A NEW CONSTRUCTION OF THE KAC JORDAN SUPERALGEBRA

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To Irving Kaplansky

Abstract. We present an elementary construction of the 10-dimensional simple Jordan superalgebra $K_{10}$ of Kac using the 3-dimensional tiny Kaplansky superalgebra. This new realization of $K_{10}$ enables us to derive many of its properties.

Introduction

The 10-dimensional Jordan superalgebra $K_{10}$ of Kac $[K]$ is the only exceptional finite-dimensional simple Jordan superalgebra over an algebraically closed field of characteristic 0. We give a new, elementary construction of it (over an arbitrary field of characteristic not 2) using the tiny Kaplansky superalgebra $K$. In our construction, $K_{10}$ appears as a direct sum $F^1 \oplus (K \otimes K)$ with unit element 1 and with a suitable multiplication (see (2.1)). In characteristic 3, $K_{10}$ is not simple but possesses a simple ideal $K_9$ of dimension 9, which is just the tensor product $K \otimes K$ (of superalgebras).

Our realization of $K_{10}$ makes it easy to deduce certain of its known properties: for example, that its Lie superalgebra of superderivations is isomorphic to a direct sum of two copies of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2)$, or that its Lie multiplication superalgebra is $\mathfrak{osp}(2,4)$ if $-1$ is a square in the field $F$.

The Kac superalgebra was originally constructed in $[K]$ by Lie theoretical methods from a 3-grading of the simple Lie superalgebra $F(4)$. No direct proof of the fact that it is a Jordan superalgebra is known. Indeed, in their work on Jordan superalgebras with semisimple even part, Racine and Zelmanov $[RZ]$ remark “That $K_{10}$, and hence $K_9$, is a Jordan superalgebra can be obtained by Lie methods as in $[K]$ but a direct proof would be desirable.” Here we provide such a proof.

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1. The Kaplansky superalgebra

Suppose $F$ is a field of characteristic not 2. The tiny Kaplansky superalgebra $(\mathbb{K}_2 | M)$ is the 3-dimensional Jordan superalgebra $K = K_0 \oplus K_1$, with $K_0 = Fe$ and $K_1 = Fx \oplus Fy$, and with multiplication given by

\begin{align}
  e^2 &= e, \quad ex = \frac{1}{2} x e, \quad ey = \frac{1}{2} y e, \\
  xy &= e = -yx, \quad x^2 = 0 = y^2.
\end{align}

(1.1)

It can be seen using (1.1) that the even supersymmetric bilinear form $(\cdot | \cdot)$ defined on $K$ by

$$(e | e) = \frac{1}{2}, \quad (x | y) = 1, \quad \text{and} \quad (K_0 | K_1) = 0$$

is associative ($(ab | c) = (a | bc)$ for all $a, b, c \in K$). Thus,

\begin{equation}
  (ab | c) = (-1)^{\bar{a}(\bar{b}+\bar{c})}(bc | a) = (-1)^{\bar{c}(\bar{a}+\bar{b})}(ca | b)
\end{equation}

for all $a, b, c \in K_0 \cup K_1$. (Our convention is that $\bar{a} = i$ whenever $a \in K_i$.) Hence, since the supertrace, $\text{str}(L_e) = \text{tr}_{K_0}(L_e) - \text{tr}_{K_1}(L_e)$, of the left multiplication operator $L_e$ of $e$ on $K$ is 0, it follows that

$$L_{K_i} = \{ \phi \in \text{End}_F(K)_i | (\phi(u) | v) = (-1)^{i\bar{u}}(u | \phi(v)) \ \forall \ u, v \in K, \ \text{and} \ \text{str}(\phi) = 0 \}$$

for $i = 0, \bar{1}$. Moreover, $[L_{K_0}, L_{K_1}] = \text{Der} K = \mathfrak{osp}(K) \cong \mathfrak{osp}(1, 2)$.

**Proposition 1.3.** For all $u, v, w \in K_0 \cup K_1$,

\begin{equation}
  (uv)w = (u | v)w + \frac{1}{2} (u(v | w) + (-1)^{\bar{a}\bar{b}} v(u | w)).
\end{equation}

(1.4)

**Proof.** Observe first that the expression in (1.4) is supersymmetric in $u$ and $v$. If $u = v = e$, then (1.4) reduces to $ew = \frac{1}{2}w + (e | w)e$, which is valid for all homogeneous elements $w$. If $u = e$ and $v \in K_1$, then (1.4) becomes

$$\frac{1}{2}vw = \frac{1}{2} (e(v | w) + v(e | w)),
$$

which is true. Finally, if $u, v \in K_1$, then $w = (u | v)e$, and in this case, (1.4) says

\begin{equation}
  (u | v)w = (u | v)w + \frac{1}{2} (u(v | w) + (-1)^{\bar{a}\bar{b}} v(u | w)).
\end{equation}

(1.6)

For $w = e$ this reduces to $(u | v)e = (u | v)e$. If instead $w \in K_1$, then (1.4) holds because $(u | v)w + (v | w)u + (w | u)v = 0$, as the expression on the left is a skew-symmetric trilinear map on a 2-dimensional vector space (namely $K_1$).

**Corollary 1.5.** For all $u, v \in K_0 \cup K_1$,

\begin{equation}
  [L_\phi, L_\psi] = \frac{1}{2} (u(v | -) - (-1)^{\bar{a}\bar{b}} v(u | -)) \quad \text{and}
\end{equation}

\begin{equation}
  L_\phi \circ L_\psi = (u | v) I + \frac{3}{2} (u(v | -) + (-1)^{\bar{a}\bar{c}} v(u | -)),
\end{equation}

(1.7)

where $\phi \circ \psi = \phi \psi + (-1)^{\bar{a}\bar{b}} \psi \phi$ for all homogeneous $\phi, \psi \in \text{End}_F(K)$.
First let us remark that

\begin{equation}
\text{(2.3)}
\end{equation}

For homogeneous elements \(\phi, \psi\) for homogeneous \(J\) specified otherwise.) By its definition \(a, b, c, d\)

\begin{equation}
\text{(2.4)}
\end{equation}

which demonstrates (1.6). The verification of (1.7) is similar. \(\Box\)

2. The Jordan superalgebra \(J\)

In this section we consider the superalgebra \(J = F1 \oplus (K \otimes K)\)

with unit element 1 and with product defined by

\begin{equation}
(a \otimes b)(c \otimes d) = (-1)^{bc} \left(ac \otimes bd - \frac{3}{4}(a | c)(b | d)1\right)
\end{equation}

for homogeneous elements \(a, b, c, d \in K\). (All tensor products are over \(F\) unless specified otherwise.) By its definition \(J\) is a (super-)commutative superalgebra. Our aim is to show that \(J\) is a Jordan superalgebra. First we establish some properties of the multiplication.

**Proposition 2.2.** For homogeneous elements \(a, b, c, d \in K\),

\begin{equation}
[L_{a \otimes b}, L_{c \otimes d}](1) = 0,
\end{equation}

\begin{equation}
[L_{a \otimes b}, L_{c \otimes d}]|_{K \otimes K} = (-1)^b c \left(L_a, L_c \otimes (b | d) I + (a | c) I \otimes L_b, L_d\right).
\end{equation}

**Proof.** First let us remark that

\((\phi \otimes \psi)(u \otimes v) = (-1)^{\tilde{b}a} \phi(u) \otimes \psi(v)\)

for homogeneous \(\phi, \psi \in \text{End}_F(K), u, v \in K\). The first equation in (2.3) is clear by supercommutativity. As for the second, note that

\begin{equation}
[L_{a \otimes b}, L_{c \otimes d}](u \otimes v)
\end{equation}

\begin{equation}
= (-1)^{\tilde{a}a}(a \otimes b) \left(cu \otimes dv - \frac{3}{4}(c | u)(d | v)1\right)
\end{equation}

\begin{equation}
- (-1)^{\tilde{a} + \tilde{b}}(\tilde{c} + \tilde{d}) \left(-1\right)^{\tilde{b}a} (c \otimes d) \left(au \otimes bv - \frac{3}{4}(a | u)(b | v)1\right)
\end{equation}

\begin{equation}
= (-1)^{\tilde{a}a} \left(-1\right)^{\tilde{b}c} \left(a(cu) \otimes b(dv) - \frac{3}{4}(a | cu)(b | dv)1 - \frac{3}{4}(a | c)(b \otimes b)(d | v)\right)
\end{equation}

\begin{equation}
- (-1)^{\tilde{a}c + \tilde{b}d} \left(c(au) \otimes d(bv) - \frac{3}{4}(c | au)(d | bv)1 - \frac{3}{4}(a | u)(c | d | v)\right).
\end{equation}
By associativity \((a \mid cu) = (-1)^{\bar{a}\bar{c}}(c \mid au)\) and \((b \mid dv) = (-1)^{\bar{b}\bar{d}}(d \mid bv)\), so that the \(F_1\)-component of (2.4) is 0. Moreover,

\[
a(cu) \otimes b(dv) - (-1)^{\bar{a}\bar{c} + \bar{b}\bar{d}}c(au) \otimes d(bv)
= \frac{1}{2} ([L_a, L_c](u) \otimes (L_b \circ L_d)(v) + (L_a \circ L_c)(u) \otimes [L_b, L_d](v)),
\]

and by (1.6),

\[
a(c \mid u) \otimes b(d \mid v) - (-1)^{\bar{a}\bar{c} + \bar{b}\bar{d}}c(a \mid u) \otimes d(b \mid v)
= \frac{1}{2} \left( a(c \mid u) - (-1)^{\bar{a}\bar{c}}c(a \mid u) \right) \otimes \left( b(d \mid v) + (-1)^{\bar{b}\bar{d}}d(b \mid v) \right)
+ \left( a(c \mid u) + (-1)^{\bar{a}\bar{c}}c(a \mid u) \right) \otimes \left( b(d \mid v) - (-1)^{\bar{b}\bar{d}}d(b \mid v) \right)
= [L_a, L_c](u) \otimes \left( b(d \mid v) + (-1)^{\bar{b}\bar{d}}d(b \mid v) \right)
+ \left( a(c \mid u) + (-1)^{\bar{a}\bar{c}}c(a \mid v) \right) \otimes [L_b, L_d](v).
\]

Therefore, it follows from (1.7) that

\[
[L_{a \otimes b}, L_{c \otimes d}]|_{K \otimes K}
= (-1)^{\bar{e}} \left( [L_a, L_c] \otimes \left( \frac{1}{2}L_b \circ L_d - \frac{3}{4} \left( b(d \mid -) + (-1)^{\bar{b}\bar{d}}d(b \mid -) \right) \right)
+ \left( \frac{1}{2}L_a \circ L_c - \frac{3}{4} \left( a(c \mid -) + (-1)^{\bar{a}\bar{c}}c(a \mid -) \right) \right) \otimes [L_b, L_d] \right)
= (-1)^{\bar{e}} \frac{1}{2} \left( [L_a, L_c] \otimes (b \mid d) \text{id} + (a \mid c) \text{id} \otimes [L_b, L_d] \right),
\]

as desired. 

\(\square\)

**Corollary 2.5.** For \(a, b, c, d \in K_\bar{0} \cup K_1\), \([L_{a \otimes b}, L_{c \otimes d}]\) is a superderivation of \(J\).

**Proof.** Because \(K\) is a Jordan superalgebra, for \(a, b \in K_\bar{0} \cup K_1\), the mapping \([L_a, L_b]\) is a superderivation of \(K\). By the associativity of \((\cdot)\), it is super-skew-symmetric relative to \((\cdot)\). Consequently, from (2.3) we can conclude that \([L_{a \otimes b}, L_{c \otimes d}]\) is a superderivation of \(J\). \(\square\)

**Theorem 2.6.** \(J = F_1 \oplus (K \otimes K)\) is a Jordan superalgebra if \(\text{char } F \neq 2\).

**Proof.** A Jordan algebra over a field \(F\) of characteristic not 2 or 3 is characterized by commutativity and the identity \(\sum_{\text{cyclic}}[L_{uv}, L_w] = 0\). Hence a Jordan superalgebra over \(F\) is characterized by supercommutativity and

\[
\sum_{\text{supercyclic}} [L_{uv}, L_w] = [L_{uv}, L_w] + (-1)^{\bar{u}(\bar{v} + \bar{w})}[L_{uv}, L_w] + (-1)^{(\bar{u} + \bar{v})\bar{w}}[L_{uv}, L_w] = 0.
\]
In the supercommutative algebra $J$, we have from (1.2) and (2.3) that for all homogeneous $a, b, c, d, u, v \in K$,

\[
\sum_{\text{supercyclic}} [L_{(a \otimes b)(c \otimes d)}, L_{u \otimes v}]
\]

\[
= \sum_{\text{supercyclic}} (-1)^{\bar{b} \bar{c}} [L_{ac \otimes bd}, L_{u \otimes v}]
\]

\[
= \sum_{\text{supercyclic}} (-1)^{\bar{b} (\bar{c}+\bar{u})+\bar{d} \bar{e}} \frac{1}{2} \left( [L_{ac}, L_u] \otimes (bd \mid v) I + (ac \mid u) I \otimes [L_{bd}, L_v] \right)
\]

\[
= (-1)^{\bar{b} (\bar{c}+\bar{u})+\bar{d} \bar{e}} \frac{1}{2} \left( \sum_{\text{supercyclic}} [L_{ac}, L_u] \right) \otimes (bd \mid v) I
\]

\[
+ (ac \mid u) I \otimes \left( \sum_{\text{supercyclic}} [L_{bd}, L_v] \right)
\]

\[= 0.
\]

Thus, $J$ is a Jordan superalgebra whenever the field has characteristic different from 2 or 3.

If $R$ is the ring of fractions $R = S^{-1} \mathbb{Z}$ relative to the multiplicative set of integers $S = \{ 2^n \mid n = 0, 1, \ldots \}$, we could have defined $K$ and $J$ over $R$ as above. (Denote the result by $J_R$.) Then $J_Q = Q \otimes_R J_R$ is a Jordan superalgebra over the rationals $Q$, and so is $J_R$ over $R$. But if $F$ is a field having characteristic not 2, then $J = F \otimes_R J_R$ is a Jordan superalgebra over $F$. \hfill $\square$

From the computations and results above, it is easy to draw the following conclusions about the Jordan superalgebra $J$. Here we use the associator $(u, v, w) = (uv)w - u(vw)$ in $J$ in the statements.

**Proposition 2.7.** For the Jordan superalgebra $J = F1 \oplus (K \otimes K)$:

(a) $(J, J, J) = K \otimes K$.

(b) $J = F1 \oplus (J, J, J)$.

(c) If char $F = 3$, then $K \otimes K = (J, J, J)$ is an ideal of $J$.

(d) $e_1 := -\frac{1}{2} I + 2e \otimes e$ and $e_2 := \frac{3}{2} I - 2e \otimes e$ are orthogonal idempotents in $J$.

Relative to $e_1$, the Peirce spaces $J^{(\lambda)} := \{ u \in J \mid e_1 u = \lambda u \}$ are given by

\[J^{(1)} = Fe_1, \quad J^{(0)} = Fe_2 \oplus (K_1 \otimes K_1), \quad J^{(\frac{1}{2})} = (e \otimes K_1) \oplus (K_1 \otimes e).
\]

Thus, $J_0 = J^{(1)} \oplus J^{(0)}$ and $J_1 = J^{(\frac{1}{2})}$. The space $J^{(0)}$ is the Jordan algebra of a nondegenerate symmetric bilinear form, and $e_2$ is its unit element.

**Theorem 2.8.** (a) If char $F \neq 2, 3$, $J = F1 \oplus (K \otimes K)$ is a simple Jordan superalgebra.

(b) If char $F = 3$, $(J, J, J) = K \otimes K$ is a simple Jordan superalgebra.

(c) Der $J$ is inner $J$ and it is isomorphic to $\mathfrak{osp}(1, 2) \oplus \mathfrak{osp}(1, 2)$.

**Proof.** We begin with (c). Any superderivation of $J$ annihilates $F1$ and leaves invariant $(J, J, J) = K \otimes K$. Therefore, using the natural identifications and the fact that $\text{Der} K = \text{Inder} K \cong \mathfrak{osp}(1, 2)$, we have by (2.3) that

\[\text{Der} K \oplus \text{Der} K \cong (\text{Der} K \otimes 1) \oplus (1 \otimes \text{Der} K) = \text{Inder} J.
\]
Now one may check that Inder $J = \text{Der} J$ as follows: First for any $D \in (\text{Der} J)_0$, $D|_{J_0} \in \text{Der}(J_0)$. Since $J_0$ is a direct sum of a 1-dimensional ideal $J^{(1)} = Fe_1$ and the Jordan algebra $J^{(0)} = Fe_2 \oplus (K_1 \otimes K_1)$ of a bilinear form, $\text{Der}(J_0) = \text{Inder} J_0$. Thus, there is a $\tilde{D} \in (\text{Inder} J)_0$ with $(D - \tilde{D})(J_0) = 0$. It follows easily that $D = \tilde{D}$. Now suppose $E \in (\text{Der} J)_1$. Then there is a $E \in (\text{Inder} J)_1$ with $E(e_1) = \tilde{E}(e_1)$ because $(\text{Inder} J)_1 e_1 = J_1$. For any $z \in J_1$, $Ez = \frac{1}{2}z$, so $(E - \tilde{E})(z) = 2(E - \tilde{E})(e_1 z) = 2e_1(E - \tilde{E})(z)$. Thus, $(E - \tilde{E})(z) \in J_0 \cap J^{(2)} = 0$, so that $(E - \tilde{E})(J_1) = 0$. As a consequence, $E = \tilde{E}$ and $(\text{Der} J)_1 = (\text{Inder} J)_1$.

Since $\text{Der} K$ acts irreducibly on $K$, so does $\text{Der} J = \text{Inder} J$ on $K \otimes K = (J, J, J)$, which is an ideal of $J$ if and only if the characteristic of $F$ is 3. From this (a) and (b) follow. \qed

The Lie multiplication superalgebra of $J$. To compute the Lie multiplication superalgebra of $J$ we need some preliminaries. Let $H$ be a unital Jordan superalgebra with a normalized trace, that is, a linear map $t : H \to F$ such that $t(1) = 1$ and $t[(H, H, H)] = 0$. Assume that $(H, H, H) \neq 0$. Let $H_0 = \ker t$. Then for any $x, y \in H_0$, $xy = t(xy)1 + x \ast y$, where $x \ast y = xy - t(xy)1$. The Lie multiplication superalgebra is the Lie superalgebra spanned by $H$, which is a direct sum of the central ideal $F1$ and the ideal $\mathfrak{L}_0(H) = L_{H_0} \oplus [L_{H_0}, L_{H_0}]$. For simplicity, let us adopt the notation $S = \text{Inder} H$ and $T = H_0$. Then $\mathfrak{L}_0(H)$ is isomorphic to the Lie superalgebra $\mathfrak{L} = S \oplus T$ with multiplication determined by

1. the natural Lie multiplication on $S$,
2. the natural action of $S$ on $T$, that is, $[s, t] = s(t)$ for any $s \in S$ and $t \in T$, and
3. $[t_1, t_2] = [L_{t_1}, L_{t_2}] \in S$, for any $t_1, t_2 \in T$.

Also, given a nonzero scalar $\nu \in F$, let $\mathfrak{L}_\nu$ be the Lie superalgebra defined over $\mathfrak{L}$, but with the new bracket given by

(i) $\nu [x, y]^\nu = [x, y]$, for $x, y$ both in $S$ or one in $S$ and the other in $T$,
(ii) $[x, y]^\nu = \nu [x, y]$, for $x, y \in T$.

Lemma 2.10. Under the hypotheses above, if $\text{Hom}_{\mathfrak{L}}(T \otimes T, T)$ is spanned by $u \otimes v \mapsto u \ast v$ and if any automorphism of the superalgebra $(T, \ast)$ is an isometry of the supersymmetric bilinear form $(u, v) \mapsto t(uv)$, then for any $0 \neq \mu, \nu \in F$, $\mathfrak{L}_\mu$ is isomorphic to $\mathfrak{L}_\nu$ if and only if $\mu^{-1}\nu \in F^2$.

Proof. It is enough to apply the ideas of [BDE, proof of Prop. 4.2]. We include the argument for completeness.

First, since any isomorphism $\mathfrak{L}_\mu \to \mathfrak{L}_\nu$ is also an isomorphism $\mathfrak{L} \to \mathfrak{L}_{\mu^{-1}\nu}$, it is enough to assume $\mu = 1$. Suppose $\Phi : \mathfrak{L} \to \mathfrak{L}_\nu$ is an isomorphism. Then the map $T \otimes T \to T$ defined by $u \otimes v \mapsto \Phi^{-1}\left(\Phi(u) \ast \Phi(v)\right)$ is $S$-invariant. Consequently, there exists a nonzero scalar $\alpha \in F$ such that $\alpha \Phi(u) \ast \Phi(v) = \Phi(u \ast v)$ for any $u, v \in T$. Then $\psi : T \to T$ given by $\psi(u) = \alpha \Phi(u)$ is an automorphism of $(T, \ast)$ which, by hypothesis, extends by means of $\psi(1) = 1$ to an automorphism of the Jordan superalgebra $H$. Now the map $\Psi : \mathfrak{L} \to \mathfrak{L}_{\nu}$, such that $\Psi(s) = \psi s \psi^{-1}$ (composition of maps) for $s \in S$, and $\Psi(t) = \psi(t)$ for $t \in T$ is an automorphism of $\mathfrak{L}$. \qed
Furthermore, $\tilde{\Phi} = \Phi \Psi^{-1} : \mathcal{L} \rightarrow \mathcal{L}_\nu$ is an isomorphism such that $\tilde{\Phi}(t) = \Phi(t_1)t_2 = \alpha^{-2}t$ for $t \in T$. Hence, for any $t_1, t_2, t_3 \in T$,
\[
\tilde{\Phi}(t_1, t_2) = \Phi(t_1, t_2) = \alpha^{-2}t_1, t_2,
\]
so that $\nu = \alpha^{-2}$ as required. (Note $[T, T], T \neq 0$ because $(H, H, H) \neq 0$.)

The converse is clear, since the map $\Phi$ given by $\Phi(s) = s$ for $s \in S$ and $\Phi(t) = \alpha t$ for $t \in T$ provides an isomorphism $\mathcal{L} \rightarrow \mathcal{L}_\alpha^2$.

In order to apply the last result to our 10-dimensional Jordan superalgebra $J$, notice first that $J$ is endowed with a normalized trace $t$ such that $t(1) = 1$ and $t((J, J, J)) = 0$, as $(J, J, J) = K \otimes K$ does not intersect $F1$. Moreover, the associated bilinear form is given by $t((a \otimes b)(c \otimes d)) = (-1)^{ld(a \mid c)}(b \mid d)$ for any homogeneous $a, b, c, d \in K$, because of (2.1). Then all we need is the following

**Lemma 2.11.** Any automorphism of $(K \otimes K, *)$ is an isometry of the bilinear trace form.

**Proof.** Write $T = J_0 = K \otimes K$ and set $\langle a \otimes b \mid c \otimes d \rangle = (-1)^{ld(a \mid c)}(b \mid d)$ for any homogeneous elements $a, b, c, d \in K$. Then $T_0 = F(e \otimes e) \oplus (K_1 \otimes K_1)$. Suppose $\tilde{e} = e \otimes e$. Then the multiplication in $T_0$ is given by
\[
\tilde{e}^2 = \tilde{e}, \quad \tilde{e} \ast v = \frac{1}{4}v, \quad v \ast w = \langle v \mid w \rangle \tilde{e}
\]
for all $v, w \in K_1 \otimes K_1$.

Let $\varphi$ be an automorphism of $(T, \ast)$ and let us first verify that $\varphi(\tilde{e}) = \tilde{e}$. This is clear if the characteristic is 3, because $\tilde{e}$ is then the unit element of $T_0$. Otherwise, $\tilde{e}$ is the only idempotent of $T_0$ such that the corresponding left multiplication operator has eigenvalue $\frac{1}{4}$ with multiplicity 4. This is because the eigenvalues of $L_{a \otimes e \ast + v}$ (for $v \in K_1 \otimes K_1$, $a \in F$) include $\frac{1}{4}$ with multiplicity at least 3, as any $w \in K_1 \otimes K_1$ which is orthogonal to $v$ relative to $\langle \rangle$ is a corresponding eigenvector. Moreover, $\tilde{e} + v$ is an idempotent only if $v = 0$.

Now, $\varphi(K_1 \otimes K_1) \subseteq K_1 \otimes K_1$ because $K_1 \otimes K_1 = \{ v \in T_0 \mid \tilde{e} \neq Fv \}$. Because of (2.12), $\varphi|_{T_0}$ is an isometry of the restriction of $\langle \rangle$ to $T_0$.

Finally, $(K_1 \otimes e) \cup (e \otimes K_1) = \{ z \in T_1 \mid z \ast z \in F\tilde{e} \}$, and $K_1 \otimes e$ and $e \otimes K_1$ are the only 2-dimensional subspaces contained in this union. Therefore, either $\varphi$ preserves $K_1 \otimes e$ and $e \otimes K_1$ or it interchanges them. Since $(u \otimes e) \ast (v \otimes e) = \frac{1}{2}(u \otimes e \mid v \otimes e)\tilde{e}$ and $(e \otimes u) \ast (e \otimes v) = \frac{1}{2}(e \otimes u \mid e \otimes v)\tilde{e}$ for any $u, v \in K_1$, it follows that $\varphi|_{T_1}$ is an isometry too.

We are ready to determine the Lie multiplication superalgebra $\mathfrak{L}_0(J)$ for $J = F1 \oplus (K \otimes K)$. When the underlying field $F$ has characteristic 0, the next result is stated in [Sh] but without the assumption of $-1$ being a square in $F$.

**Theorem 2.13.** $\mathfrak{L}_0(J) \cong \mathfrak{osp}(K \otimes K)$ ($\cong \mathfrak{osp}(2, 4)$). In particular, $\mathfrak{L}_0(J)$ is a form of $\mathfrak{osp}(2, 4)$ and $\mathfrak{L}_0(J) \cong \mathfrak{osp}(2, 4)$ if and only if $-1$ is a square in $F$. 

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orthogonal sum) by means of $K$.

This can be extended to a linear bijection $\Phi : L_a \otimes b \to \mathfrak{osp}(K \oplus K)$ by defining

$$\Phi(L_a \otimes b) = \frac{1}{2} \gamma(a,0), (0, b)$$

for $a, b \in K$. Since

$$\begin{align*}
\left[ \gamma(a,0), (0, b) \right], \gamma(c,0), (0, d) \\
= \gamma(a,0), (0, b) \gamma(c,0), (0, d) + (-1)^{(a+b)c} \gamma(c,0), (a,0) (0, d) \\
= -(-1)^{(a+b)c} (a | c) \gamma(0, b), (0, d) + (-1)^{(a+b)c} \gamma(c,0), (a,0) (b | d) \\
= -(-1)^{(a+b)c} \left( (a | c) \gamma(0, b), (0, d) + \gamma(a,0), (c,0) (b | d) \right),
\end{align*}$$

it follows that $\Phi$ is an isomorphism of Lie superalgebras. The last part is a consequence of Lemmata 2.10 and 2.11.

3. The Kac superalgebra

We may identify the simple Jordan superalgebra $J$ with the 10-dimensional simple Jordan superalgebra $K_{10}$ discovered by Kac [K] (see also [HK]) when the characteristic is not 2 or 3, and $(J, J, J) = K \oplus K$ with the 9-dimensional degenerate Kac superalgebra $K_9$ if char $F = 3$. This realization may be described explicitly as follows. In the notation of [MZ] or [RZ], we suppose $\{ \hat{e}, v_1, v_2, v_3, v_4, \hat{f}, x_1, y_1, x_2, y_2 \}$ is a basis of $K_{10}$ with $(K_{10})_0 = \text{span}_F \{ \hat{e}, v_1, v_2, v_3, v_4, \hat{f} \}$ and $(K_{10})_1 = \text{span}_F \{ x_1, y_1, x_2, y_2 \}$ and with multiplication given by

$$\begin{align*}
\hat{e}^2 &= \hat{e}, & \hat{e} v_1 &= v_1, & v_1 v_2 &= 2 \hat{e} = v_3 v_4, & \hat{f}^2 &= \hat{f}, \\
\hat{e} x_j &= \frac{1}{2} x_j = \hat{f} x_j, & \hat{e} y_j &= \frac{1}{2} y_j = \hat{f} y_j, \\
v_1 y_1 &= x_2, & v_1 y_2 &= -x_1, & v_2 x_1 &= -y_2, & v_2 x_2 &= y_1, \\
v_3 x_2 &= x_1, & v_3 y_1 &= y_3, & v_4 x_1 &= x_2, & v_4 y_2 &= y_1, \\
x_j y_j &= \hat{e} - 3 \hat{f}, & x_1 x_3 &= v_1, & x_1 y_2 &= v_3, & x_2 y_1 &= v_4, & y_1 y_2 &= v_2 \\
& \text{for } i = 1, 2, 3, 4 \text{ and } j = 1, 2. \text{ All other products are either 0 or obtained from those in (3.1) by supercommutativity. Then a straightforward computation using (3.1) shows that the assignment}
\end{align*}$$

$$\begin{align*}
\hat{e} &\leftrightarrow \frac{3}{2} 1 - 2e \otimes e = e_2, & x_1 &\leftrightarrow 4x \otimes e \\
\hat{f} &\leftrightarrow -\frac{3}{2} 1 + 2e \otimes e = e_1, & x_2 &\leftrightarrow -4e \otimes x \\
v_1 &\leftrightarrow -4x \otimes x, & y_1 &\leftrightarrow -2y \otimes e \\
v_2 &\leftrightarrow -y \otimes y, & y_2 &\leftrightarrow 2e \otimes y \\
v_3 &\leftrightarrow 2x \otimes y, & v_4 &\leftrightarrow -2y \otimes x
\end{align*}$$

Proof. On any vector superspace $V$ equipped with a supersymmetric bilinear form $(\cdot | \cdot)$, the transformation $\gamma_{u,v} = u(v | -) - (-1)^{|u||v|} v(u | -)$ belongs to $\mathfrak{osp}(V)$ for all homogeneous $u, v \in V$. Because of (2.3) and (1.7), Der $J$ is embedded in $\mathfrak{osp}(K \oplus K)$ (with the natural supersymmetric bilinear form on $K \oplus K$ that makes this an orthogonal sum) by means of

$$\Phi \left( [L_a \otimes b, L_c \otimes d] \right) = (-1)^{|b|} \frac{1}{4} \left( \gamma(a,0), (c,0) (b | d) + (a | c) \gamma(0, b), (0, d) \right) \in \mathfrak{osp}(K \oplus K).$$

This can be extended to a linear bijection $\Phi : \mathfrak{L}_0(J)_{-1} \to \mathfrak{osp}(K \oplus K)$ by defining
A NEW CONSTRUCTION OF THE KAC JORDAN SUPeralgebra

3217

gives an isomorphism between $K_{10}$ and $J$. Additionally, if $\text{char} F = 3$, then $\mathfrak{e} \leftrightarrow -2e \otimes e$, so that the isomorphism restricts to an isomorphism $K_9 \leftrightarrow K \otimes K$.

Concluding remarks. As the reader may have guessed, the actual process for deriving our results was the reverse path of the presentation above. First it was verified that $g := \text{Der } K_{10} = \mathfrak{osp}(1, 2) \oplus \mathfrak{osp}(1, 2)$, (compare Shtern [Sh]), and that as module for the Lie superalgebra $g$, $K_{10} = F_1 \oplus (V \otimes V)$, where $V$ is the natural 3-dimensional module for $\mathfrak{osp}(1, 2)$. The first copy of $\mathfrak{osp}(1, 2)$ operates on the first tensor slot of $V \otimes V$, and the second copy on the second slot. The spaces $\text{Hom}_{\mathfrak{osp}(1, 2)}(V \otimes V, F)$ and $\text{Hom}_{\mathfrak{osp}(1, 2)}(V \otimes V, \mathfrak{osp}(1, 2))$ are 1-dimensional. By identifying $V$ with the tiny Kaplansky superalgebra $K$, we may describe the generators of these spaces nicely by the maps $u \otimes v \mapsto (u | v)$ and $u \otimes v \mapsto [L_u, L_v] \in \mathfrak{osp}(1, 2)$ for $u, v \in K$. From this, the multiplication in the Kac superalgebra $K_{10}$ is almost uniquely determined in terms of $K$.

In [S1], Shestakov proposed quite a different realization of the superalgebra $K_{10}$ in characteristic $\neq 2, 3$, but the same description of $K_9$; however, he has not published this work [S2].

References


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