g) \tag{2.8}$$

for all u in $L^p(G)$ and v in $L^{p'}(G)$, where

$$(u, v) = \int_G u(g)\overline{v(g)} d\mu(g), \quad u \in L^p(G), v \in L^{p'}(G).$$

Proposition 2.2. *Let $F \in L^1(G)$. Then for $1 \le p \le \infty$, the localization operator $L_{F,\varphi} : L^p(G) \to L^p(G)$ is a bounded linear operator and*

$$\|L_{F,\varphi}\|_{B(L^p(G))} \le \frac{1}{c_\varphi} \|\varphi\|_{L^p(G)} \|\varphi\|_{L^{p'}(G)} \|F\|_{L^1(G)},$$

where $\| \|_{B(L^p(G))}$ is the norm in $B(L^p(G))$.

Proof. By (2.8), Hölder's inequality and the fact that π is the left regular representation of G, we get

$$\begin{aligned} |(L_{F,\varphi}u, v)| &= \frac{1}{c_\varphi} \int_G |F(g)||(u, \pi(g)\varphi)||(\pi(g)\varphi, v)| d\mu(g) \\ &\le \frac{1}{c_\varphi} \|\varphi\|_{L^p(G)} \|\varphi\|_{L^{p'}(G)} \|F\|_{L^1(G)} \|u\|_{L^p(G)} \|v\|_{L^{p'}(G)} \end{aligned}$$

for all u in $L^p(G)$ and v in $L^{p'}(G)$. This completes the proof. ∎

Proposition 2.3. *Let $F \in L^\infty(G)$. Then for $1 \le p \le \infty$, the localization operator $L_{F,\varphi} : L^p(G) \to L^p(G)$ is a bounded linear operator and*

$$\|L_{F,\varphi}\|_{B(L^p(G))} \le \frac{1}{c_\varphi} \|\varphi\|_{L^1(G)}^2 \|F\|_{L^\infty(G)}.$$

Proof. By (2.8), we get

$$|(L_{F,\varphi}u,v)| = \frac{1}{c_\varphi}\int_G |F(g)||(u,\pi(g)\varphi)||(\pi(g)\varphi,v)|d\mu(g) \tag{2.9}$$

for all u in $L^p(G)$ and v in $L^{p'}(G)$. Now, using the fact that π is the left regular representation of G and Young's inequality on, say, page 52 of the book [7] by Folland again, we get

$$\left\{\int_G |(u,\pi(g)\varphi)|^p d\mu(g)\right\}^{\frac{1}{p}} \le \|u\|_{L^p(G)}\|\varphi\|_{L^1(G)} \tag{2.10}$$

and

$$\left\{\int_G |(\pi(g)\varphi,v)|^{p'} d\mu(g)\right\}^{\frac{1}{p'}} \le \|\varphi\|_{L^1(G)}\|v\|_{L^{p'}(G)} \tag{2.11}$$

for all u in $L^p(G)$ and v in $L^{p'}(G)$. So, by (2.9), Hölder's inequality, (2.10) and (2.11), we get

$$|(L_{F,\varphi}u,v)| \quad \le \quad \frac{1}{c_\varphi}\|\varphi\|^2_{L^1(G)}\|F\|_{L^\infty(G)}\|u\|_{L^p(G)}\|v\|_{L^{p'}(G)}$$

for all u in $L^p(G)$ and v in $L^{p'}(G)$. This completes the proof. □

3. L^p BOUNDEDNESS

We begin with the following result, which is known as the Riesz-Thorin theorem. References on this theorem include [17] by Stein and Weiss and [18] by Wong.

Theorem 3.1. *Let (X,μ) be a measure space and (Y,ν) a σ-finite measure space. Let T be a linear transformation with domain $\mathcal{D}$ consisting of all μ-simple functions f on X such that*

$$\mu\{x \in X : f(x) \neq 0\} < \infty$$

and such that the range of T is contained in the set of all ν-measurable functions on Y. Suppose that α_1, α_2, β_1 and β_2 are real numbers in $[0,1]$ and there exist positive constants M_1 and M_2 such that

$$\|Tf\|_{L^{\frac{1}{\beta_j}}(Y)} \le M_j\|f\|_{L^{\frac{1}{\alpha_j}}(X)}, \quad f \in \mathcal{D},\ j = 1,2.$$

Then for $0 < \theta < 1$,

$$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$$

and

$$\beta = (1-\theta)\beta_1 + \theta\beta_2,$$

we have

$$\|Tf\|_{L^{\frac{1}{\beta}}(Y)} \le M_1^{1-\theta}M_2^{\theta}\|f\|_{L^{\frac{1}{\alpha}}(X)}, \quad f \in \mathcal{D}.$$

Theorem 3.2. *Let $F \in L^r(G)$, $1 \le r \le \infty$. Then for $1 \le p \le \infty$, there exists a unique bounded linear operator $L_{F,\varphi} : L^p(G) \to L^p(G)$ such that formula (2.2) is valid for all u in $L^p(G)$, v in $L^{p'}(G)$ and μ-simple functions F on G for which*

$$\mu\{g \in G : F(g) \neq 0\} < \infty.$$

Moreover,

$$\|L_{F,\varphi}\|_{B(L^p(G))} \le \frac{1}{c_\varphi}\|\varphi\|_{L^p(G)}^{\frac{1}{r}}\|\varphi\|_{L^{p'}(G)}^{\frac{1}{r}}\|\varphi\|_{L^1(G)}^{\frac{2}{r'}}\|F\|_{L^r(G)}.$$

Proof. We only need to prove the theorem for $1 < r < \infty$. To this end, let $u \in L^p(G)$. Let T_u be the linear transformation with domain $\mathcal{D}$ consisting of all μ-simple functions on G with the property that

$$\mu\{g \in G : F(g) \neq 0\} < \infty,$$

and defined by

$$T_u F = L_{F,\varphi} u, \quad u \in \mathcal{D}.$$

Then by Proposition 2.2,

(3.1)

$$\|T_u F\|_{L^p(G)} = \|L_{F,\varphi}u\|_{L^p(G)} \le \frac{1}{c_\varphi}\|\varphi\|_{L^p(G)}\|\varphi\|_{L^{p'}(G)}\|u\|_{L^p(G)}\|F\|_{L^1(G)},$$

and by Proposition 2.3,

$$\|T_u F\|_{L^p(G)} = \|L_{F,\varphi}u\|_{L^p(G)} \le \frac{1}{c_\varphi}\|\varphi\|_{L^1(G)}^2\|u\|_{L^p(G)}\|F\|_{L^\infty(G)} \tag{3.2}$$

for all F in $\mathcal{D}$. In order to apply the Riesz-Thorin theorem, we let $\alpha_1 = 1$, $\alpha_2 = 0$ and $\beta_1 = \beta_2 = \frac{1}{p}$. Let $\alpha = \frac{1}{r}$. Then $\theta = \frac{1}{r'}$, where r' is the conjugate index of r. Hence $\alpha = \frac{1}{r}$ and $\beta = \frac{1}{p}$, and we get by (3.1), (3.2) and the Riesz-Thorin theorem,

$$\begin{aligned}\|L_{F,\varphi}u\|_{L^p(G)} &= \|T_u F\|_{L^p(G)} \\ &\le \frac{1}{c_\varphi}\|\varphi\|_{L^p(G)}^{\frac{1}{r}}\|\varphi\|_{L^{p'}(G)}^{\frac{1}{r}}\|\varphi\|_{L^1(G)}^{\frac{2}{r'}}\|F\|_{L^r(G)}\|u\|_{L^p(G)}\end{aligned}$$

for all F in $\mathcal{D}$. Since $\mathcal{D}$ is dense in $L^r(G)$, we can use a density argument and the proof is complete. □

4. Wavelet multipliers

For $1 \le p \le \infty$, we let $\pi : \mathbb{R}^n \to B(L^p(\mathbb{R}^n))$ be the left regular representation of the additive group $\mathbb{R}^n$ on $L^p(\mathbb{R}^n)$, i.e.,

$$(\pi(y)u)(x) = u(x - y), \quad x, y \in \mathbb{R}^n.$$

Let $\varphi \in \bigcap_{1\le p\le\infty} L^p(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then, by Plancherel's theorem,

$$\begin{aligned}c_\varphi &= \int_{\mathbb{R}^n} |\langle\varphi, \pi(y)\varphi\rangle|^2 dy \\ &= \int_{\mathbb{R}^n} |\langle\hat\varphi, M_y\hat\varphi\rangle|^2 dy, \end{aligned}\tag{4.1}$$

where

$$(M_y\hat\varphi)(\xi) = e^{iy\cdot\xi}\hat\varphi(\xi), \quad \xi \in \mathbb{R}^n. \tag{4.2}$$

So, by (4.1), (4.2) and Plancherel's theorem,

$$\begin{aligned}c_\varphi &= \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{-iy\cdot\xi}|\hat\varphi(\xi)|^2 d\xi\right|^2 dx \\ &= (2\pi)^n\|\hat\varphi\|_{L^4(\mathbb{R}^n)}^4. \end{aligned}\tag{4.3}$$

The following result follows immediately from Theorem 3.2.

Theorem 4.1. *Let $\sigma \in L^r(\mathbb{R}^n)$, $1 \le r \le \infty$, and let $\varphi \in \bigcap_{1\le p\le\infty} L^p(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^2(G)} = 1$. Then for $1 \le p \le \infty$, there exists a unique localization operator $L_{\sigma,\varphi} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ such that*

$$\langle L_{\sigma,\varphi}u, v\rangle = (2\pi)^{-n}\|\hat{\varphi}\|^{-4}_{L^4(\mathbb{R}^n)} \int_{\mathbb{R}^n} \sigma(y)\langle u, \pi(y)\varphi\rangle\langle \pi(y)\varphi, v\rangle dy$$

for all u and v in $\mathcal{S}$. Moreover,

$$\|L_{\sigma,\varphi}\|_{B(L^p(\mathbb{R}^n))} \le (2\pi)^{-n}\|\hat{\varphi}\|^{-4}_{L^4(\mathbb{R}^n)}\|\varphi\|^{\frac{1}{r}}_{L^p(\mathbb{R}^n)}\|\varphi\|^{\frac{1}{r}}_{L^{p'}(\mathbb{R}^n)}\|\varphi\|^{\frac{2}{r'}}_{L^1(\mathbb{R}^n)}\|\sigma\|_{L^r(\mathbb{R}^n)}.$$

Now, for all σ in $L^r(\mathbb{R}^n)$, $1 \le r \le \infty$, and u and v in the Schwartz space $\mathcal{S}$, we get by (2.2), (4.3), and Plancherel's theorem,

$$\begin{aligned}
\langle L_{\sigma,\varphi}u, v\rangle &= \frac{1}{c_\varphi}\int_{\mathbb{R}^n} \sigma(y)\langle u, \pi(y)\varphi\rangle\langle \pi(y)\varphi, v\rangle dy \\
&= (2\pi)^{-n}\|\hat{\varphi}\|^{-4}_{L^4(\mathbb{R}^n)}\int_{\mathbb{R}^n} \sigma(y)\langle \hat{u}, M_y\hat{\varphi}\rangle\langle M_y\hat{\varphi}, \hat{v}\rangle dy \\
&= \|\hat{\varphi}\|^{-4}_{L^4(\mathbb{R}^n)}\int_{\mathbb{R}^n} \sigma(y)(\widehat{\hat{u}\hat{\varphi}})(y)\overline{(\widehat{\hat{v}\hat{\varphi}})(y)}dy \\
&= (2\pi)^{\frac{n}{2}}\langle T_\sigma(\hat{\varphi}\hat{u}), \hat{\varphi}\hat{v}\rangle \\
&= (2\pi)^{\frac{n}{2}}\|\hat{\varphi}\|^{-4}_{L^4(\mathbb{R}^n)}\langle \mathcal{F}^{-1}\overline{\hat{\varphi}}T_\sigma\hat{\varphi}\mathcal{F}u, v\rangle.
\end{aligned} \tag{4.4}$$

Thus, in the case when $p = 2$, (4.4) tells us that the localization operator $L_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitarily equivalent to the wavelet multiplier $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

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