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A NOTE CONCERNING THE INDEX OF THE SHIFT

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Abstract. Let μ be a finite, positive Borel measure with support in $\{z : |z| \leq 1\}$ such that $P^2(\mu)$ – the closure of the polynomials in $L^2(\mu)$ – is irreducible and each point in $\mathbb{D} := \{z : |z| < 1\}$ is a bounded point evaluation for $P^2(\mu)$. We show that if $\mu(\partial\mathbb{D}) > 0$ and there is a nontrivial subarc γ of $\partial\mathbb{D}$ such that

$$\int_\gamma log(\tfrac{d\mu}{dm})dm > -\infty,$$

then $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for the shift M_z on $P^2(\mu)$.

1. Introduction

Given a finite, positive Borel measure μ with compact support in the complex plane $\mathbb{C}$ and $1 \leq t < \infty$, we let $P^t(\mu)$ denote the closure of the polynomials in $L^t(\mu)$. J. Thomson has established (see [T]) a direct sum decomposition of $P^t(\mu)$ that involves the components of $abpe(P^t(\mu))$ – the set of analytic bounded point evaluations for $P^t(\mu)$. In this brief paper we restrict our attention to the case $t = 2$ and assume that the support of μ is contained in $\overline{\mathbb{D}}$ ($\mathbb{D} := \{z : |z| < 1\}$), that $abpe(P^2(\mu)) = \mathbb{D}$ and that $P^2(\mu)$ is irreducible (which means that $P^2(\mu)$ contains no nontrivial characteristic functions). A consequence of these assumptions is that $\mu|_{\partial\mathbb{D}} \ll m$, where m denotes normalized Lebesgue measure on $\partial\mathbb{D}$. Another consequence is that multiplication by the independent variable z is a bounded operator on $P^2(\mu)$, with closed range; we call this operator the shift and denote it by M_z. If $\mu(\partial\mathbb{D}) = 0$, then work of C. Apostol, H. Bercovici, C. Foias and C. Pearcy in [ABFP] shows that for any natural number n, and for $n = \infty$, there is a closed invariant subspace $\mathcal{M}$ for the shift on $P^2(\mu)$ such that $dim(\mathcal{M} \ominus z\mathcal{M}) = n$. Information concerning the lattice of invariant subspaces for M_z on such $P^2(\mu)$ spaces – in particular, the classical Bergman space $L^2_a(\mathbb{D})$ – contributes to a better understanding of bounded operators on separable Hilbert spaces in general (see [ABFP] or [HRS]). In contrast, if $\mu(\partial\mathbb{D}) > 0$, then the lattice of invariant subspaces for M_z on $P^2(\mu)$ appears to be somewhat limited and indeed the results to date have led the authors of [CY] to conjecture that (in this setting) the outcome follows that of the classical Hardy space $H^2(\mathbb{D})$, and $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for the shift on $P^2(\mu)$. Seminal work in support of

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this conjecture was done by R. Olin and J. Thomson in [OT] who established it in the special case that the support of μ has an "outer hole". The work of L. Miller, J. Thomson and L. Yang, who make use of results in [OT], all but dispatches with the conjecture in the case $\mu|_{\mathbb{D}}$ is area measure; see [M], [Y] and [TY]. Recently, the author of this paper has given an analytic condition that defines what it means for a measure to be so-called "strongly inscribed" and has shown that if a measure μ is such, then there is a measure μ_o whose support has an outer hole and for which the shifts on $P^2(\mu)$ and $P^2(\mu_o)$ are similar as operators (see [A1]). From this it follows that if μ is strongly inscribed, then $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for the shift on $P^2(\mu)$. Furthermore, the existence of a measure μ_o (as described above) is basically determined by whether or not μ is strongly inscribed. In this paper we use an idea or two from [A1] to show that the above conjecture holds whenever there is a nontrivial subarc γ of $\partial\mathbb{D}$ such that

$$\int_\gamma log(\tfrac{d\mu}{dm})dm > -\infty;$$

no special assumption is made concerning $\mu|_{\mathbb{D}}$ here.

2. An index theorem for the shift

Our first result of this section is well-known. Among the references that could be cited in its support is [OY1] (Lemma 2.6).

Lemma 2.1. *Let μ be a finite, positive Borel measure with compact support in $\mathbb{C}$ and let K be a compact subset of abpe$(P^t(\mu))$. With $\nu := \mu - \mu|_K$, there is a positive constant M such that*

$$||p||_{L^t(\mu)} \leq M||p||_{L^t(\nu)}$$

for all polynomials p.

For the sake of completeness, we now recall what it means for a measure to be strongly inscribed (cf. [A1], Definition 2.3).

Definition 2.2. Let μ be a finite, positive Borel measure with support in $\overline{\mathbb{D}}$ such that $P^2(\mu)$ is irreducible and $abpe(P^2(\mu)) = \mathbb{D}$. We say that μ is *strongly inscribed* if there is a Jordan curve Γ in $\overline{\mathbb{D}}$ ($\Omega := inside(\Gamma)$ and ω_Ω denotes harmonic measure on Γ for evaluation at some z_o in Ω) having the properties:

1) $\omega_\Omega(\partial\mathbb{D}) > 0$, and
2) there exists ψ, $0 \leq \psi \in L^\infty(\omega_\Omega)$, such that $log(\psi) \in L^1(\omega_\Omega)$ and $\int_\Gamma |p|^2\psi d\omega_\Omega \leq \int |p|^2 d\mu$ for all polynomials p.

The next result is a straightforward consequence of the above definition; we state it without proof.

Lemma 2.3. *Let μ and ν be finite, positive Borel measures with support in $\overline{\mathbb{D}}$ such that $P^2(\mu)$ and $P^2(\nu)$ are irreducible, and abpe$(P^2(\mu)) =$ abpe$(P^2(\nu)) = \mathbb{D}$. Suppose $0 \not\equiv f \in H^\infty(\mathbb{D})$ and define η by $d\eta = |f|d\mu$. Then $P^2(\eta)$ is irreducible and abpe$(P^2(\eta)) = \mathbb{D}$. Furthermore, if μ is strongly inscribed, then so is η, and so also is ν, if $\mu \leq \nu$.*

Theorem 2.4. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbb{D}}$ such that $P^2(\mu)$ is irreducible and $abpe(P^2(\mu)) = \mathbb{D}$. If $\mu(\partial\mathbb{D}) > 0$ and there is a nontrivial subarc γ of $\partial\mathbb{D}$ such that*

$$\int_\gamma log(\tfrac{d\mu}{dm})dm > -\infty,$$

then $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for the shift M_z on $P^2(\mu)$.

Proof. Our objective is to show that μ is strongly inscribed; by [A1] (Corollary 2.5) this will establish the theorem. We begin by observing that we have some freedom to modify $\mu|_\gamma$. Indeed, define η by

$$d\eta = d\mu - d\mu|_\gamma + wdm|_\gamma,$$

where $0 \leq w \in L^1(m|_\gamma)$ and $\int_\gamma log(w)dm > -\infty$. Then η is mutually absolutely continuous with respect to μ and there is a nonzero bounded outer function f_1 such that $|f_1|d\mu \leq d\eta$. So, by Lemma 2.3, $P^2(\eta)$ is irreducible and $abpe(P^2(\eta)) = \mathbb{D}$. Moreover, since $\int_\gamma log\left(\frac{d\mu}{dm}\right) dm > -\infty$, there is another nonzero bounded outer function f_2 such that $|f_2|d\eta \leq d\mu$. Applying Lemma 2.3 once again, we now see that if η is strongly inscribed, then so is μ.

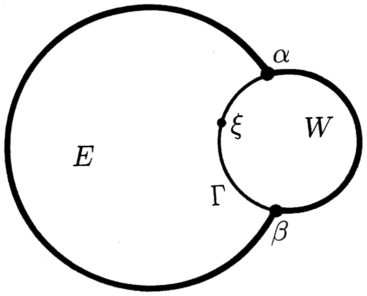
Our next step is to show that $abpe(P^2(\mu_o)) = \mathbb{D}$ for μ_o of the form

$$\mu_o := \mu - \mu|_S,$$

where $S = \mathbb{D} \cap \{z : |z - e^{i\theta}| < r\}$, $e^{i\theta}$ is in the relative interior of γ and $r > 0$ is chosen so that $(\partial\mathbb{D}) \cap \{z : |z - e^{i\theta}| \leq r\}$ is contained in the relative interior of γ. Toward this objective, first observe that we may assume γ is a proper (nontrivial) subarc of $\partial\mathbb{D}$. Let φ be a conformal mapping from $\mathbb{D}$ one-to-one and onto $E := \mathbb{D} \setminus \{z : |z - 1| \leq \frac{1}{2}\}$ such that $\Gamma := \varphi(\gamma) = \overline{\mathbb{D}} \cap \{z : |z - 1| = \frac{1}{2}\}$, and let $\nu = \mu \circ \varphi^{-1}$. Let $W = \{z : |z - 1| < \frac{1}{2}\}$ and let ω_W denote harmonic measure on ∂W for evaluation at 1. Since we have the freedom to modify $\mu|_\gamma$ (within the parameters discussed earlier), we may assume that $\nu|_\Gamma = \omega_W|_\Gamma$. In what follows, let $\sigma = \nu + \omega_W$.

Claim 1. $abpe(P^2(\sigma)) = \mathbb{D} \cup W$.

Now, by our hypothesis and a standard conformal mapping argument, $E = abpe(P^2(\nu))$ $(\subseteq abpe(P^2(\sigma)))$. Furthermore, since $|p|^2$ is subharmonic for any polynomial p, Harnack's Inequality gives $W = abpe(P^2(\omega_W))$ $(\subseteq abpe(P^2(\sigma)))$. What remains to be shown in establishing Claim 1 is that $\Gamma \setminus \{\alpha, \beta\} \subseteq abpe(P^2(\sigma))$, where α and β are the endpoints of Γ; $Im(\alpha) > 0$ and $Im(\beta) < 0$ – see the figure. Since $abpe(P^2(\sigma))$ is an open subset of $\mathbb{C}$ and its components are simply connected, if



there exists ξ in $\Gamma \setminus \{\alpha, \beta\}$ such that $\xi \notin abpe(P^2(\sigma))$, then one of the two components of $\Gamma \setminus \{\xi\}$ has empty intersection with $abpe(P^2(\sigma))$; without loss, we may assume that the subarc of Γ that has endpoints α and ξ – call this subarc Γ_α – has empty intersection with $abpe(P^2(\sigma))$. Let $\mathcal{C}$ be the chord of W that has endpoints ξ and $1 - \frac{i}{2}$ and let V be the component of $W \setminus \mathcal{C}$ that contains 1. Let ω_V denote harmonic measure on ∂V for evaluation at 1 and let $\tau = \nu + \omega_V$. Since $|p|^2$ is subharmonic (for any polynomial p) and $V \subseteq W$,

$$||p||_{L^2(\tau)} \leq ||p||_{L^2(\sigma)}$$

for all polynomials p. Therefore, $\Gamma_\alpha \cap abpe(P^2(\tau)) = \emptyset$ and so it follows that $abpe(P^2(\tau)) = E \cup V$. However, $P^2(\nu)$ and $P^2(\omega_V)$ are irreducible, and the measures ν and ω_V are nonzero and mutually absolutely continuous on their shared support (i.e., Γ_α). By [T], Theorem 5.8, this outcome is not possible, and so we have a contradiction. Therefore, $\Gamma \setminus \{\alpha, \beta\} \subseteq abpe(P^2(\sigma))$, and so Claim 1 holds. As a footnote, we mention that there are other ways of establishing this claim, at least one of which uses results found in [OY2]. For convenience, we let $U = \mathbb{D} \cup W$ $(= abpe(P^2(\sigma)))$. Let Ω be a Jordan region such that $E \cap \Omega \neq \emptyset$ and $\overline{\Omega} \subseteq U$. Let $\nu^* = \nu - \nu|_{(E \cap \Omega)}$ and let $\sigma^* = \nu^* + \omega_W$. Since we have reduced our proof to the case that $\nu|_\Gamma = \omega_W|_\Gamma$, we have $\sigma^* = 2\omega_W$ on Γ. By Claim 1 and Lemma 2.1, $abpe(P^2(\sigma^*)) = U$. Let $\mathcal{P}$ denote the collection of polynomials and let $\mathcal{Q} = \{p(\frac{1}{z-1}) : p \text{ is a polynomial and } p(0) = 0\}$. Let G denote the complement of $\overline{W}$ in the Riemann sphere and let Σ denote the sweep of ν in $\overline{G}$ to ∂G $(= \partial W)$. Notice that $\Sigma \ll \omega_W$, and so we can find a nonzero function h in $H^\infty(G)$, whose (conformal) pull-back to $\mathbb{D}$ is an outer function, such that

$$\int |q|^2 |h| d\nu \leq \int |q|^2 d\omega_W$$

for all q in $\mathcal{Q}$. Since $h|_E \circ \varphi$ is itself a nonzero bounded outer function, we can argue as we did at the beginning of this proof (via φ^{-1}, replacing $d\nu$ by $|h| d\nu$ if need be) and make one last reduction to the special case: there is a positive constant c such that

$$\int |q|^2 d\sigma \leq c \cdot \int |q|^2 d\omega_W$$

for all q in $\mathcal{Q}$. By our preliminary observation, we may still assume that $\nu|_\Gamma = \omega_W|_\Gamma$. Choose λ in $E \cap \Omega$ and let $K = \overline{E} \cup (\partial W)$.

Claim 2. $z \longrightarrow \frac{1}{z-\lambda} \notin R^2(K, \sigma^*)$– the closure in $L^2(\sigma^*)$ of the rational functions with poles off K.

To see this, let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be sequences in $\mathcal{P}$ and $\mathcal{Q}$ respectively such that $||p_n + q_n||_{L^2(\sigma^*)} \longrightarrow 0$, as $n \to \infty$. Then $||p_n + q_n||_{L^2(\omega_W)} \longrightarrow 0$ (as $n \to \infty$) and so it follows from a theorem of M. Riesz (see [H], page 151) that $||q_n||_{L^2(\omega_W)} \longrightarrow 0$. By this and our reduction, we have

(a) $\{q_n\}_{n=1}^\infty$ converges to 0 uniformly on compact subsets of G, and
(b) $||q_n||_{L^2(\sigma^*)} \longrightarrow 0$, as $n \to \infty$.

Since $||p_n + q_n||_{L^2(\sigma^*)} \longrightarrow 0$, (b) implies that $||p_n||_{L^2(\sigma^*)} \longrightarrow 0$, as $n \to \infty$. So, by (a) and since $\lambda \in abpe(P^2(\sigma^*))$, we can now find $r > 0$ such that $\{p_n + q_n\}_{n=1}^\infty$ converges to 0 uniformly on $\{z : |z - \lambda| \leq r\}$. From Runge's Theorem it now follows that λ is an analytic bounded point evaluation for $R^2(K, \sigma^*)$, and therefore Claim 2 holds. Now by Claim 2, there exists g in $L^2(\sigma^*)$ such that $\int g f d\sigma^* = 0$ for all f in $R^2(K,\sigma^*)$ (i.e., $\overline{g} \perp R^2(K,\sigma^*)$) and yet $\int \frac{g(z)}{z-\lambda} d\sigma^*(z) \neq 0$. Recall that the Cauchy transform

$$\hat{g}(\zeta) := \int \frac{g(z)}{z-\zeta} d\sigma^*(z)$$

is defined and analytic off the support of σ^* and, since $\overline{g} \perp R^2(K,\sigma^*)$, $\hat{g} \equiv 0$ on both W and $\mathbb{C} \setminus \overline{U}$. Applying a well-known technique (see the proof of Lemma 6 in [OT] or the proof of Lemma 7 in [A2]), we find $g = 0$ on $(\partial W) \setminus \Gamma$. Evidently, therefore, $z \longrightarrow \frac{1}{z-\lambda} \notin P^2(\nu^*)$. From this it follows that $z \longrightarrow \frac{1}{z-\kappa} \notin P^2(\mu^*)$, where $\mu^* := \nu^* \circ \varphi$ $(= \mu - \mu|_{\varphi^{-1}(E\cap\Omega)})$ and κ is any point in $\varphi^{-1}(E \cap \Omega)$. Since this holds for all Ω as described above, if we select a particular μ_o as defined in the early stages of this proof, then we necessarily have $abpe(P^2(\mu_o)) = \mathbb{D}$. Now by [T], Theorem 5.8, there is a Borel partition $\{\Delta_0, \Delta_1\}$ of the support of μ_o such that

$$P^2(\mu_o) = L^2(\mu_o|_{\Delta_0}) \oplus P^2(\mu_o|_{\Delta_1}),$$

where $P^2(\mu_o|_{\Delta_1})$ is irreducible and $abpe(P^2(\mu_o|_{\Delta_1})) = abpe(P^2(\mu_o))$ $(= \mathbb{D})$. We can proceed with $\mu_o|_{\Delta_1}$, or bypass this direct sum decomposition and argue as above to show that $\chi_B \notin P^2(\mu_o)$ for any Borel subset B of $\partial\mathbb{D}$ such that $\mu(B) > 0$. Consequently, $P^2(\mu_o)$ is irreducible, and so $L^2(\mu_o|_{\Delta_0})$ is trivial. Notice that the support of μ_o has an outer hole and the boundary of this outer hole contains a nontrivial subarc of $\partial\mathbb{D}$. Therefore, by Remark 3.5 of [A1], μ_o is strongly inscribed. Since $\mu_o \leq \mu$, Lemma 2.3 now tells us that μ is strongly inscribed, and the proof is complete. $\square$

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References

[A1] J. Akeroyd, Another look at some index theorems for the shift, Indiana Univ. Math. J., vol. 50, no. 2 (2001), 705-718.

[A2] J. Akeroyd, An Extension of Szegö's Theorem II, Indiana Univ. Math. J., vol. 45, no. 1 (1996), 241-252. MR **97h:**30055

[ABFP] C. Apostol, H. Bercovici, C. Foias, C. Pearcy, Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra, I, J. of Funct. Analysis 63 (1985), 369-404. MR **87i:**47004a

[C] J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monographs, Vol. 36 (1991), Amer. Math. Soc., Providence, RI. MR **92h:**47026

[CY] J. B. Conway, L. Yang, Some open problems in the theory of subnormal operators, *Holomorphic spaces*, Cambridge University Press, vol. 33, 1998, 201-209. MR **99e:**47027

[H] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962. MR **24:**A2844

[HRS] H. Hedenmalm, S. Richter, K. Seip, Interpolating sequences and invariant subspaces of given index in the Bergman spaces, J. Reine Angew. Math., 477 (1996), 13-30. MR **97i:**46044

[KM] T. L. Kriete, B. D. MacCluer, Mean-square approximation by polynomials on the unit disk, Trans. Amer. Math. Soc., vol. 322, no. 1 (1990), 1-34. MR **91b:**30119

[M] T. L. Miller, Some subnormal operators not in $\mathbf{A_2}$, J. Functional Analysis, 82 (1989), 296-302. MR **90c:**47040

[OT] R. F. Olin, J. E. Thomson, Some index theorems for subnormal operators, J. Operator Theory, 3 (1980), 115-142. MR **81f:**47031

[OY1] R. F. Olin, L. Yang, A subnormal operator and its dual, Canad. J. Math., vol. 48, no. 2 (1996), 381-396. MR **98j:**47055

- [OY2] R. F. Olin, L. Yang, The commutant of multiplication by z on the closure of the polynomials in $L^t(\mu)$, J. of Funct. Analysis, vol. 134, no. 2 (1995), 297-320. MR **97m:**47023
- [T] J. E. Thomson, Approximation in the mean by polynomials, Ann. of Math. (2) 133 (1991), 477-507. MR **93g:**47026
- [TY] J. E. Thomson, L. Yang, Invariant subspaces with the codimension one property in $L^t(\mu)$, Indiana Univ. Math. J., vol. 44, no. 4 (1995), 1163-1173. MR **97c:**47036
- [Y] L. Yang, Invariant subspaces of the Bergman space and some subnormal operators in $\mathbf{A_1} \backslash \mathbf{A_2}$, Mich. Math. J. 42 (1995), 301-310. MR **96f:**47013

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