$L^p$ BOUNDEDNESS OF LOCALIZATION OPERATORS ASSOCIATED TO LEFT REGULAR REPRESENTATIONS

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Abstract. We prove an $L^p$ boundedness result for localization operators associated to left regular representations of locally compact and Hausdorff groups and give an application to wavelet multipliers.

1. Introduction

Let $G$ be a locally compact and Hausdorff group on which the left Haar measure is denoted by $\mu$. Let $X$ be an infinite dimensional, separable and complex Hilbert space in which the inner product and the norm are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$ respectively. Let $B(X)$ be the $C^*$-algebra of all bounded linear operators on $X$. Let $\pi : G \to B(X)$ be an irreducible and unitary representation of $G$ on $X$ such that there exists a nonzero element $\varphi$ in $X$ for which

\begin{equation}
\int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) < \infty.
\end{equation}

Then we call the representation $\pi : G \to B(X)$ a square-integrable representation of $G$ on $X$. We call any $\varphi$ in $X$, such that $\|\varphi\| = 1$ and (1.1) is valid, an admissible wavelet for the square-integrable representation $\pi : G \to B(X)$ of $G$ on $X$ and we define the constant $c_\varphi$ by

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g).$$

The formula in the following theorem is known as the resolution of the identity formula.

Theorem 1.1. Let $\varphi$ be an admissible wavelet for the square-integrable representation $\pi : G \to B(X)$ of $G$ on $X$. Then

$$(x, y) = \frac{1}{c_\varphi} \int_G (x, \pi(g)\varphi)(\pi(g)\varphi, y)d\mu(g)$$

for all $x$ and $y$ in $X$.

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Then we define the linear operator $T$ follows from Plancherel’s theorem that $T\hat{\chi}$ to localize on the group $G$ operator.

Let $F \in L^1(G) \cup L^\infty(G)$. Then it is easy to prove that the linear operator $L_{F,\varphi} : X \rightarrow X$ defined by

$$(1.2) \quad (L_{F,\varphi}x, y) = \frac{1}{c_\varphi} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y) d\mu(g), \quad x, y \in X,$$

is a bounded linear operator on $X$. Using the Riesz-Thorin theorem, which we recall in Section 3, it is proved in the paper [9] by He and Wong, and in the book [20] by Wong that if $F \in L^p(G), 1 \leq p \leq \infty$, then there exists a unique bounded linear operator $L_{F,\varphi} : X \rightarrow X$ such that the formula (1.1) is valid for all $F$ in $L^1(G) \cup L^\infty(G)$, and $x$ and $y$ in $X$. Furthermore,

$$\|L_{F,\varphi}\|_{B(X)} \leq c_\varphi^{-\frac{1}{2}} \|F\|_{L^p(G)},$$

where $\| \cdot \|_{B(X)}$ is the norm in $B(X)$.

Remark 1.2. The bounded linear operators $L_{F,\varphi} : X \rightarrow X$ are known as localization operators and have been studied in [5] by Du and Wong, in [9] by He and Wong, and in the book [20] by Wong. The origin of these operators can be traced back to the study of a class of bounded linear operators introduced by Daubechies in [2, 3] in signal analysis. The rationale for the terminology lies in the observation that by Theorem 1.1, the bounded linear operator $L_{F,\varphi} : X \rightarrow X$ is simply the identity operator $I$ on $X$ if $F(g) = 1, g \in G$. Thus, the role of the symbol $F$ is to localize on the group $G$ so as to produce a nontrivial bounded linear operator with various applications in signal analysis and quantization among others. In fact, it is proved in the paper [9] by He and Wong, and the book [20] by Wong that if $F \in L^p(G), 1 \leq p \leq \infty$, then the localization operator $L_{F,\varphi} : X \rightarrow X$ is in the Schatten-von Neumann class $S_p$.

In another direction, guided by the Landau-Pollak-Slepian operator in signal analysis, a theory of wavelet multipliers has been introduced and studied in [6] by Daubechies and in the book [20] by Wong. To recall, let $\sigma \in L^\infty(\mathbb{R}^n)$. Then we define the linear operator $T_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by

$$T_\sigma u = \mathcal{F}^{-1}\sigma \mathcal{F} u, \quad u \in L^2(\mathbb{R}^n),$$

where $\mathcal{F}^{-1}$ and $\mathcal{F}$ are the inverse Fourier transform and the Fourier transform respectively. The Fourier transform $\mathcal{F} u$, sometimes denoted by $\hat{u}$, of a function $u$ in $L^2(\mathbb{R}^n)$ is given by

$$\mathcal{F} u = \lim_{R \rightarrow \infty} \hat{\chi}_R u,$$

where $\chi_R$ is the characteristic function of the ball with center at the origin and radius $R$,

$$\hat{\chi}_R u(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \chi_R(x) u(x) dx, \quad \xi \in \mathbb{R}^n,$$

and the convergence of $\hat{\chi}_R u$ to $\mathcal{F} u$ as $R \rightarrow \infty$ is understood to be in $L^2(\mathbb{R}^n)$. It follows from Plancherel’s theorem that $T_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a bounded linear operator.
Let $B(L^2(\mathbb{R}^n))$ be the $C^*$-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$. Let $\pi : \mathbb{R}^n \to B(L^2(\mathbb{R}^n))$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix\cdot \xi}u(x), \quad x, \xi \in \mathbb{R}^n,$$

for all $u$ in $L^2(\mathbb{R}^n)$. Let $\varphi$ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then it is proved in the paper [10] by He and Wong, and in [20] by Wong that

$$\langle \varphi u, \varphi v \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \langle u, \pi(\xi)\varphi \rangle \langle \pi(\xi)\varphi, v \rangle d\xi$$

for all $u$ and $v$ in the Schwartz space $S$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^n)$.

Let $\sigma \in L^1(\mathbb{R}^n) \cup L^\infty(\mathbb{R}^n)$. Then it is easy to prove that the linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$(1.3) \quad \langle P_{\sigma,\varphi}u, v \rangle = \int_{\mathbb{R}^n} \sigma(\xi) \langle u, \pi(\xi)\varphi \rangle \langle \pi(\xi)\varphi, v \rangle d\xi$$

for all $u$ and $v$ in $L^2(\mathbb{R}^n)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. Using the Riesz-Thorin theorem again, it is proved in [10] by He and Wong, and in [20] by Wong that if $\sigma \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then there exists a unique bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that the formula (1.3) is valid for all $\sigma$ in $L^1(\mathbb{R}^n) \cup L^\infty(\mathbb{R}^n)$, and $u$ and $v$ in $L^2(\mathbb{R}^n)$. Moreover,

$$\|P_{\sigma,\varphi}\|_{B(L^2(\mathbb{R}^n))} \leq (2\pi)^{-\frac{n}{2}} \|\varphi\|_{L^\infty(\mathbb{R}^n)} \|\varphi\|_{L^p(\mathbb{R}^n)},$$

where $p'$ is the conjugate index of $p$ and $\| \cdot \|_{B(L^2(\mathbb{R}^n))}$ is the norm in $B(L^2(\mathbb{R}^n))$.

Remark 1.3. In order to understand the linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ better, let $\sigma \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$. Then it is proved in [10] by He and Wong, and in [20] by Wong that the bounded linear operators $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are equal. Thus, the function $\varphi$ plays the role of the admissible wavelet in a localization operator. Had the admissible wavelet $\varphi$ in the linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ been replaced by the function $\varphi_0$ on $\mathbb{R}^n$ given by

$$\varphi_0(x) = 1, \quad x \in \mathbb{R}^n,$$

we would have obtained the pseudo-differential operator or Fourier multiplier $T_{\sigma} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$(T_{\sigma}u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot \xi} \sigma(\xi) \hat{u}(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

See, for instance, the books [11] and [19] by Kumano-go and Wong for detailed expositions of pseudo-differential operators. Therefore it is reasonable to call the bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a wavelet multiplier. That the Landau-Pollak-Slepian operator in signal analysis (studied in the fundamental papers [12] [13] by Landau and Pollak, [14] [15] by Slepian, and [16] by Slepian and Pollak) is in fact a wavelet multiplier is a result proved in [10] by He and Wong, and in [20] by Wong. As in the case of localization operators, it is proved in [10] by He and Wong, and in [20] by Wong that if $\sigma \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then the wavelet multiplier $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in the Schatten-von Neumann class $S_p$. 
In this paper, we let $B(L^p(G))$, $1 \leq p \leq \infty$, be the Banach algebra of all bounded linear operators on $L^p(G)$ and look at admissible wavelets associated to left regular representations $\pi : G \to B(L^p(G))$ of unimodular, locally compact and Hausdorff groups $G$ on $L^p(G)$, and define localization operators from $L^p(G)$ into $L^p(G)$ in terms of these admissible wavelets and symbols $F$ in $L^r(G)$, $1 \leq r \leq \infty$. An $L^p$ boundedness result for all these localization operators is established for $1 \leq p \leq \infty$. As an application, we show that wavelet multipliers can be looked at as such localization operators when the underlying group is taken to be the additive group $\mathbb{R}^n$ and we obtain the $L^p$ boundedness of wavelet multipliers from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.

2. $L^1$ BOUNDEDNESS AND $L^\infty$ BOUNDEDNESS

Let $G$ be a unimodular, locally compact and Hausdorff group on which the left Haar measure is denoted by $\mu$. Let $\pi : G \to B(L^p(G))$ be the left regular representation of $G$ on $L^p(G)$, $1 \leq p \leq \infty$, i.e., $(\pi(g)u)(h) = u(g^{-1}h)$, $g, h \in G$, for all $u$ in $L^p(G)$). Let $\varphi \in \bigcap_{1 \leq p \leq \infty} L^p(G)$ be such that $\|\varphi\|_{L^2(G)} = 1$. By Young’s inequality on, say, page 52 of the book \cite{14} by Folland, we get

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2d\mu(g) \leq \|\varphi\|^2_{L^1(G)} < \infty. \tag{2.1}$$

In view of (2.1), the fact that $\|\varphi\|_{L^2(G)} = 1$ and the notion of admissible wavelets given in Section 1, we call the function $\varphi$ an admissible wavelet for the left regular representation $\pi$ of $G$ on $L^p(G)$.

Remark 2.1. In order to obtain an explicit formula for $c_\varphi$, we assume that $G$ is a second countable, unimodular, type I, locally compact and Hausdorff group. We let $\hat{G}$ be the set of all equivalence classes of irreducible and unitary representations of $G$ on $L^2(G)$. Let $\nu$ be the Plancherel measure on $\hat{G}$. Then by Plancherel’s theorem,

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2d\mu(g) = \int_G |(\varphi*\varphi^*)(g)|^2d\mu(g) = \int_{\hat{G}} \text{tr}\{\hat{\psi}(\omega)\hat{\varphi}(\omega)^*\}d\nu(\omega), \tag{2.2}$$

where $\varphi^*$ is the function on $G$ defined by

$$\varphi^*(g) = \overline{\varphi(g^{-1})}, \quad g \in G, \tag{2.3}$$

$$\psi(g) = (\varphi*\varphi^*)(g), \quad g \in G, \tag{2.4}$$

$\text{tr}\{\cdots\}$ is the trace of the trace class operator $\{\cdots\}$ and $\{\cdots\}^*$ denotes the adjoint of the bounded linear operator $\{\cdots\}$. Now,

$$\hat{\psi}(\omega) = \hat{\varphi}(\omega)^*\hat{\varphi}(\omega), \quad \omega \in \hat{G}.$$  

Thus, by (2.2)-(2.4),

$$c_\varphi = \int_{\hat{G}} \text{tr}\{|\hat{\varphi}(\omega)|^4\}d\nu(\omega), \tag{2.5}$$

where

$$|\hat{\varphi}(\omega)| = (\hat{\varphi}(\omega)^*\hat{\varphi}(\omega))^{\frac{1}{2}}. \tag{2.6}$$
Therefore by (2.5) and (2.6), we get

\[ (2.7) \quad c_\varphi = \int_\hat{G} \| \hat{\varphi}(\omega) \|_{S_4}^4 \omega d\mu(\omega), \]

where \( \| \cdot \|_{S_4} \) is the norm in the Schatten-von Neumann class \( S_4 \). Finally, we can write (2.7) as

\[ c_\varphi = \| \hat{\varphi} \|_{L^4(\hat{G}, S_4)}^4, \]

where \( L^4(\hat{G}, S_4) \) is the Banach space of all \( S_4 \)-valued functions \( f \) on \( \hat{G} \) for which

\[ \int_\hat{G} \| f(\omega) \|_{S_4}^4 \omega d\mu(\omega) < \infty \]

and the norm \( \| \cdot \|_{L^4(\hat{G}, S_4)} \) in it is given by

\[ \| f \|_{L^4(\hat{G}, S_4)} = \left\{ \int_\hat{G} \| f(\omega) \|_{S_4}^4 \omega d\mu(\omega) \right\}^{\frac{1}{4}} \]

for all \( f \) in \( L^4(\hat{G}, S_4) \).

Let \( F \in L^1(G) \cup L^\infty(G) \). Then for \( 1 \leq p \leq \infty \), we define the localization operator \( L_{F,\varphi} : L^p(G) \to L^p(G) \) associated to the symbol \( F \) and the admissible wavelet \( \varphi \) by

\[ (2.8) \quad (L_{F,\varphi}u, v) = \frac{1}{c_\varphi} \int_G (u(\pi(g)\varphi)(\pi(g)\varphi), v) d\mu(g) \]

for all \( u \) in \( L^p(G) \) and \( v \) in \( L^{p'}(G) \), where

\[ (u, v) = \int_G u(g)v(g)d\mu(g), \quad u \in L^p(G), v \in L^{p'}(G). \]

**Proposition 2.2.** Let \( F \in L^1(G) \). Then for \( 1 \leq p \leq \infty \), the localization operator \( L_{F,\varphi} : L^p(G) \to L^p(G) \) is a bounded linear operator and

\[ \| L_{F,\varphi} \|_{B(L^p(G))} \leq \frac{1}{c_\varphi} \| \varphi \|_{L^p(G)} \| \varphi \|_{L^{p'}(G)} \| F \|_{L^1(G)}, \]

where \( \| \cdot \|_{B(L^p(G))} \) is the norm in \( B(L^p(G)) \).

**Proof.** By (2.8), Hölder’s inequality and the fact that \( \pi \) is the left regular representation of \( G \), we get

\[ |(L_{F,\varphi}u, v)| = \frac{1}{c_\varphi} \int_G |F(g)||(u(\pi(g)\varphi)||(\pi(g)\varphi), v)| d\mu(g) \leq \frac{1}{c_\varphi} \| \varphi \|_{L^p(G)} \| \varphi \|_{L^{p'}(G)} \| F \|_{L^1(G)} \| u \|_{L^p(G)} \| v \|_{L^{p'}(G)} \]

for all \( u \) in \( L^p(G) \) and \( v \) in \( L^{p'}(G) \). This completes the proof. \( \square \)

**Proposition 2.3.** Let \( F \in L^\infty(G) \). Then for \( 1 \leq p \leq \infty \), the localization operator \( L_{F,\varphi} : L^p(G) \to L^p(G) \) is a bounded linear operator and

\[ \| L_{F,\varphi} \|_{B(L^p(G))} \leq \frac{1}{c_\varphi} \| \varphi \|_{L^1(G)}^2 \| F \|_{L^\infty(G)}. \]
Proof. By (2.8), we get
\begin{align}
(L_F \varphi, u, v) = \frac{1}{c_\varphi} \int_G \left| \varphi \right|^p d\mu(g)
\end{align}
for all $u$ in $L^p(G)$ and $v$ in $L^p(G)$. Now, using the fact that $\pi$ is the left regular representation of $G$ and Young’s inequality on, say, page 52 of the book [7] by Folland again, we get
\begin{align}
\left\{ \int_G |(u, \pi(g)\varphi)|^p d\mu(g) \right\}^{\frac{1}{p}} \leq \|u\|_{L^p(G)} \|\varphi\|_{L^1(G)}
\end{align}
and
\begin{align}
\left\{ \int_G |(\pi(g)\varphi, v)|^p d\mu(g) \right\}^{\frac{1}{p}} \leq \|\varphi\|_{L^1(G)} \|v\|_{L^p(G)}
\end{align}
for all $u$ in $L^p(G)$ and $v$ in $L^p(G)$. So, by (2.9), Hölder’s inequality, (2.10) and (2.11), we get
\begin{align}
|(L_F \varphi, u, v)| \leq \frac{1}{c_\varphi} \|\varphi\|_{L^1(G)}^2 \|F\|_{L^\infty(G)} \|u\|_{L^p(G)} \|v\|_{L^p(G)}
\end{align}
for all $u$ in $L^p(G)$ and $v$ in $L^p(G)$. This completes the proof.

3. $L^p$ boundedness

We begin with the following result, which is known as the Riesz-Thorin theorem. References on this theorem include [17] by Stein and Weiss and [18] by Wong.

**Theorem 3.1.** Let $(X, \mu)$ be a measure space and $(Y, \nu)$ a $\sigma$-finite measure space. Let $T$ be a linear transformation with domain $D$ consisting of all $\mu$-simple functions $f$ on $X$ such that
\begin{align}
\mu\{ x \in X : f(x) \neq 0 \} < \infty
\end{align}
and such that the range of $T$ is contained in the set of all $\nu$-measurable functions on $Y$. Suppose that $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are real numbers in $[0,1]$ and there exist positive constants $M_1$ and $M_2$ such that
\begin{align}
\|Tf\|_{L^\beta(Y)} \leq M_j \|f\|_{L^\alpha(X)}, \quad f \in D, \quad j = 1, 2.
\end{align}
Then for $0 < \theta < 1$,
\begin{align}
\alpha = (1 - \theta)\alpha_1 + \theta \alpha_2
\end{align}
and
\begin{align}
\beta = (1 - \theta)\beta_1 + \theta \beta_2,
\end{align}
we have
\begin{align}
\|Tf\|_{L^\beta(Y)} \leq M_1^{1-\theta} M_2^\theta \|f\|_{L^\alpha(X)}, \quad f \in D.
\end{align}

**Theorem 3.2.** Let $F \in L^r(G)$, $1 \leq r \leq \infty$. Then for $1 \leq p \leq \infty$, there exists a unique bounded linear operator $L_{F, \varphi} : L^p(G) \rightarrow L^p(G)$ such that formula (2.2) is valid for all $u$ in $L^p(G)$, $v$ in $L^p(G)$ and $\mu$-simple functions $F$ on $G$ for which
\begin{align}
\mu\{ g \in G : F(g) \neq 0 \} < \infty.
\end{align}
Moreover,
\[ \|L_{F_\varphi}\|_{B(L^p(G))} \leq \frac{1}{c_\varphi} \|\varphi\|_{L^p(G)} \|\varphi\|_{L^{r'}(G)} \|\varphi\|_{L^1(G)} \|F\|_{L^r(G)}. \]

**Proof.** We only need to prove the theorem for \(1 < r < \infty\). To this end, let \(u \in L^p(G)\). Let \(T_u\) be the linear transformation with domain \(D\) consisting of all \(\mu\)-simple functions on \(G\) with the property that
\[ \mu\{g \in G : F(g) \neq 0\} < \infty, \]
and defined by
\[ T_u F = L_{F_\varphi} u, \quad u \in D. \]
Then by Proposition 2.2,
\[ \|T_u F\|_{L^p(G)} = \|L_{F_\varphi} u\|_{L^p(G)} \leq \frac{1}{c_\varphi} \|\varphi\|_{L^p(G)} \|\varphi\|_{L^{r'}(G)} \|u\|_{L^p(G)} \|F\|_{L^1(G)}, \]
and by Proposition 2.3,
\[ \|T_u F\|_{L^p(G)} = \|L_{F_\varphi} u\|_{L^p(G)} \leq \frac{1}{c_\varphi} \|\varphi\|_{L^p(G)} \|\varphi\|_{L^{r'}(G)} \|u\|_{L^p(G)} \|F\|_{L^\infty(G)} \]
for all \(F \in D\). In order to apply the Riesz-Thorin theorem, we let \(\alpha_1 = 1\), \(\alpha_2 = 0\) and \(\beta_1 = \beta_2 = \frac{1}{p}\). Let \(\alpha = \frac{1}{p}\) and \(\beta = \frac{1}{r}\), where \(r'\) is the conjugate index of \(r\). Hence \(\alpha = \frac{1}{p}\) and \(\beta = \frac{1}{r}\), and we get by (3.1), (3.2) and the Riesz-Thorin theorem,
\[ \|L_{F_\varphi} u\|_{L^p(G)} = \|T_u F\|_{L^p(G)} \leq \frac{1}{c_\varphi} \|\varphi\|_{L^p(G)} \|\varphi\|_{L^{r'}(G)} \|u\|_{L^p(G)} \|F\|_{L^1(G)} \|
\]
for all \(F \in D\). Since \(D\) is dense in \(L^r(G)\), we can use a density argument and the proof is complete. \(\square\)

4. **Wavelet multipliers**

For \(1 \leq p \leq \infty\), let \(\pi : \mathbb{R}^n \to B(L^p(\mathbb{R}^n))\) be the left regular representation of the additive group \(\mathbb{R}^n\) on \(L^p(\mathbb{R}^n)\), i.e.,
\[ (\pi(y)u)(x) = u(x - y), \quad x, y \in \mathbb{R}^n. \]
Let \(\varphi \in \bigcap_{1 \leq \mu < \infty} L^p(\mathbb{R}^n)\) be such that \(\|\varphi\|_{L^2(\mathbb{R}^n)} = 1\). Then, by Plancherel’s theorem,
\[ c_\varphi = \int_{\mathbb{R}^n} |\langle \varphi, \pi(y)\varphi \rangle|^2 dy = \int_{\mathbb{R}^n} |\langle \hat{\varphi}, M_y \hat{\varphi} \rangle|^2 dy, \]
where
\[ (M_y \hat{\varphi})(\xi) = e^{iy \cdot \xi} \hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n. \]
So, by (4.1), (4.2) and Plancherel’s theorem,
\[ c_\varphi = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-iy \cdot \xi} |\hat{\varphi}(\xi)|^2 d\xi \right|^2 dx = (2\pi)^n \|\hat{\varphi}\|_{L^4(\mathbb{R}^n)}^4. \]
The following result follows immediately from Theorem 3.2.

**Theorem 4.1.** Let \( \sigma \in L^r(\mathbb{R}^n), 1 \leq r \leq \infty \), and let \( \varphi \in \bigcap_{1 \leq p \leq \infty} L^p(\mathbb{R}^n) \) be such that \( \| \varphi \|_{L^2(G)} = 1 \). Then for \( 1 \leq p \leq \infty \), there exists a unique localization operator \( L_{\sigma, \varphi} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \) such that

\[
\langle L_{\sigma, \varphi} u, v \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(y) \langle u, \pi(y) \varphi \rangle \langle \pi(y) \varphi, v \rangle dy
\]

for all \( u \) and \( v \) in \( S \). Moreover,

\[
\| L_{\sigma, \varphi} \|_{B(L^p(\mathbb{R}^n))} \leq (2\pi)^{-n} \| \varphi \|_{L^4(\mathbb{R}^n)} \| \varphi \|_{L^p(\mathbb{R}^n)} \| \varphi \|_{L^p(\mathbb{R}^n)} \| \varphi \|_{L^1(\mathbb{R}^n)} \| \sigma \|_{L^r(\mathbb{R}^n)}.
\]

Now, for all \( \sigma \in L^r(\mathbb{R}^n), 1 \leq r \leq \infty \), and \( u \) and \( v \) in the Schwartz space \( S \), we get by (2.2), (4.3), and Plancherel’s theorem,

\[
\langle L_{\sigma, \varphi} u, v \rangle = \frac{1}{c_\varphi} \int_{\mathbb{R}^n} \sigma(y) \langle u, \pi(y) \varphi \rangle \langle \pi(y) \varphi, v \rangle dy
\]

\[
= (2\pi)^{-n} \| \varphi \|_{L^4(\mathbb{R}^n)} \int_{\mathbb{R}^n} \sigma(y) \langle u, M_y \hat{\varphi} \rangle \langle M_y \hat{\varphi}, v \rangle dy
\]

\[
= \| \varphi \|_{L^4(\mathbb{R}^n)} \int_{\mathbb{R}^n} \sigma(y) \langle \hat{u} \hat{\varphi} \rangle \langle \hat{v} \hat{\varphi} \rangle \langle \hat{y} \hat{\varphi} \rangle \langle \hat{y} \hat{\varphi} \rangle dy
\]

\[
= (2\pi)^{\frac{n}{2}} \langle T_\sigma \langle \hat{u} \hat{\varphi} \rangle, \hat{v} \hat{\varphi} \rangle
\]

(4.4)

Thus, in the case when \( p = 2 \), (4.4) tells us that the localization operator \( L_{\sigma, \varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is unitarily equivalent to the wavelet multiplier \( P_{\sigma, \varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \).

**References**


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