A NEW CONSTRUCTION OF THE KAC JORDAN SUPERALGEBRA

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To Irving Kaplansky

ABSTRACT. We present an elementary construction of the 10-dimensional simple Jordan superalgebra $K_{10}$ of Kac using the 3-dimensional tiny Kaplansky superalgebra. This new realization of $K_{10}$ enables us to deduce many of its properties.

INTRODUCTION

The 10-dimensional Jordan superalgebra $K_{10}$ of Kac [K] is the only exceptional finite-dimensional simple Jordan superalgebra over an algebraically closed field of characteristic 0. We give a new, elementary construction of it (over an arbitrary field of characteristic not 2) using the tiny Kaplansky superalgebra $K$. In our construction, $K_{10}$ appears as a direct sum $F^1 \oplus (K \otimes K)$ with unit element 1 and with a suitable multiplication (see (2.1)). In characteristic 3, $K_{10}$ is not simple but possesses a simple ideal $K_9$ of dimension 9, which is just the tensor product $K \otimes K$ (of superalgebras).

Our realization of $K_{10}$ makes it easy to deduce certain of its known properties: for example, that its Lie superalgebra of superderivations is isomorphic to a direct sum of two copies of the orthosymplectic Lie superalgebra $osp(1, 2)$, or that its Lie superalgebra of superalgebra is $osp(2, 4)$ if $-1$ is a square in the field $F$.

The Kac superalgebra was originally constructed in [K] by Lie theoretical methods from a 3-grading of the simple Lie superalgebra $F(4)$. No direct proof of the fact that it is a Jordan superalgebra is known. Indeed, in their work on Jordan superalgebras with semisimple even part, Racine and Zelmanov [RZ] remark “That $K_{10}$, and hence $K_9$, is a Jordan superalgebra can be obtained by Lie methods as in [K] but a direct proof would be desirable.” Here we provide such a proof.
1. The Kaplansky superalgebra

Suppose $F$ is a field of characteristic not 2. The tiny \textit{Kaplansky superalgebra} \((\mathbb{K}^2 | \mathbb{M})\) is the 3-dimensional Jordan superalgebra $K = K_0 \oplus K_1$, with $K_0 = F e$ and $K_1 = F x \oplus F y$, and with multiplication given by

\begin{equation}
\begin{aligned}
e^2 &= e, \quad ex = \frac{1}{2} x e = xe, \quad ey = \frac{1}{2} y = ye, \\
x y &= e = -yx, \quad x^2 = 0 = y^2.
\end{aligned}
\end{equation}

(1.1)

It can be seen using (1.1) that the even supersymmetric bilinear form \(( \ | \ )\) defined on $K$ by

\[ (e \ | \ e) = \frac{1}{2}, \quad (x \ | \ y) = 1, \quad \text{and} \quad (K_0 \ | \ K_1) = 0 \]

is associative ((\(ab \ | \ c\)) = (\(a \ | \ bc\)) for all \(a, b, c \in K\)). Thus,

\begin{equation}
(\begin{aligned}
ab & | c = (-1)^{\bar{a}\bar{b}\bar{c}}(bc | a) = (-1)^{\bar{a}\bar{b}}(ca | b)
\end{aligned})
\end{equation}

(1.2)

for all \(a, b, c \in K_0 \cup K_1\). (Our convention is that \(\bar{a} = i\) whenever \(a \in K_0\).) Hence, since the supertrace, \(\text{str}(L_e) = \text{tr}_{K_0}(L_e) - \text{tr}_{K_1}(L_e)\), of the left multiplication operator \(L_e\) of \(e\) on \(K\) is 0, it follows that

\[ L_{K_i} = \{ \phi \in \text{End}_F(K) \mid (\phi(u) \ | \ v) \}
\]

\[ = \{ (-1)^{\bar{a}}(u \ | \ \phi(v)) \forall u, v \in K, \text{ and str}(\phi) = 0 \}\]

for \(i = 0, \bar{1}\). Moreover, \([L_{K_i}, L_{K_j}] = \text{Der} \ K = \mathfrak{osp}(K) \cong \mathfrak{osp}(1, 2)\).

**Proposition 1.3.** For all \(u, v, w \in K_0 \cup K_1\),

\begin{equation}
(uw)v = (u \ | \ v)w + \frac{1}{2}(u(v \ | \ w) + (-1)^{\bar{a}\bar{b}}v(u \ | \ w)).
\end{equation}

(1.4)

**Proof.** Observe first that the expression in (1.4) is supersymmetric in \(u\) and \(v\). If \(u = v = e\), then (1.4) reduces to \(ew = \frac{1}{2}w + (e \ | \ w)e\), which is valid for all homogeneous elements \(w\). If \(u = e\) and \(v \in K_1\), then (1.4) becomes

\[ \frac{1}{2}vw = \frac{1}{2}(e(v \ | \ w) + v(e \ | \ w)), \]

which is true. Finally, if \(u, v \in K_1\), then \(w = (u \ | \ v)e\), and in this case, (1.4) says

\[ (u \ | \ v)ew = (u \ | \ v)w + \frac{1}{2}(u(v \ | \ w) + (-1)^{\bar{a}\bar{b}}v(u \ | \ w)). \]

For \(w = e\) this reduces to \((u \ | \ v)e = (u \ | \ v)e\). If instead \(w \in K_1\), then (1.4) holds because \((u \ | \ v)w + (v \ | \ w)u + (w \ | \ u)v = 0\), as the expression on the left is a skew-symmetric trilinear map on a 2-dimensional vector space (namely \(K_1\)). \qed

**Corollary 1.5.** For all \(u, v \in K_0 \cup K_1\),

\begin{equation}
[L_u, L_v] = \frac{1}{2}(u(v \ | \ -) - (-1)^{\bar{a}\bar{b}}v(u \ | \ -)) \quad \text{and}
\end{equation}

(1.6)

\[ L_u \circ L_v = (u \ | \ v)I + \frac{3}{2}(u(v \ | \ -) + (-1)^{\bar{a}\bar{b}}v(u \ | \ -)), \]

(1.7)

where \(\phi \circ \psi = \phi \psi + (-1)^{\bar{a}\bar{b}}\psi \phi\) for all homogeneous \(\phi, \psi \in \text{End}_F(K)\).
Proof. From (1.4) we have
\[ u(vw) - (-1)^{\bar{v}\bar{w}} v(uw) = u(v | w) + \frac{1}{2} (u | v)w + (-1)^{\bar{v}\bar{w}} v(u | w) \]
\[ - (-1)^{\bar{v}\bar{w}} (v(u | w) + \frac{1}{2} (v | u)w + (-1)^{\bar{v}\bar{w}} (v | w)a) \]
\[ = \frac{1}{2} (u(v | w) - (-1)^{\bar{v}\bar{w}} v(u | w)), \]
which demonstrates (1.6). The verification of (1.7) is similar. \qed

2. The Jordan superalgebra J

In this section we consider the superalgebra
\[ J = F1 \oplus (K \otimes K) \]
with unit element 1 and with product defined by
\[ (a \otimes b)(c \otimes d) = (-1)^{bc} \left( ac \otimes bd - \frac{3}{4} (a | c)(b | d)1 \right) \]
for homogeneous elements \( a, b, c, d \in K \). (All tensor products are over \( F \) unless specified otherwise.) By its definition \( J \) is a (super-)commutative superalgebra. Our aim is to show that \( J \) is a Jordan superalgebra. First we establish some properties of the multiplication.

Proposition 2.2. For homogeneous elements \( a, b, c, d \in K \),
\[ [L_{a\otimes b}, L_{c\otimes d}](1) = 0, \]
\[ [L_{a\otimes b}, L_{c\otimes d}]_{K\otimes K} = (-1)^{bc} \frac{1}{2} ([L_a, L_c] \otimes (b | d) I + (a | c) I \otimes [L_b, L_d]). \]

Proof. First let us remark that
\[ (\phi \otimes \psi)(u \otimes v) = (-1)^{\bar{u}\bar{v}} \phi(u) \otimes \psi(v) \]
for homogeneous \( \phi, \psi \in \text{End}_F(K) \), \( u, v \in K \). The first equation in (2.3) is clear by supercommutativity. As for the second, note that
\[ [L_{a\otimes b}, L_{c\otimes d}](u \otimes v) \]
\[ = (-1)^{\bar{a}}(a \otimes b) \left( cu \otimes dv - \frac{3}{4} (c | u)(d | v)1 \right) \]
\[ - (-1)^{\bar{a}+\bar{b}(\bar{c}+\bar{d})}(-1)^{\bar{c}}(c \otimes d) \left( au \otimes bv - \frac{3}{4} (a | u)(b | v)1 \right) \]
\[ = (-1)^{\bar{a}}(-1)^{\bar{b}+\bar{c}} \left( a(cu) \otimes b(dv) - \frac{3}{4} (a | cu)(b | dv)1 - \frac{3}{4} (c | u) \otimes b(d | v) \right) \]
\[ - (-1)^{\bar{a}+\bar{c}+\bar{d}} \left( c(au) \otimes d(bv) - \frac{3}{4} (c | au)(d | bv)1 - \frac{3}{4} (a | u) \otimes d(b | v) \right). \]
By associativity \((a \mid cu) = (-1)^{\bar{a}\bar{c}}(c \mid au)\) and \((b \mid dv) = (-1)^{\bar{b}\bar{d}}(d \mid bv)\), so that the \(F1\)-component of (2.4) is 0. Moreover,

\[
a(cu) \otimes b(dv) - (-1)^{\bar{a}\bar{c} + \bar{b}\bar{d}}c(au) \otimes d(bv)
= \frac{1}{2} \left( [L_a, L_c](u) \otimes (L_b \circ L_d)(v) + (L_a \circ L_c)(u) \otimes [L_b, L_d](v) \right),
\]

and by (1.6),

\[
a(c \mid u) \otimes b(d \mid v) - (-1)^{\bar{a}\bar{c} + \bar{b}\bar{d}}c(a \mid u) \otimes d(b \mid v)
= \frac{1}{2} \left( a(c \mid u) - (-1)^{\bar{a}\bar{c}}c(a \mid u) \right) \otimes \left( b(d \mid v) + (-1)^{\bar{b}\bar{d}}d(b \mid v) \right) \\
+ \left( a(c \mid u) + (-1)^{\bar{a}\bar{c}}c(a \mid u) \right) \otimes \left( b(d \mid v) - (-1)^{\bar{b}\bar{d}}d(b \mid v) \right)
= [L_a, L_c](u) \otimes \left( b(d \mid v) + (-1)^{\bar{b}\bar{d}}d(b \mid v) \right) \\
+ \left( a(c \mid u) + (-1)^{\bar{a}\bar{c}}c(a \mid v) \right) \otimes [L_b, L_d](v).
\]

Therefore, it follows from (1.7) that

\[
[L_{a \otimes b}, L_{c \otimes d}]|_{K \otimes K}
= (-1)^{\bar{e}} \left( [L_a, L_c] \otimes \left( \frac{1}{2} L_b \circ L_d - \frac{3}{4} \left( b(d \mid -) + (-1)^{\bar{b}\bar{d}}d(b \mid -) \right) \right) \\
+ \left( \frac{1}{2} L_a \circ L_c - \frac{3}{4} \left( a(c \mid -) + (-1)^{\bar{a}\bar{c}}c(a \mid -) \right) \right) \otimes [L_b, L_d] \right)
= (-1)^{\bar{e}} \frac{1}{2} \left( [L_a, L_c] \otimes (b \mid d) 1 + (a \mid c) 1 \otimes [L_b, L_d] \right),
\]

as desired. \(\square\)

**Corollary 2.5.** For \(a, b, c, d \in K_0 \cup K_1\), \([L_{a \otimes b}, L_{c \otimes d}]\) is a superderivation of \(J\).

**Proof.** Because \(K\) is a Jordan superalgebra, for \(a, b \in K_0 \cup K_1\), the mapping \([L_a, L_b]\) is a superderivation of \(K\). By the associativity of (\(\bigcirc\)), it is super-skew-symmetric relative to (\(\bigcirc\)). Consequently, from (2.3) we can conclude that \([L_{a \otimes b}, L_{c \otimes d}]\) is a superderivation of \(J\). \(\square\)

**Theorem 2.6.** \(J = F1 \oplus (K \otimes K)\) is a Jordan superalgebra if \(\text{char } F \neq 2\).

**Proof.** A Jordan algebra over a field \(F\) of characteristic not 2 or 3 is characterized by commutativity and the identity \(\sum_{\text{cyclic}} [L_{uw}, L_w] = 0\). Hence a Jordan superalgebra over \(F\) is characterized by supercommutativity and

\[
\sum_{\text{supercyclic}} [L_{uw}, L_w] = [L_{uv}, L_u] + (-1)^{\bar{u}(\bar{v} + \bar{w})}[L_{uw}, L_v] + (-1)^{\bar{u}(\bar{v} + \bar{w})}[L_{uw}, L_v] = 0.
\]
In the supercommutative algebra $J$, we have from (1.2) and (2.3) that for all homogeneous $a, b, c, d, u, v \in K$,

\[
\sum_{\text{supercyclic}} [L_{(a \otimes b)(c \otimes d)}, L_{u \otimes v}]
\]

\[
= \sum_{\text{supercyclic}} (-1)^{\bar{b} \bar{e}}[L_{ac \otimes bd}, L_{u \otimes v}]
\]

\[
= \sum_{\text{supercyclic}} (-1)^{\bar{b}(\bar{e} + \bar{u}) + \bar{d} \bar{e}} \frac{1}{2} \left( [L_{ac}, L_u] \otimes (bd \mid v) + (ac \mid u) I \otimes [L_{bd}, L_v] \right)
\]

\[
= (-1)^{\bar{b}(\bar{e} + \bar{u}) + \bar{d} \bar{e}} \frac{1}{2} \left\{ \sum_{\text{supercyclic}} [L_{ac}, L_u] \right\} \otimes (bd \mid v) I
\]

\[
+ (ac \mid u) I \otimes \left( \sum_{\text{supercyclic}} [L_{bd}, L_v] \right)
\]

\[= 0.\]

Thus, $J$ is a Jordan superalgebra whenever the field has characteristic different from 2 or 3.

If $R$ is the ring of fractions $R = S^{-1}Z$ relative to the multiplicative set of integers $S = \{2^n \mid n = 0, 1, \ldots\}$, we could have defined $K$ and $J$ over $R$ as above. (Denote the result by $J_R$.) Then $J_Q = Q \otimes_R J_R$ is a Jordan superalgebra over the rationals $Q$, and so is $J_R$ over $R$. But if $F$ is a field having characteristic not 2, then $J = F \otimes_R J_R$ is a Jordan superalgebra over $F$.

□

From the computations and results above, it is easy to draw the following conclusions about the Jordan superalgebra $J$. Here we use the associator $(u, v, w) = (uv)w - u(vw)$ in $J$ in the statements.

**Proposition 2.7.** For the Jordan superalgebra $J = F1 \oplus (K \otimes K)$:

(a) $(J, J, J) = K \otimes K$.

(b) $J = F1 \oplus (J, J, J)$.

(c) If char $F = 3$, then $K \otimes K = (J, J, J)$ is an ideal of $J$.

(d) $e_1 := \frac{1}{2} - 2e \otimes e$ and $e_2 := \frac{3}{4}I - 2e \otimes e$ are orthogonal idempotents in $J$.

Relative to $e_1$ the Peirce spaces $J^{(\lambda)} := \{u \in J \mid e_1 u = \lambda u\}$ are given by

\[
J^{(1)} = Fe_1, \quad J^{(0)} = Fe_2 \oplus (K_1 \otimes K_1), \quad J^{(\frac{1}{2})} = (e \otimes K_1) \oplus (K_1 \otimes e).
\]

Thus, $J_0 = J^{(1)} \oplus J^{(0)}$ and $J_1 = J^{(\frac{1}{2})}$. The space $J^{(0)}$ is the Jordan algebra of a nondegenerate symmetric bilinear form, and $e_2$ is its unit element.

**Theorem 2.8.** (a) If char $F \neq 2, 3$, then $J = F1 \oplus (K \otimes K)$ is a simple Jordan superalgebra.

(b) If char $F = 3$, then $J = K \otimes K$ is a simple Jordan superalgebra.

(c) Der $J$ = Inder $J$ and it is isomorphic to $osp(1,2) \oplus osp(1,2)$.

Proof. We begin with (c). Any superderivation of $J$ annihilates $F1$ and leaves invariant $(J, J, J) = K \otimes K$. Therefore, using the natural identifications and the fact that Der $K$ = Inder $K \cong osp(1,2)$, we have by (2.3) that

\[\text{(2.9)} \quad \text{Der } K \oplus \text{Der } K \cong (\text{Der } K \otimes 1) \oplus (I \otimes \text{Der } K) = \text{Inder } J.\]
Now one may check that $\text{Inder} J = \text{Der} J$ as follows: First for any $D \in (\text{Der} J)_{0}$, $D|_{J_0} \in \text{Der}(J_0)$. Since $J_0$ is a direct sum of a 1-dimensional ideal $J^{(1)} = F e_1$ and the Jordan algebra $J^{(0)} = F e_2 \oplus (K_1 \otimes K_1)$ of a bilinear form, $\text{Der}(J_0) = \text{Inder} J_0$. Thus, there is a $\tilde{D} \in (\text{Inder} J)_{0}$ with $(D - \tilde{D})(J_0) = 0$. It follows easily that $D = \tilde{D}$. Now suppose $E \in (\text{Der} J)_{1}$. Then there is a $\tilde{E} \in (\text{Inder} J)_{1}$ with $E(e_1) = \tilde{E}(e_1)$ because $(\text{Inder} J)_{1} e_1 = J_1$. For any $z \in J_1$, $e_1 z = \frac{1}{2} z$, so $(E - \tilde{E})(z) = 2(E - \tilde{E})(e_1 z) = 2e_1 (E - \tilde{E})(z)$. Thus, $(E - \tilde{E})(z) \in J_{0} \cap J^{(2)} = 0$, so that $(E - \tilde{E})(J_1) = 0$. As a consequence, $E = \tilde{E}$ and $(\text{Der} J)_{1} = (\text{Inder} J)_{1}$.

Since $\text{Der} K$ acts irreducibly on $K$, so does $\text{Der} J$ on $K \otimes K = (J, J)$, which is an ideal of $J$ if and only if the characteristic of $F$ is 3. From this (a) and (b) follow. \hfill \Box

The Lie multiplication superalgebra of $J$. To compute the Lie multiplication superalgebra of $J$ we need some preliminaries. Let $H$ be a unital Jordan superalgebra with a normalized trace, that is, a linear map $t : H \to F$ such that $t(1) = 1$ and $t((H, H, H)) = 0$. Assume that $(H, H, H) \not= 0$. Let $H_0 = \ker t$. Then for any $x, y \in H_0$, $xy = t(xy)1 + x \cdot y$, where $x \cdot y = xy - t(xy)1$. The Lie multiplication superalgebra is the Lie superalgebra spanned by $L_{H}$, which is a direct sum of the central ideal $F 1$ and the ideal $\mathfrak{I}_{0}(H) = L_{H_0} \oplus [L_{H_0}, L_{H_0}]$. For simplicity, let us adopt the notation $S = \text{Inder} H$ and $T = H_0$. Then $\mathfrak{I}_{0}(H)$ is isomorphic to the Lie superalgebra $\mathfrak{L} = S \oplus T$ with multiplication determined by

1. the natural Lie multiplication on $S$,
2. the natural action of $S$ on $T$, that is, $[s, t] = s(t)$ for any $s \in S$ and $t \in T$, and
3. $[t_1, t_2] = [L_{t_1}, L_{t_2}] \in S$, for any $t_1, t_2 \in T$.

Also, given a nonzero scalar $\nu \in F$, let $\mathfrak{L}_{\nu}$ be the Lie superalgebra defined over $\mathfrak{L}$, but with the new bracket given by

1. $[x, y]^\nu = [x, y]$, for $x, y$ both in $S$ or one in $S$ and the other in $T$,
2. $[x, y]^\nu = \nu [x, y]$, for $x, y \in T$.

**Lemma 2.10.** Under the hypotheses above, if $\text{Hom}_{\mathfrak{L}}(T \otimes T, T)$ is spanned by $u \otimes v \mapsto u \cdot v$ and if any automorphism of the superalgebra $(T, \cdot)$ is an isometry of the supersymmetric bilinear form $(u, v) \mapsto t(uv)$, then for any $0 \not= \mu, \nu \in F$, $\mathfrak{L}_{\mu}$ is isomorphic to $\mathfrak{L}_{\nu}$ if and only if $\mu^{-1} \nu \in F^2$.

**Proof.** It is enough to apply the ideas of ([BDE], proof of Prop. 4.2). We include the argument for completeness.

First, since any isomorphism $\mathfrak{L}_{\mu} \to \mathfrak{L}_{\nu}$ is also an isomorphism $\mathfrak{L} \to \mathfrak{L}_{\mu^{-1} \nu}$, it is enough to assume $\mu = 1$. Suppose $\Phi : \mathfrak{L} \to \mathfrak{L}_{\nu}$ is an isomorphism. Then the map $T \otimes T \to T$ defined by $u \otimes v \mapsto \Phi^{-1}(\Phi(u) \ast \Phi(v))$ is $S$-invariant. Consequently, there exists a nonzero scalar $\alpha \in F$ such that $\alpha \Phi(u) \ast \Phi(v) = \Phi(u \ast v)$ for any $u, v \in T$. Then $\psi : T \to T$ given by $\psi(u) = \alpha \Phi(u)$ is an automorphism of $(T, \ast)$ which, by hypothesis, extends by means of $\psi(1) = 1$ to an automorphism of the Jordan superalgebra $H$. Now the map $\Psi : \mathfrak{L} \to \mathfrak{L}_{\nu}$, such that $\Psi(s) = \psi \psi^{-1}(s)$ (composition of maps) for $s \in S$, and $\Psi(t) = \psi(t)$ for $t \in T$ is an automorphism of $\mathfrak{L}$. 

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Furthermore, \( \tilde{\Phi} = \Phi \Psi^{-1} : \mathcal{L} \rightarrow \mathcal{L}_\nu \) is an isomorphism such that \( \tilde{\Phi}(t) = \Phi(\psi^{-1}(t)) = \alpha^{-t} t \) for \( t \in T \). Hence, for any \( t_1, t_2, t_3 \in T \),

\[
\tilde{\Phi}(t_1, t_2) = [\tilde{\Phi}(t_1), \tilde{\Phi}(t_2)]^\nu = \alpha^{-2 \nu} [t_1, t_2],
\]

so that \( \nu = \alpha^{-2} \) as required. (Note \( [T, T], T \neq 0 \) because \( (H, H, H) \neq 0 \).)

The converse is clear, since the map \( \Phi \) given by \( \Phi(s) = s \) for \( s \in S \) and \( \Phi(t) = \alpha t \) for \( t \in T \) provides an isomorphism \( \mathcal{L} \rightarrow \mathcal{L}_{\alpha^2} \).

In order to apply the last result to our 10-dimensional Jordan superalgebra \( J \), notice first that \( J \) is endowed with a normalized trace \( t \) such that \( t(1) = 1 \) and \( t((J, J, J)) = 0 \), as \( (J, J, J) = K \otimes K \) does not intersect \( F \). Moreover, the associated bilinear form is given by \( t((a \otimes b)(c \otimes d)) = (-1)^{\tilde{\delta} c d} (a \mid c)(b \mid d) \) for any homogeneous \( a, b, c, d \in K \), because of (2.1). Then all we need is the following

**Lemma 2.11.** Any automorphism of \( (K \otimes K, *) \) is an isometry of the bilinear trace form.

**Proof.** Write \( T = J_0 = K \otimes K \) and set \( \langle a \otimes b \mid c \otimes d \rangle = (-1)^{\tilde{\delta} c d} (a \mid c)(b \mid d) \) for any homogeneous elements \( a, b, c, d \in K \). Then \( T_0 = F(e \otimes e) \oplus (K_1 \otimes K_1) \). Suppose \( \tilde{e} = e \otimes e \). Then the multiplication in \( T_0 \) is given by

\[
\langle \tilde{e}^2 \rangle = \tilde{e}, \quad \tilde{e} * v = \frac{1}{4} v, \quad v * w = \langle v \mid w \rangle \tilde{e}
\]

for all \( v, w \in K_1 \otimes K_1 \).

Let \( \varphi \) be an automorphism of \( (T, *) \) and let us first verify that \( \varphi(\tilde{e}) = \tilde{e} \). This is clear if the characteristic is 3, because \( \tilde{e} \) is then the unit element of \( T_0 \). Otherwise, \( \tilde{e} \) is the only idempotent of \( T_0 \) such that the corresponding left multiplication operator has eigenvalue \( \frac{1}{4} \) with multiplicity 4. This is because the eigenvalues of \( L_{a \otimes e + v} \) (for \( v \in K_1 \otimes K_1, a \in F \)) include \( \frac{1}{4} \) with multiplicity at least 3, as any \( w \in K_1 \otimes K_1 \) which is orthogonal to \( v \) relative to \( \langle \mid \rangle \) is a corresponding eigenvector. Moreover, \( \tilde{e} + v \) is an idempotent only if \( v = 0 \).

Now, \( \varphi(K_1 \otimes K_1) \subseteq K_1 \otimes K_1 \) because \( K_1 \otimes K_1 = \{ v \in T_0 \mid \tilde{e} \notin Fv \text{ and } v + v \in F\tilde{e} \} \).

Because of (2.12), \( \varphi|_{T_0} \) is an isometry of the restriction of \( \langle \mid \rangle \) to \( T_0 \).

Finally, \( (K_1 \otimes e) \cup (e \otimes K_1) = \{ z \in T_1 \mid z \otimes z \in F\tilde{e} \} \), and \( K_1 \otimes e \) and \( e \otimes K_1 \) are the only 2-dimensional subspaces contained in this union. Therefore, either \( \varphi \) preserves \( K_1 \otimes e \) and \( e \otimes K_1 \) or it interchanges them. Since \( (u \otimes e) * (v \otimes e) = \frac{1}{2} (u \otimes e \mid v \otimes e) \tilde{e} \) and \( (e \otimes u) * (e \otimes v) = \frac{1}{2} (e \otimes u \mid e \otimes v) \tilde{e} \) for any \( u, v \in K_1 \), it follows that \( \varphi|_{T_1} \) is an isometry too.

We are ready to determine the Lie multiplication superalgebra \( \mathcal{L}_0(J) \) for \( J = F1 \oplus (K \otimes K) \). When the underlying field \( F \) has characteristic 0, the next result is stated in [Sh] but without the assumption of \(-1 \) being a square in \( F \).

**Theorem 2.13.** \( \mathcal{L}_0(J)_{-1} \cong osp(K \oplus K) \) \( \cong osp(2, 4) \). In particular, \( \mathcal{L}_0(J) \) is a form of \( osp(2, 4) \) and \( \mathcal{L}_0(J) \cong osp(2, 4) \) if and only if \(-1 \) is a square in \( F \).
Proof. On any vector superspace \( V \) equipped with a supersymmetric bilinear form \( (\cdot,\cdot) \), the transformation \( \gamma_{u,v} = u(v|\cdot) - (-1)^{|v||u|}v(u|\cdot) \) belongs to \( \mathfrak{osp}(V) \) for all homogeneous \( u,v \in V \). Because of (2.3) and (1.7), \( \text{Der} J \) is embedded in \( \mathfrak{osp}(K \oplus K) \) (with the natural supersymmetric bilinear form on \( K \oplus K \) that makes this an orthogonal sum) by means of

\[
\Phi \left( [L_{a\otimes b},L_{c\otimes d}] \right) = (-1)^{|b|} \frac{1}{4} \left( \gamma_{(a,0),(c,0)}(b|d) + (a|c)\gamma_{(0,b),(0,d)} \right) \in \mathfrak{osp}(K \oplus K).
\]

This can be extended to a linear bijection \( \Phi : \Lambda_0(J) \rightarrow \mathfrak{osp}(K \oplus K) \) by defining \( \Phi(L_{a\otimes b}) = \frac{1}{2} \gamma_{(a,0),(0,b)} \) for \( a,b \in K \). Since

\[
\left[ \gamma_{(a,0),(0,b)}, \gamma_{(c,0),(0,d)} \right] = \gamma_{(a,0),(0,b)} \gamma_{(c,0),(0,d)} + (-1)^{(a+b)} \gamma_{(c,0),(a,b)}(0,d) + (-1)^{(a+b)} \gamma_{(a,0),(0,d)}(b|d) + (-1)^{(a+b)} \gamma_{(a,0),(c,d)}(b|d),
\]

it follows that \( \Phi \) is an isomorphism of Lie superalgebras. The last part is a consequence of Lemmata 2.10 and 2.11. \( \square \)

3. The Kac superalgebra

We may identify the simple Jordan superalgebra \( J \) with the 10-dimensional simple Jordan superalgebra \( K_{10} \) discovered by Kac [K] (see also [HK]) when the characteristic is not 2 or 3, and \( (J,J,J) = K \otimes K \) with the 9-dimensional degenerate Kac superalgebra \( K_9 \) if \( \text{char} F = 3 \). This realization may be described explicitly as follows. In the notation of [MZ] or [RZ], we suppose \( \{ \hat{e},v_1,v_2,v_3,f,x_1,x_2,y_1,y_2 \} \) and \( \{ \hat{e},v_1,v_2,v_3,f,x_1,x_2,y_1,y_2 \} \) is a basis of \( K_{10} \) with \( (K_{10})_0 = \text{span}_F \{ \hat{e},v_1,v_2,v_3,f \} \) and \( (K_{10})_1 = \text{span}_F \{ x_1,y_1,x_2,y_2 \} \) and with multiplication given by

\[
\begin{align*}
\hat{e}^2 &= \hat{e}, & \hat{e}v_1 &= v_1, & v_1v_2 &= 2\hat{e} = v_3v_4, & \hat{f}^2 &= \hat{f}, \\
\hat{e}x_j &= \frac{1}{2} x_j = \hat{f}x_j, & \hat{e}y_j &= \frac{1}{2} y_j = \hat{f}y_j, \\
v_1y_1 &= x_2, & v_1y_2 &= -x_1, & v_2x_1 &= -y_2, & v_2x_2 &= y_1, \\
v_3x_2 &= x_1, & v_3y_1 &= y_3, & v_4x_1 &= x_2, & v_4y_2 &= y_1, \\
x_jy_j &= \hat{e} - 3\hat{f}, & x_1x_3 &= v_1, & x_1y_2 &= v_3, & x_2y_1 &= v_4, & y_1y_2 &= v_2
\end{align*}
\]

for \( i = 1,2,3,4 \) and \( j = 1,2, \). All other products are either 0 or obtained from those in (3.1) by supercommutativity. Then a straightforward computation using (3.1) shows that the assignment

\[
\begin{align*}
\hat{e} &\leftrightarrow \frac{3}{2} 1 - 2e \otimes e = e_2 \\
\hat{f} &\leftrightarrow -\frac{3}{2} 1 + 2e \otimes e = e_1 \\
v_1 &\leftrightarrow -4x \otimes x \\
v_2 &\leftrightarrow -y \otimes y \\
v_3 &\leftrightarrow 2x \otimes y \\
v_4 &\leftrightarrow -2y \otimes x \\
x_1 &\leftrightarrow 4x \otimes e \\
x_2 &\leftrightarrow -4e \otimes x \\
y_1 &\leftrightarrow -2y \otimes e \\
y_2 &\leftrightarrow 2e \otimes y \\
v_1 &\leftrightarrow -2y \otimes x
\end{align*}
\]
gives an isomorphism between $K_{10}$ and $J$. Additionally, if $\text{char} F = 3$, then $\hat{e} \leftrightarrow -2e \otimes e$, so that the isomorphism restricts to an isomorphism $K_9 \leftrightarrow K \otimes K$.

Concluding remarks. As the reader may have guessed, the actual process for deriving our results was the reverse path of the presentation above. First it was verified that $\mathfrak{g} := \text{Der} K_{10} = \mathfrak{osp}(1,2) \oplus \mathfrak{osp}(1,2)$, (compare Shtern [Sh]), and that as module for the Lie superalgebra $\mathfrak{g}$, $K_{10} = F^1 \oplus (V \otimes V)$, where $V$ is the natural 3-dimensional module for $\mathfrak{osp}(1,2)$. The first copy of $\mathfrak{osp}(1,2)$ operates on the first tensor slot of $V \otimes V$, and the second copy on the second slot. The spaces $\text{Hom}_{\mathfrak{osp}(1,2)}(V \otimes V, F)$ and $\text{Hom}_{\mathfrak{osp}(1,2)}(V \otimes V, \mathfrak{osp}(1,2))$ are 1-dimensional. By identifying $V$ with the tiny Kaplansky superalgebra $K$, we may describe the generators of these spaces nicely by the maps $u \otimes v \mapsto (u \mid v)$ and $u \otimes v \mapsto [L_u, L_v] \in \mathfrak{osp}(1,2)$ for $u, v \in K$. From this, the multiplication in the Kac superalgebra $K_{10}$ is almost uniquely determined in terms of $K$.

In [S1], Shestakov proposed quite a different realization of the superalgebra $K_{10}$ in characteristic $\neq 2, 3$, but the same description of $K_9$; however, he has not published this work [S2].

References


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