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A $D_E[0,1]$ REPRESENTATION OF RANDOM UPPER SEMICONTINUOUS FUNCTIONS

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Abstract. In this paper a representation of random upper semicontinuous functions in terms of $D_E[0,1]$-valued random elements is stated. This fact allows us to consider for the first time a complete and separable metric, the Skorohod one, on a wide class of upper semicontinuous functions. Finally, different relevant concepts of measurability for random upper semicontinuous functions are studied and the relationships between them are analyzed.

1. Introduction

The upper semicontinuous functions are very useful in many contexts such as in optimization theory, image processing, spatial statistics, and so on (see, for instance, [1], [2], [3], [6], [11], [26]). The notion of convergence of sequences of such functions is especially interesting. One of the most useful metrics on the class of functions we will deal with is the supremum one (see [22]). This metric is complete, but a big inconvenience in handling it is the lack of separability. To avoid this problem the d_p metrics, which are separable (although not complete), were introduced in [17]. In this paper we introduce an intermediate metric which preserves the uniformity idea and it is both separable and complete. This metric is prompted by an identification of the upper semicontinuous functions with certain elements of $D_E[0,1]$.

On the other hand, different concepts of measurability have been considered for random upper semicontinuous functions (see, for instance, [17], [9], [24]). In this paper, and due to the new introduced metric, we establish the relationships which exist between these concepts of measurability. We conclude that the most suitable measurability concept for these random elements seems to be the one based on the Skorohod metric. In particular, we will check that there exist some relevant nonmeasurable mappings with respect to the Borel σ-field associated with the supremum metric which are measurable with respect to the Borel σ-field associated with the Skorohod one.

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2. Preliminaries

Let (E, d) be a complete and separable metric space and let $D_E[0,1]$ denote the class of corresponding cadlag functions, that is, the class of functions $f : [0,1] \to E$ which are right-continuous, left-continuous at $t = 1$, and having left-hand limits everywhere (see [27]).

If there exists an order relation on E, we will denote by $D_E^{\uparrow}$ the class of non-decreasing elements of $D_E[0,1]$. The most common metrics defined on this space are the following ones:

$$m_\infty(x, y) = \sup_{t \in [0,1]} d(x(t), y(t)),$$
$$m_S(x, y) = \inf_{\lambda \in \Lambda} \max \left\{ \sup_t |\lambda(t) - t| \, , \, \sup_t d(x(t), y(\lambda(t))) \right\},$$

where Λ is the class of (strict) increasing continuous functions $\lambda : [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$.

The topology of the uniform convergence is too strong (in fact, the space endowed with the sup-metric m_∞ becomes nonseparable) and, therefore, this space is usually considered to be endowed with the Skorohod topology, which is not too weak and preserves important properties under passage to the limit (see [18], [27]). Billingsley [4] defines a metric compatible with the Skorohod topology m_S, under which $D_E[0,1]$ is a Polish space.

From now on, if r is a metric, $\mathcal{A}_r$ will denote the Borel σ-field generated by the topology associated with r. It is well-known that $\sigma(\pi_t | t \in [0,1]) = \mathcal{A}_{m_S} \subset \mathcal{A}_{m_\infty}$ (see [10] and [13]) where the projection $\pi_t : D_E[0,1] \to E$ is defined so that $\pi_t(x) = x(t)$, $x \in D_E[0,1], t \in [0,1]$. Given a probability space $(\Omega, \mathcal{A}, P)$, a $D_E[0,1]$-*valued random element* is usually defined as an $\mathcal{A}|\mathcal{A}_{m_S}$-measurable function $X : \Omega \to D_E[0,1]$.

On the other hand, let $(B, |\cdot|)$ be a separable Banach space and let $\mathcal{K}(B)$ denote the collection of the nonempty bounded and closed subsets of B. The Hausdorff metric on this space is given by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b| \right\}$$

for all $A, B \in \mathcal{K}(B)$. The space $(\mathcal{K}(B), d_H)$ is complete and separable (see [8]). A mapping $X : \Omega \to \mathcal{K}(B)$ is said to be a *compact random set* if it is $\mathcal{A}|\mathcal{A}_{d_H}$-measurable.

Let $\mathcal{F}(B)$ be the set of normal upper semicontinuous functions $u : B \to [0,1]$ so that $u_0 = \text{cl}\{b \in B \,|\, u(b) > 0\}$ is bounded. The usual metrics employed in this context (see [17], [22], [23]) are

$$d_p(u, v) = \left(\int_0^1 d_H(u_\alpha, v_\alpha)^p d\alpha \right)^{1/p} \quad \text{if } 1 \le p < \infty,$$
$$d_\infty(u, v) = \sup_{\alpha \in [0,1]} d_H(u_\alpha, w_\alpha),$$

for all $u, v \in \mathcal{F}(B)$, where $u_\alpha = \{b \in B \,|\, u(b) \ge \alpha\}$.

Let $X : \Omega \to \mathcal{F}(B)$ be a mapping. There are several concepts of measurability to formalize the notion of the $\mathcal{F}(B)$-valued random element (*random upper semicontinuous function*). The most used one requires that the α-level mappings $X_\alpha : \Omega \to \mathcal{K}(B)$ are random sets for all $\alpha \in [0,1]$ (see [24]). The $\mathcal{A}|\mathcal{A}_{d_\infty}$-measurability (see [17]) and the $\mathcal{A}|\mathcal{A}_{d_p}$-measurability (see [9]) are also employed.

3. The results

In this section we will define a representation of upper semicontinuous functions in terms of cadlag functions which will be used to define the Skorohod metric in $\mathcal{F}(B)$. We will also study the relationships between different concepts of measurability.

3.1. **Cadlag representation.**

We are first going to define the cadlag representation of elements in the space $\mathcal{F}(B)$, where $(B, |\cdot|)$ is a separable Banach space.

Thus, if $u \in \mathcal{F}(B)$ we define the *cadlag representation* of u as the mapping $x_u : [0,1] \to \mathcal{K}(B)$ such that $x_u(\alpha) = u_{1-\alpha}$.

The next proposition states the identification between $\mathcal{F}(B)$ and $D^{\uparrow}_{\mathcal{K}(B)}$ through the cadlag representation.

Theorem 3.1. *If $u \in \mathcal{F}(B)$, then $x_u \in D^{\uparrow}_{\mathcal{K}(B)}$, and conversely if $x \in D^{\uparrow}_{\mathcal{K}(B)}$ there is $u \in \mathcal{F}(B)$ such that $x_u = x$.*

As a consequence of Theorem 3.1, it makes sense to define the Skorohod metric on $\mathcal{F}(B)$, and since $D^{\uparrow}_{\mathcal{K}(B)}$ is closed in the Skorohod topology, some interesting properties will then be preserved. Let d_S denote the Skorohod metric on $\mathcal{F}(B)$. On the basis of the comments above for general spaces $D_E[0,1]$, this topology is neither as strong as the one associated with d_∞ nor as weak as the one associated with d_p $(1 \le p < \infty)$ metrics. Besides, whereas $(\mathcal{F}(B), d_\infty)$ is complete and nonseparable (see [24], [17]), and $(\mathcal{F}(B), d_p)$ $(1 \le p < \infty)$ is separable and noncomplete (see [17], [9]), the space $(\mathcal{F}(B), d_S)$ is Polish.

3.2. **Measurability.**

In the following theorem we are going to prove the equivalence between the measurability induced by d_S, by d_p for all $p \in [0, +\infty)$, and the measurability of the α-level mappings.

Theorem 3.2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to D^{\uparrow}_{\mathcal{K}(B)}$ be a mapping. The following measurability conditions are equivalent:*

i) *X is $\mathcal{A}|\mathcal{A}_{d_S}$-measurable;*
ii) *X is $\mathcal{A}|\mathcal{A}_{d_p}$-measurable for all $p \in [1, +\infty)$;*
iii) *X_α is a compact random set for all $\alpha \in [0,1]$.*

Proof. As we have commented, in general spaces $D_E[0,1]$, $\sigma(\pi_t | t \in [0,1]) = \mathcal{A}_{m_S} \subset \mathcal{A}_{m_\infty}$, so that we just need to prove the equivalence between i) and ii).

The Skorohod topology is finer than the one given by the d_p metric for all $p \in [0, +\infty)$ (see [4]), so that to guarantee the equivalence it is sufficient to verify that $\mathcal{A}_{d_S} \subset \mathcal{A}_{d_p}$. Since the space $(D^{\uparrow}_{\mathcal{K}(B)}, d_S)$ is complete and separable, $(D^{\uparrow}_{\mathcal{K}(B)}, d_p)$ is separable, and $\mathcal{A}_{d_p} \subset \mathcal{A}_{d_S}$ implies that the identity mapping from $(D^{\uparrow}_{\mathcal{K}(B)}, d_S)$ to $(D^{\uparrow}_{\mathcal{K}(B)}, d_p)$ is measurable, then on the basis of Kuratowski Theorem (see [20]) we conclude that $\mathcal{A}_{d_S} \subset \mathcal{A}_{d_p}$. □

Finally, we will check the connection between the above measurability conditions and the $\mathcal{A}|\mathcal{A}_{d_\infty}$-measurability.

Theorem 3.3. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space and* $X : \Omega \to D^{\uparrow}_{\mathcal{K}(B)}$ *be a mapping. If* X *is* $\mathcal{A}|\mathcal{A}_{d_\infty}$*-measurable, then* X *is* $\mathcal{A}|\mathcal{A}_{d_S}$*-measurable, although the converse implication can fail.*

Proof. In Section 2 we have recalled that $\mathcal{A}_{m_S} \subset \mathcal{A}_{m_\infty}$ is verified for $D_E[0,1]$, which leads to the first implication.

To prove that both concepts of measurability are not equivalent, we can consider an example inspired by [21]. Let $X : \Omega \to D^{\uparrow}_{\mathcal{K}(R)}$ be a mapping based on the so-called change-level random variable $\beta : \Omega \to [0,1]$ (see [7]) uniformly distributed and defined as follows:

$$X(\omega)(t) = \begin{cases} [1,2] & \text{if } t \in [0, 1-\beta(\omega)), \\ [0,2] & \text{if } t \in [1-\beta(\omega), 1]. \end{cases}$$

The mapping X is a $D^{\uparrow}_{\mathcal{K}(R)}$-valued mapping $\mathcal{A}|\mathcal{A}_{d_S}$-measurable, since

$$X_\alpha(\omega) = \begin{cases} \beta^{-1}([\alpha, 1,]) \cup \beta^{-1}([0,\alpha)) & \text{if } \max\{d_H(K,[0,2]), d_H(K,[1,2])\} < r, \\ \beta^{-1}([\alpha,1]) & \text{if } d_H(K,[0,2]) < r \le d_H(K,[1,2]), \\ \beta^{-1}([0,\alpha)) & \text{if } d_H(K,[1,2]) < r \le d_H(K,[0,2]), \\ \emptyset & \text{otherwise} \end{cases}$$

for all $t \in R$ and $K \in \mathcal{K}(R)$, and hence the measurability of β implies that X_α is a compact random set for all $\alpha \in [0,1]$.

Furthermore, we can check that X is not $\mathcal{A}|\mathcal{A}_{d_\infty}$-measurable. Thus, if A is an arbitrary subset of $[0,1]$, let

$$G_A = \{v \in D^{\uparrow}_{\mathcal{K}(R)} \,|\, v \text{ is discontinuous at some point of } \alpha \in A\}.$$

The set G_A defined above is open with respect to the topology induced by the metric d_∞, whatever $A \subset [0,1]$ may be. Indeed, if $v \in G_A$ there exists $\alpha \in A$ such that v has a jump at α and, hence, there exists $\epsilon > 0$ so that for all $\delta > 0$ there is an α_δ with $|\alpha_\delta - \alpha| < \delta$ and $d_H(v(\alpha_\delta), v(\alpha)) > \epsilon$. To prove that $B_{d_\infty}(v, \epsilon/3) \subset G_A$, we assume that there is $u \in B_{d_\infty}(v, \epsilon/3)$ which does not belong to G_A, whence u must be continuous at α and given $\epsilon/3$ there exists $\delta_0 > 0$ such that for all γ with $0 \le |\gamma - \alpha| < \delta_0$ we have that $d_H(u(\gamma), u(\alpha)) < \epsilon/3$. Consequently, we would conclude under the above assumption that

$$d_H(v(\alpha), v(\alpha_{\delta_0})) \le d_H(v(\alpha), u(\alpha)) + d_H(u(\alpha), u(\alpha_{\delta_0})) + d_H(u(\alpha_{\delta_0}), v(\alpha_{\delta_0}))$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

so we reach a contradiction.

Since G_A is open, then if we assume that X is $\mathcal{A}|\mathcal{A}_{d_\infty}$-measurable, we can ensure that $X^{-1}(G_A)$ would be measurable. However, $X^{-1}(G_A) = \{\omega \in \Omega \,|\, X(\omega) \in G_A\} = \{\omega \in \Omega \,|\, \beta'(\omega) = 1 - \beta(\omega) \in A\}$, and therefore for any $A \subset (0,1)$ the set $\{\omega \in \Omega \,|\, \beta'(\omega) \in A\}$ would belong to $\mathcal{A}$, whence it would be possible to define a probability measure $\mu(A) = P(\beta' \in A)$ extending the uniform distribution to the power set $\mathcal{P}((0,1))$. □

4. Concluding remarks

The fact that the Borel σ-field $\mathcal{A}_{d_\infty}$ is too large is stressed by showing that a mapping $\mathcal{A}|\mathcal{A}_{d_S}$-measurable as elementary as the one considered in Theorem 3.3 is not $\mathcal{A}|\mathcal{A}_{d_\infty}$-measurable. Therefore, in this context, it is quite convenient to consider the Skorohod metric on $\mathcal{F}(B)$ under which this space is separable and topologically complete. This fact entails some valuable consequences, like the possibility of applying without difficulties the classical theory of weak convergence (for other approaches see [28]). Besides, the measurability induced for the Skorohod metric has been proved to be equivalent to the measurability of the α-level mappings and to the $\mathcal{A}|\mathcal{A}_{d_p}$-measurability for all $p \in [1, +\infty)$.

Finally, we also point out that the cadlag representation form introduced in this paper implies that many studies developed for the spaces $D_E[0,1]$ (see, for instance, [10], [12], [15], [16], [25]) can be used to investigate the space $\mathcal{F}(B)$, since a random upper semicontinuous function can be considered now as a particular case of the $D_E^{\uparrow}$-valued random element. There are also studies developed for the space $D[0,1]$ (see, for instance, [5], [14], [19], [21]) which can be taken as basis, in the sense that some arguments are valid for $D_{\mathcal{K}(B)}^{\uparrow}$, too.

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