

## FIRST STABILITY EIGENVALUE CHARACTERIZATION OF CLIFFORD HYPERSURFACES

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ABSTRACT. The stability operator of a compact oriented minimal hypersurface  $M^{n-1} \subset S^n$  is given by  $J = -\Delta - \|A\|^2 - (n-1)$ , where  $\|A\|$  is the norm of the second fundamental form. Let  $\lambda_1$  be the first eigenvalue of  $J$  and define  $\beta = -\lambda_1 - 2(n-1)$ . In 1968 Simons proved that  $\beta \geq 0$  for any non-equatorial minimal hypersurface  $M \subset S^n$ . In this paper we will show that  $\beta = 0$  only for Clifford hypersurfaces. For minimal surfaces in  $S^3$ , let  $|M|$  denote the area of  $M$  and let  $g$  denote the genus of  $M$ . We will prove that  $\beta|M| \geq 8\pi(g-1)$ . Moreover, if  $M$  is embedded, then we will prove that  $\beta \geq \frac{g-1}{g+1}$ . If in addition to the embeddeness condition we have that  $\beta < 1$ , then we will prove that  $|M| \leq \frac{16\pi}{1-\beta}$ .

### 1. INTRODUCTION AND PRELIMINARIES

In 1968, James Simons [S] proved an estimate for the first eigenvalue of the stability operator on any minimal hypersurface  $M^{n-1} \subset S^n$ . In this paper we will show that this estimate is sharp *only* for the minimal products:

$$S^k \left( \sqrt{\frac{k}{n-1}} \right) \times S^l \left( \sqrt{\frac{l}{n-1}} \right) \subset S^n \subset \mathbf{R}^{n+1} \quad \text{with } k+l=n-1.$$

In the case  $k=l=1$  the resulting minimal surface is called the Clifford torus. We will refer to all the products above as Clifford hypersurfaces.

Let  $M$  be a compact, oriented minimal hypersurface immersed in the  $n$ -dimensional sphere  $S^n$ . Let  $\nu$  be a unit normal vector field along  $M$ . For any tangent vector  $v \in T_m M$ ,  $m \in M$ , the shape operator  $A$  is given by  $A(v) = -\bar{\nabla}_v \nu$ , where  $\bar{\nabla}$  denotes the Levi Civita connection in  $S^n$ . We will denote by  $\Delta$  the Laplacian on  $M$ . Given any function  $f : M \rightarrow \mathbf{R}^1$  we can form the 1-parameter variational family defined by

$$M_t = \{ \exp(m, tf(m)\nu) : m \in M \}$$

where  $\exp(m, \cdot)$  is the exponential map at  $m \in S^n$ .

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It is well known (see e.g. [SL]) that the  $n - 1$ -dimensional volume of  $M_t$  satisfies

$$\begin{aligned}\frac{d}{dt}(\text{Vol}(M_t))\Big|_{t=0} &= 0 && \text{(minimality of } M), \\ \frac{d^2}{dt^2}(\text{Vol}(M_t))\Big|_{t=0} &= \int_M J(f)f && \text{(second variation formula)}\end{aligned}$$

where  $J$  is the Jacobi or stability operator on  $M$ , given by

$$J = -\Delta - \|A\|^2 - (n - 1).$$

We will denote the first eigenvalue of  $J$  by  $\lambda_1$ . This eigenvalue has the following characterization [C]:

$$\lambda_1 = \min\left\{\frac{\int_M J(f)f}{\int_M f^2} : f \in C^\infty(M), f \not\equiv 0\right\}$$

and it is known that its multiplicity is 1. Let  $\rho$  be an eigenfunction of  $J$  associated with  $\lambda_1$ .

The easiest minimal hypersurfaces to describe are the equators, i.e. the totally geodesic  $S^{n-1}$ 's in  $S^n$ , and the Clifford hypersurfaces defined above.

Because of the symmetries of these minimal hypersurfaces, equators and Clifford hypersurfaces have  $\|A\|^2$  constant. Therefore, the stability operator and the laplacian differ by a constant, hence, it is not difficult to show that  $\lambda_1 = -(n - 1)$  for the equators and  $\lambda_1 = -2(n - 1)$  for the Clifford hypersurfaces.

In this paper we will show that the only minimal hypersurfaces with  $\lambda_1 = -2(n - 1)$  are the Clifford hypersurfaces. For minimal surfaces in  $S^3$ , we will give an additional identity that relates the genus  $g$  of  $M$ , the area  $|M|$  of  $M$ ,  $\lambda_1$ , and the simple invariant  $\alpha = \int_M \frac{\|\nabla\rho\|^2}{\rho^2}$ . Notice that this invariant is independent of the choice of  $\rho$  because the multiplicity of  $\lambda_1$  is 1. We also have that  $\alpha$  is defined not only for surfaces but for any minimal hypersurface in  $S^n$  and that  $\alpha = 0$  if and only if  $\|A\|$  is constant.

In [S] Simons studied the function  $\|A\|$  and he deduced that if  $M$  is not an equator, then  $\lambda_1 \leq -2(n - 1)$ . This result allowed him to deduce that the only stable cones in  $\mathbf{R}^n$ ,  $n \leq 7$ , are the ones that come from equators, i.e. hyperplanes. The result we just mentioned and the main result in this paper use the following elliptic equation for the shape operator,  $A$ , found by Simons [S]:

$$(1.1) \quad \Delta A = (n - 1)A - \|A\|^2 A.$$

The following theorem, proven by Chern, DoCarmo and Kobayashi [C-D-K] and independently by Lawson [L1], gives another consequence of this elliptic equation:

**Theorem 1.2.** *If  $M$  is a compact orientable minimal hypersurface on  $S^n$  with  $\|A\|^2 \equiv n - 1$ , then  $M$  is a Clifford hypersurface.*

In section §2 we find an elliptic inequality for the function  $f = \|A\|\rho^{-1}$ , that will help us, after applying the maximum principle, to deduce that  $\lambda_1 = -2(n - 1)$  implies that  $\|A\|$  is a first eigenfunction of the stability operator.

In section §3 we compute the laplacian of the function  $h = \ln(\rho)$ , then we deduce, after applying Stokes' theorem, the identity for minimal surfaces we mentioned earlier.

2.  $\lambda_1$ -CHARACTERIZATION OF CLIFFORD HYPERSURFACES

In this section we characterize the Clifford hypersurfaces as the only minimal immersions whose first stability eigenvalue,  $\lambda_1$ , equals  $-2(n - 1)$ .

Before we state and prove our main theorem of this section we will make some computations. Choose a first eigenfunction,  $\rho$ , of the stability operator with  $\rho > 0$ . Then we have

$$-\Delta\rho - \|A\|^2\rho - (n - 1)\rho = \lambda_1\rho.$$

For any  $v, w \in T_mM$ , denote by  $D_vA(w)$  the covariant tensor derivative of the shape operator  $A$ . Using that  $\Delta A = (n - 1)A - \|A\|^2A$  (equation (1.1)), we obtain, assuming  $\|A\|(m) \neq 0$ ,

$$\begin{aligned} \Delta\|A\| &= \operatorname{div}(\nabla\|A\|) \\ &= \operatorname{div}\left(\frac{1}{2}\|A\|^{-1}\nabla\|A\|^2\right) \\ &= \frac{1}{2}(\langle\nabla\|A\|^{-1}, \nabla\|A\|^2\rangle + \|A\|^{-1}\Delta\langle A, A\rangle) \\ &= -\|A\|^{-1}|\nabla\|A\|^2| + \|A\|^{-1}(\langle\Delta A, A\rangle + |DA|^2) \\ &= (n - 1)\|A\| - \|A\|^3 + \|A\|^{-3}(\|A\|^2\langle DA, DA\rangle - \|A\|^2|\nabla\|A\|^2). \end{aligned}$$

Taking an orthonormal basis  $\{e_1, \dots, e_{n-1}\}$  of  $T_mM$  we have

$$\begin{aligned} (\|A\|^2\langle DA, DA\rangle - \|A\|^2|\nabla\|A\|^2) &= \|A\|^2 \sum_{i=1}^{n-1} \langle D_{e_i}A, D_{e_i}A\rangle - \frac{1}{4}\langle\nabla\|A\|^2, \nabla\|A\|^2\rangle \\ &= \|A\|^2 \sum_{i=1}^{n-1} \langle D_{e_i}A, D_{e_i}A\rangle - \frac{1}{4} \sum_{i=1}^{n-1} (e_i\|A\|^2)^2 \\ &= \|A\|^2 \sum_{i=1}^{n-1} \langle D_{e_i}A, D_{e_i}A\rangle - \sum_{i=1}^{n-1} \langle A, D_{e_i}A\rangle^2. \end{aligned}$$

Therefore using the Cauchy-Schwarz inequality we get

**Lemma 2.1.**  $\Delta\|A\| \geq (n - 1)\|A\| - \|A\|^3$  and equality holds if and only for any vector  $v \in T_mM$   $D_vA = \beta(v)A$ , for some linear function  $\beta$  on  $T_mM$ .

Define  $f = \|A\|\rho^{-1}$ . Let  $f(m_0)$  be the maximum of  $f$  and let  $\Omega$  be a region around  $m_0$  in which  $f$  is greater than some positive constant.

Given the computations above, if we also assume that  $\lambda_1 = -2(n - 1)$  we get on  $\Omega$ ,

$$\begin{aligned} \Delta f &= \Delta(\rho^{-1}\|A\|) = \|A\|\Delta\rho^{-1} + 2\langle\nabla\rho^{-1}, \nabla\|A\|\rangle + \rho^{-1}\Delta\|A\| \\ &\geq \|A\|(2\rho^{-3}|\nabla\rho|^2 - (n - 1)\rho^{-1} + \|A\|^2\rho^{-1}) \\ (2.2) \quad &+ \rho^{-1}((n - 1)\|A\| - \|A\|^3) + 2\langle\nabla\rho^{-1}, \nabla\|A\|\rangle \\ &= 2\rho^{-3}|\nabla\rho|^2\|A\| - 2\rho^{-2}\langle\nabla\rho, \nabla\|A\|\rangle \\ &= -2\rho^{-1}\langle\nabla(\rho^{-1}\|A\|), \nabla\rho\rangle = -2\rho^{-1}\langle\nabla f, \nabla\rho\rangle. \end{aligned}$$

This puts us in the position to prove:

**Theorem 2.3.** *If  $M \subset S^n$  is a compact oriented immersed minimal hypersurface with  $\lambda_1 = -2(n - 1)$ , then  $M$  is a Clifford hypersurface.*

*Proof.* Assume  $\lambda_1 = -2(n - 1)$ . Then  $M$  is not totally geodesic, and therefore the function  $f = \rho^{-1}\|A\|$  reaches a positive maximum. Letting  $\Omega$  be defined as above, we have by (2.2) that

$$\Delta f + 2\rho^{-1}\langle \nabla f, \nabla \rho \rangle \geq 0 \quad \text{on} \quad \Omega.$$

Since the maximum of  $f$  in  $\Omega$  is obtained in the interior of  $\Omega$  we get by the maximum principle that  $f$  is constant in  $\Omega$ . Since  $M$  is connected, we deduce that  $f$  is constant in all  $M$ , i.e.  $\|A\| = c\rho$  is itself a first eigenfunction of the stability operator. Now, since  $\lambda_1 = -2(n - 1)$ , we get  $\Delta\|A\| = (n - 1)\|A\| - \|A\|^3$ , hence Lemma 2.1 gives us a 1-form  $\beta$  on  $M$  such that

$$(2.4) \quad D_v A = \beta(v)A.$$

We now prove that  $A$  is parallel. Fix  $m \in M$  and choose  $\{e_1, \dots, e_{n-1}\} \in T_m M$  an orthonormal basis that diagonalizes  $A$ . Then the Codazzi equations in  $S^n$  give us

$$\left\langle \sum_{i=1}^{n-1} D_{e_i} A(e_k), e_i \right\rangle = \left\langle \sum_{i=1}^{n-1} D_{e_k} A(e_i), e_i \right\rangle = \beta(e_k) \sum_{i=1}^{n-1} \langle A(e_i), e_i \rangle = 0.$$

On the other hand, using (2.4) we conclude

$$\left\langle \sum_{i=1}^{n-1} D_{e_i} A(e_k), e_i \right\rangle = \sum_{i=1}^{n-1} \langle \beta(e_i)A(e_k), e_i \rangle = \beta(e_k) \langle A(e_k), e_k \rangle.$$

If  $\langle A(e_k), e_k \rangle \neq 0$ , then  $\beta(e_k) = 0$  and  $D_{e_k} A = 0$ . If  $\langle A(e_k), e_k \rangle = 0$ , then for any  $w \in T_m M$ ,

$$D_{e_k} A(w) = D_w A(e_k) = \beta(w)A(e_k) = 0$$

and we get again that  $D_{e_k} A = 0$ .

Since this holds for all  $k$ ,  $A$  is parallel. It follows that  $\|A\|$  is constant. The stability equation then shows  $\|A\|^2 = n - 1$ . Theorem 1.2 then implies the result.  $\square$

*Remark 2.5.* For  $n = 7$  we showed that  $\lambda_1 < -12$  if  $M$  is not an equator or a Clifford hypersurface. An improvement of the previous estimate to  $\lambda_1 \leq -12.25$  would be of great interest, as it would show that the only stable cones in  $\mathbf{R}^8$  are hyperplanes (cones over equators) and cones over Clifford hypersurfaces in  $S^7$  [S]. Together with the results in [SS], this would yield a complete classification of all area-minimizing hypersurfaces in  $\mathbf{R}^8$ .

### 3. MINIMAL SURFACES IN $S^3$

In this section, for the case  $n = 3$ , i.e. for  $M$  an oriented minimal immersed surface of  $S^3$ , we will find an identity relating the genus of the surface, its area, the value  $\alpha$  defined in §1, and  $\lambda_1$ . This identity will give us a different proof of the result in §2 and of Simons' result in [S] which states that  $\lambda_1 \leq -4$  if  $M$  is not an equator.

Let  $\rho$  be as in §2, we have

$$-\Delta\rho - \|A\|^2\rho - (n - 1)\rho = \lambda_1\rho.$$

Let us compute  $\Delta \ln \rho$ :

$$\begin{aligned} \Delta \ln \rho &= \operatorname{div}(\nabla(\ln \rho)) \\ &= \operatorname{div}(\rho^{-1} \nabla(\rho)) \\ &= \{\langle \nabla \rho^{-1}, \nabla \rho \rangle + \rho^{-1} \Delta \rho\} \\ &= \{(-1)\rho^{-2} |\nabla \rho|^2 + (-\lambda_1 - \|A\|^2 - (n - 1))\}. \end{aligned}$$

Integrating the equation above we find

$$(3.1) \quad \int_M \rho^{-2} |\nabla \rho|^2 = (-\lambda_1 - (n - 1))|M| - \int_M \|A\|^2.$$

In the case where  $M$  is a minimal surface, the Gauss equation, gives us a relation between the norm of the shape operator,  $\|A\|^2$ , and the Gauss curvature of the surface,  $K$ . Namely,

$$K = 1 - \frac{\|A\|^2}{2}.$$

If we integrate the relation above and use Gauss-Bonnet, we get

$$8\pi(1 - g) = 2|M| - \int_M \|A\|^2.$$

Now combining the equation above with (3.1) we obtain the following proposition.

**Proposition 3.2.** *Let  $M$  be a compact oriented minimal immersed in  $S^3$ . If  $\rho$  is an eigenfunction associated to the first eigenvalue of the stability operator  $\lambda_1$  and we define  $\alpha = \int_M \frac{\|\nabla \rho\|^2}{\rho^2}$ , then*

$$\alpha + 8\pi(g - 1) = (-\lambda_1 - 4)|M|.$$

**Corollary 3.3.** *Let  $M$  be a compact non-totally geodesic oriented minimal surface in  $S^3$ . Then the first eigenvalue of the stability operator,  $\lambda_1$ , satisfies  $\lambda_1 \leq -4$ . Moreover,  $\lambda_1 = -4$  if and only if  $M$  is a Clifford torus.*

*Proof.* Since  $M$  is non-totally geodesic, then the genus,  $g$ , of  $M$  is greater than 0, because the equator is the only minimal immersion of a sphere in  $S^3$ , [A]. Now, since  $g \geq 1$ , we get from the proposition above that  $\lambda_1 \leq -4$ , with equality only if  $g = 1$  and  $\rho$  is a constant function. The stability equation gives us that if  $\rho$  is constant and  $\lambda_1 = -4$ , then  $\|A\|^2 \equiv 2$ . Therefore  $M$  is a Clifford torus by Theorem 1.2. □

*Remark 3.4.* If we drop  $\alpha$  in Proposition 3.2 and we define  $\beta = (-\lambda_1 - 4)$ , then we get

$$(3.5) \quad 8\pi(g - 1) \leq \beta|M|.$$

This inequality can also be achieved by plugging the test function  $f \equiv 1$  into the Rayleigh-quotient (see characterization of  $\lambda_1$  in the introduction). If  $M$  is embedded, then Choi and Wang [C-W] proved that  $|M| \leq 8\pi(g + 1)$ . Combining this inequality with (3.5) above we get

$$\beta \geq \frac{g - 1}{g + 1} \quad \text{and if } \beta < 1, \text{ then } |M| \leq \frac{16\pi}{1 - \beta}.$$

Moreover, if  $M$  is embedded by the first eigenfunctions of the Laplacian, i.e. if Yau's conjecture were true, then Yang and Yau [Y-Y] proved that  $|M| \leq 4\pi(g+1)$ , therefore the inequalities above can be improved to

$$\beta \geq 2\frac{g-1}{g+1} \quad \text{and if } \beta < 2, \text{ then } |M| \leq \frac{16\pi}{2-\beta}.$$

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