TOPOLOGICALLY TRANSVERSAL
REVERSIBLE HOMOCLINIC SETS

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Abstract. An $R$-reversible diffeomorphism on $\mathbb{R}^{2N}$ is studied possessing a hyperbolic fixed point. If the stable manifold of the hyperbolic fixed point and the fixed point set $\text{Fix} R$ of $R$ have a nontrivial local topological crossing, then an infinite number of $R$-symmetric periodic orbits of the diffeomorphism is shown. A perturbed problem is also studied by showing the relationship between a corresponding Melnikov function and the nontriviality of a local topological crossing of the set $\text{Fix} R$ and the stable manifold for the perturbed diffeomorphism.

1. Introduction

Let $R : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be a linear involution, i.e. $R^2 = I$, such that $\dim \text{Fix} R = N$, where $\text{Fix} R = \{ x \in \mathbb{R}^{2N} \mid Rx = x \}$. Consider a $C^1$-smooth diffeomorphism $f : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ which is $R$-reversible, i.e. $Rf(x) = f^{-1}(Rx)$, $\forall x \in \mathbb{R}^{2N}$, and possessing an $R$-symmetric hyperbolic fixed point $p \in \text{Fix} R$. Any subset of $\mathbb{R}^{2N}$ invariant under the action of $R$ is called $R$-symmetric. Reversible diffeomorphisms naturally come from mechanics [5] as the time flow mappings of second order gradient differential equations. Let $W^s_p, W^u_p$ be the global stable and unstable manifolds of $p$, respectively. Let $\tilde{W}^s_p$ be an open subset of $W^s_p$ which is a submanifold of $\mathbb{R}^{2N}$, i.e. the immersed and induced topologies on $\tilde{W}^s_p$ coincide, and such that $\tilde{W}^s_p \setminus \{ p \} \cap \text{Fix} R \neq \emptyset$, i.e. there is an $R$-symmetric point $q$ homoclinic to $p$ [10]. Since $RW^s_p = W^u_p$, we put $\tilde{W}^u_p = R\tilde{W}^s_p$. We also suppose the existence of a compact component $K \ni q$ of the set $\tilde{W}^s_p \cap \text{Fix} R$, that is a compact subset $K \subset \tilde{W}^s_p \setminus \{ p \} \cap \text{Fix} R$ such that $q \in K$ and there exists an open bounded subset $U \subset \tilde{W} \subset \mathbb{R}^{2N} \setminus \{ p \}$ satisfying $U \cap \tilde{W}^s_p \cap \text{Fix} R = K$. By shrinking $U$, we can assume that $\tilde{W}^s_p \cap U = \tilde{W}^s_p \cap \tilde{U}$. We note that $\tilde{W}^s_p \cap U$ is an orientable submanifold of $\mathbb{R}^{2N}$. Then we can define the local intersection number $\#(\tilde{W}^s_p \cap U, \text{Fix} R \cap U)$ of the stable manifold $\tilde{W}^s_p$ and the plain $\text{Fix} R$ in $U \subset \mathbb{R}^{2N}$ [7]. The main purpose of this note is to prove the following result.

Theorem 1.1. If $\#(\tilde{W}^s_p \cap U, \text{Fix} R \cap U) \neq 0$, then there is an $\omega_0 \in \mathbb{N}$ such that for any $N \ni \omega \geq \omega_0$ the diffeomorphism $f$ possesses a $2\omega$-periodic orbit $\{ x_\omega \}_{\omega \in \mathbb{Z}}$ such

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that \( Rx_n^ω = x_n^ω, \ n \in \mathbb{Z} \). Moreover, \( x_0^ω \in \text{Fix } R \) is near the set \( K \), while \( x_n^ω \in \text{Fix } R \) is near the point \( p \).

When \( q \) is a transversal intersection of \( W^s_p \) and \( \text{Fix } R \), then Theorem 1.1 is proved in \([3, 4, 5, 9, 10]\). Then clearly \( \#(\tilde{W}^s_p \cap U, \text{Fix } R \cap U) \neq 0 \) for a small open neighbourhood \( U \) of \( q \). Furthermore, we study the case where \( W^s_p \) and \( \text{Fix } R \) intersect on a compact manifold. Then we consider a \( C^2 \)-smooth \( R \)-reversible perturbation of \( f \). Associated to such a perturbation there is a Melnikov function. We show that if the Brouwer degree \([6]\) of this Melnikov function is not zero, and the perturbation is small, then the perturbed stable manifold \( W^s_{p, \text{per}} \) and the plain \( \text{Fix } R \) satisfy \( \#(\tilde{W}^s_{p, \text{per}} \cap U, \text{Fix } R \cap U) \neq 0 \) with the corresponding infinitely many \( R \)-symmetric periodic orbits of the perturbed diffeomorphism. Finally, we note that any accumulation point of the set \( \{x^ω_0\}_{ω>ω_0} \subset \text{Fix } R \) from Theorem 1.1 is a starting point of an \( R \)-symmetric homoclinic orbit of \( f \) to \( p \).

If \( p \) is a hyperbolic fixed point of \( f \) but not \( R \)-symmetric, then \( Rp \) is also a hyperbolic fixed point of \( f \). If \( q \in W^s_p \cap \text{Fix } R \), then \( q \in W^s_p \cap W^s_{R \bar{p}} [5] \), hence \( q \) lies on an \( R \)-symmetric heteroclinic orbit connecting \( p \) and \( Rp \). Consequently, as for Theorem 1.1, we can prove the following result.

**Theorem 1.2.** Suppose \( f \) has a non-\( R \)-symmetric hyperbolic fixed point \( p \). If \( W^s_p \) and \( W^u_p \) meet \( \text{Fix } R \) locally topologically transversally, then \( f \) has an infinite number of \( R \)-symmetric periodic orbits with periods tending to infinity.

We note that for the case \( N = 1 \) it is elementary to show that if \( p \) is a hyperbolic fixed point of \( f \) and \( W^s_p \) (or \( W^u_p \)) meets \( \text{Fix } R \), then a local intersection number of \( W^s_p \) (or \( W^u_p \)) with \( \text{Fix } R \) is nonzero. Then Theorems 1.1 and 1.2 can be applied. Indeed, let \( q \in W^s_p \cap \text{Fix } R \) be the first intersection starting on \( W^s_p \) from \( p \). Then the points \( f^{-1}(q) \), \( f(q) \in W^s_p \) lie on the opposite half-planes separated by \( \text{Fix } R \). Hence an open bounded connected part \( \tilde{W}^s_p \) of \( W^s_p \) such that \( f^{-1}(q) \), \( f(q) \in \tilde{W}^s_p \) topologically nontrivially crosses \( \text{Fix } R \). Similarly for \( W^u_p \).

The paper is finished with an example of a perturbed second order differential equation in \( \mathbb{R}^2 \) with a topologically transversal, but non-\( C^1 \)-transversal, intersection of the stable manifold and \( \text{Fix } R \).

### 2. Preliminary results

Let \( \langle \cdot, \cdot \rangle \) be an inner product on \( \mathbb{R}^{2N} \). By following \([10]\), we set

\[
\langle x, y \rangle = \frac{1}{2}(x, y) + (Rx, Ry).
\]

Then \( \langle Rx, Ry \rangle = \langle x, y \rangle, \ x, y \in \mathbb{R}^{2N} \), and so \( ||R|| = ||R^{-1}|| = 1 \). Since \( RK = K \), we can assume that \( RU = U \).

For any \( ξ \in \tilde{W}^s_p \cap U \) we set \( ξ_n = f^n(ξ), \ n \in \mathbb{Z}_+, \ η_n = f^n(η), \ n \in \mathbb{Z}_-, \ η = Rξ \), where \( \mathbb{Z}_+ = \{0, 1, 2, \cdots \} \) and \( \mathbb{Z}_- = \{\cdots, -2, -1, 0\} \). We note that \( η_{-n} = Rξ_n, \ n \in \mathbb{Z}_+ \). Then the linear system

\[
ρ_{n+1} = Df(ξ_n)ρ_n, \quad n \in \mathbb{Z}_+,
\]

has an exponential dichotomy on \( \mathbb{Z}_+ [3] \), i.e. there are positive constants \( L, δ \in (0, 1) \) and the orthogonal projection \( P_ξ : \mathbb{R}^{2N} \to T_ξ\tilde{W}^s_p \) such that the fundamental
We note $n \in L$.

Moreover, (5)

Hence, the fundamental solution (2) satisfies the following:

$$\|V_\xi(n)P_\xi V_\xi(m)^{-1}\| \leq L\delta^{n-m}, \quad m \leq n, \quad m, n \in \mathbb{Z}_+,$$

$$\|V_\xi(n)(I - P_\xi)V_\xi(m)^{-1}\| \leq L\delta^{m-n}, \quad n \leq m, \quad m, n \in \mathbb{Z}_+.$$ 

We note that $L$ and $\delta$ can be chosen to be independent of $\xi \in \tilde{W}_p \cap \tilde{U}$.

By defining $Rv_n = w_n$ in (1), the reversibility of $f$ implies

$$w_{n+1} = Df(\eta_n)w_n, \quad n \in \mathbb{Z}_-, \quad n \neq 0.$$ 

Hence, the fundamental solution $W_\xi(n)$ of (2) is given by $W_\xi(n) = RV_\xi(-n)R^{-1}$, $n \in \mathbb{Z}_-$, and since $\|R\| = \|R^{-1}\| = 1$, then (2) has an exponential dichotomy on $\mathbb{Z}_-$ with the constants $L$, $\delta$ and the orthogonal projection $I - Q_\eta$, where $Q_\eta = RP_\xi R^{-1}$.

We note $\eta = R\xi$.

Now we fix $\omega \in \mathbb{N}$ large and put

$$J_\omega = \{-\omega, -\omega + 1, \cdots, \omega - 1, \omega\},$$

$$J_\omega^- = \{-\omega, -\omega + 1, \cdots, -1, 0\}, \quad I_\omega^- = \{-\omega, -\omega + 1, \cdots, -1\},$$

$$J_\omega^+ = \{0, 1, \cdots, \omega - 1, \omega\}, \quad I_\omega^+ = \{0, 1, \cdots, \omega - 2, \omega - 1\}.$$ 

We note that the family $\{P_\xi | \xi \in \tilde{W}_p \cap U\}$ is continuous on $\tilde{W}_p \cap U$. In this paper, $\mathcal{R}L$ and $\mathcal{N}L$ denote, respectively, the range and the kernel of a linear operator $L$.

**Theorem 2.1** (1). There exist $\omega_0 \in \mathbb{N}$ and a constant $c > 0$ such that, for any $\omega \in \mathbb{N}$, $\omega \geq \omega_0$, and $\xi \in \tilde{W}_p \cap U$, there exist unique $\{x_n^+(\omega, \xi)\}_{n \in J_\omega^+}$ and $\{x_n^-(\omega, \xi)\}_{n \in J_\omega^-}$ which satisfy $x_{n+1} = f(x_n)$ separately on $I_\omega^+$ and $I_\omega^-$ such that

$$P_\xi x_0^+(\omega, \xi) = P_\xi \xi, \quad Q_{R\xi} x_0^-(\omega, \xi) = Q_{R\xi} R\xi,$$$$
x_0^+(\omega, \xi) = x_0^-(\omega, \xi),$$

together with

$$\max_{\xi \in \mathcal{J}_\omega} |x_n^+(\omega, \xi) - \xi_n| \leq c\delta^\omega,$$

$$\max_{\xi \in \mathcal{J}_\omega} |x_n^-(\omega, \xi) - \eta_n| \leq c\delta^\omega.$$ 

Moreover, $x_n^+(\omega, \xi)$ are continuous with respect to $\xi$.

**Proof.** We study the nonlinear system

$$x_{n+1} = f(x_n)$$

near $\{\xi_n\}_{n \in \mathcal{J}_\omega^+}$ and $\{\eta_n\}_{n \in \mathcal{J}_\omega^-}$. By putting $x_n^+ = \xi_n + v_n$, $n \in J_\omega^+$ and $x_n^- = \eta_n + w_n$, $n \in J_\omega^-$, we get the systems

$$v_{n+1} = Df(\xi_n)v_n + f(\xi_n + v_n) - f(\xi_n) - Df(\xi_n)v_n$$

(4)

$$= Df(\xi_n)v_n + o(|v_n|), \quad n \in I_\omega^+,$$

and

$$w_{n+1} = Df(\eta_n)w_n + f(\eta_n + w_n) - f(\eta_n) - Df(\eta_n)w_n$$

(5)

$$= Df(\eta_n)w_n + o(|w_n|), \quad n \in I_\omega^-.$$
Since we are looking for solutions of equation (3) such that \( x^+_\omega = x^-_\omega \), we add the boundary value conditions

\[
v_\omega - w_\omega = \eta_\omega - \xi_\omega = O(\delta^a), \quad P_\xi v_0 = 0, \quad Q_{R\xi} w_0 = 0.
\]

Let \( v = (v_0, \ldots, v_\omega) \in \mathbb{R}^{N(\omega + 1)}, w = (w_\omega, \ldots, w_0) \in \mathbb{R}^{N(\omega + 1)} \). To solve equations (4)–(6), we take the mapping \( \Gamma_\omega : W^s_p \cap U \times \mathbb{R}^{2N(\omega + 1)} \to \mathbb{R}^{2N(\omega + 1)} \) defined by

\[
\Gamma_\omega(\xi, v, w) = \begin{pmatrix}
(v_{n+1} - f(\zeta_n + v_n) + f(\xi_n))_{n \in I^+} \\
(w_{n+1} - f(\eta_n + w_n) + f(\xi_n))_{n \in I^-} \\
v_\omega - w_\omega - (\eta_\omega - \xi_\omega) \\
P_\xi v_0 \\
Q_{R\xi} w_0
\end{pmatrix}.
\]

Arguing as in Lemma 2.1 of [1] or Lemma 2 of [2], the map \( D_{(v,w)} \Gamma_\omega(x,0,0) \) is invertible and that its inverse is bounded uniformly with respect to \( \xi \). Hence from the implicit function theorem we get that \( c > 0 \) and \( \omega_0 \gg 1 \) exist such that for \( \omega \geq \omega_0 \), the equation \( \Gamma_\omega(x,v,w) = 0 \) can be solved uniquely for \( (v,w) \) in a neighborhood of \( (0,0) \) in terms of \( (\xi,\omega) \). Moreover \( \max_{x \in \{ |v|, |w| \}} < c\delta^a \), and the solution is continuous in \( \xi \) for any fixed \( \omega \geq \omega_0 \). The proof is finished.

Since \( Q_{R\xi} = RP_\xi R^{-1}, \eta_n = R\xi_n, n \in \mathbb{Z}_+ \), we see that the sequences given by \( y^+_n(\omega,\xi) = Rx^+_n(\omega,\xi), n \in J^+_\omega, y^-_n(\omega,\xi) = Rx^-_n(\omega,\xi), n \in J^-_\omega \) also satisfy the statement of Theorem 2.1. The uniqueness of such orbits implies that

\[
x^+_n(\omega,\xi) = y^+_n(\omega,\xi) = Rx^+_n(\omega,\xi), \quad n \in J^+_\omega,
\]

\[
x^-_n(\omega,\xi) = y^-_n(\omega,\xi) = Rx^-_n(\omega,\xi), \quad n \in J^-_\omega.
\]

Hence \( Rx^+_n(\omega,\xi) = x^+_n(\omega,\xi), n \in J^+_\omega \). So the orbit of \( f \) in Theorem 2.1 is \( R \)-symmetric.

### 3. \( R \)-Symmetric Periodic Orbits

In this section we prove Theorem 1.1. If \( x^+_0(\omega,\xi) \in \text{Fix} R \), then

\[
x^+_0(\omega,\xi) = Rx^+_0(\omega,\xi) = x^-_0(\omega,\xi).
\]

Consequently, the orbit of Theorem 2.1 becomes an \( R \)-symmetric periodic orbit of \( f \). Hence we have to solve the equation

\[
(I - R)x^+_0(\omega,\xi) = 0, \quad \xi \in W^s_p \cap U.
\]
Let $V$ be an open subset such that $K \subset V \subset \bar{V} \subset U$ and let $\omega_0$ be as in Theorem 2.1. Note that the solution $x_0^+(\omega, \xi)$ is defined for $\xi \in \widetilde{W}_p^s \cap V$ and
\[
\#(\widetilde{W}_p^s \cap V, \text{Fix } R \cap V) = \#(\widetilde{W}_p^s \cap U, \text{Fix } R \cap U) \neq 0.
\]
To solve (7), we put $F_\omega(\xi) = (I - R)x_0^+(\omega, \xi)$, $F_\omega : \widetilde{W}_p^s \cap \bar{V} \to R_\omega = R(I - R)$. We note that $\dim R_\omega = \dim \widetilde{W}_p^s$.

Now we put $F_\omega$ into the homotopy
\[
\begin{align*}
H_\omega : \widetilde{W}_p^s \cap \bar{V} \times [0, 1] & \to R_\omega, \\
H_\omega(\xi, \lambda) &= \lambda F_\omega(\xi) + (1 - \lambda)(I - R)\xi.
\end{align*}
\]

Theorem 2.1 gives
\[
|H_\omega(\xi, \lambda) - (I - R)\xi| = |\lambda(F_\omega(\xi) - (I - R)\xi)| \leq c\delta^2.
\]
Consequently, $H_\omega(\cdot, \lambda) \neq 0$ on the boundary $\partial(\widetilde{W}_p^s \cap V)$ for any $0 \leq \lambda \leq 1$. This gives for the Brouwer degree [6],
\[
\deg (F_\omega, \widetilde{W}_p^s \cap V, 0) = \pm \#(\widetilde{W}_p^s \cap V, \text{Fix } R \cap V) \neq 0.
\]

Summarizing, we see that $F_\omega(\xi) = 0$ has a solution $\xi \in \widetilde{W}_p^s \cap V$ for any $\omega \geq \omega_0$, where $\omega_0$ is sufficiently large. This proves Theorem 1.1.

4. Perturbation theory

In this section, we consider a $C^2$-smooth perturbation $f(x, \varepsilon)$ of $f$, i.e. we suppose that $f(x, 0) = f(x)$ and $Rf(x, \varepsilon) = f^{-1}(Rx, \varepsilon)$, $\forall x \in \mathbb{R}^{2N}$, $\varepsilon$ small. Then Theorem 2.1 gives a $C^1$-mapping $x_0^+(\omega, \xi, \varepsilon)$ and we are led to the equation
\[
(I - R)x_0^+(\omega, \xi, \varepsilon) = 0, \quad \xi \in \widetilde{W}_p^s \cap U.
\]
By taking $\omega \to \infty$ in the above equation, we get $F(\xi, \varepsilon) = (I - R)x_0^+ (\infty, \xi, \varepsilon) = 0$ which is precisely the equation of $R$-symmetric homoclinic solutions to the hyperbolic symmetric fixed point $p_\varepsilon$ of $f(x, \varepsilon)$ near $p$ [10]. We assume that this equation has a compact nondegenerate solution manifold, i.e.

(H1) There is an embedded compact $C^2$-smooth submanifold $\mathcal{M} \subset \widetilde{W}_p^s \setminus \{p\} \cap \text{Fix } R$ for an open subset $\widetilde{W}_p^s$ of $W_p^s$ which is a submanifold of $\mathbb{R}^{2N}$ and such that $\dim N(I - R)D_\xi x_0^+(\infty, \xi, 0) = \dim \mathcal{M}$ for any $\xi \in \mathcal{M}$. Furthermore, let $O \subset \widetilde{W}_p^s \setminus \{p\}$ be an open bounded subset such that $\mathcal{M} \subset O$. Then $O$ can be oriented. We suppose that $\mathcal{M}$ is orientable embedded into $O$.

We note that always $T_\xi \mathcal{M} \subset N(I - R)D_\xi x_0^+(\infty, \xi, 0)$, $\forall \xi \in \mathcal{M}$ and $\dim \mathcal{M} = \dim T_\xi \mathcal{M}$, hence (H1) implies $T_\xi \mathcal{M} = N(I - R)D_\xi x_0^+(\infty, \xi, 0)$, $\forall \xi \in \mathcal{M}$. Since $x_0^+(\infty, \xi, 0) = \xi$, we have $D_\xi x_0^+(\infty, \xi, 0)v = v, v \in T_\xi \widetilde{W}_p^s$. Hence
\[
N(I - R)D_\xi x_0^+(\infty, \xi, 0) = \text{Fix } R \cap T_\xi \widetilde{W}_p^s.
\]
Now we take a tubular neighbourhood $\mathcal{V}$ of $\mathcal{M}$ in $\widetilde{W}_p^s$, i.e. any $\xi \in \mathcal{V}$ can be uniquely expressed as a pair $(\tau, v)$, where $\tau \in \mathcal{M}$ and $v \in T_\tau \widetilde{W}_p^s / (\text{Fix } R \cap T_\tau \widetilde{W}_p^s) = T_\tau \widetilde{W}_p^s / (\text{Fix } R \cap T_\tau \widetilde{W}_p^s) = N_\tau$—the fiber of the normal vector bundle of $\mathcal{M}$ in $\widetilde{W}_p^s$, and $|v| < \Delta$ for some $\Delta > 0$. Hence we identify $\mathcal{V}$ with an open neighbourhood of the zero section of the normal vector bundle of $\mathcal{M}$ in $\widetilde{W}_p^s$. Let $S_\tau : \text{Fix } (-R) \to RD_\tau F(\tau, 0, 0)$ be the orthogonal projection. We note that the assumption (H1)
implies the invertibility of the linear mapping $D_{\tau}F(\tau,0,0) : N_\tau \to RF(\tau,0,0)$, since $D_{\tau}F(\tau,0,0)w = (I - R)w$, $w \in N_\tau$.

From $F(\tau,0,0) = 0$, we get $F(\tau,v,\varepsilon) = D_{\tau}F(\tau,0,0)v + \varepsilon D_{\tau}F(\tau,0,0) + o(|v|) + o(\varepsilon)$. We consider the homotopy

$$H(\tau,v,\varepsilon,\lambda) = S_{\tau}(\lambda F(\tau,v,\varepsilon) + (1 - \lambda)D_{\tau}F(\tau,0,0)v) + (I - S_{\tau})\left(\lambda F(\tau,v,\varepsilon) + (1 - \lambda)\varepsilon D_{\tau}F(\tau,0,0)\right).$$

Then $H(\tau,v,\varepsilon,0) = S_{\tau}D_{\tau}F(\tau,0,0)v + \varepsilon(I - S_{\tau})D_{\tau}F(\tau,0,0)$ and $H(\tau,v,\varepsilon,1) = F(\tau,v,\varepsilon)$.

Now we suppose

(H2) There is an open connected subset $\Omega \subset \mathcal{M}$ such that $B(\tau) \neq 0$, $\forall \tau \in \partial \Omega$, where $B(\tau) = (I - S_{\tau})D_{\tau}F(\tau,0,0)$.

Since $\mathcal{M}$ is orientable embedded into $\mathcal{O}$ and $\mathcal{O}$ is orientable, the tangent vector bundle $TM$ and the normal vector bundle $\bigcup_{\tau \in \mathcal{M}}N_{\tau}$ are both orientable. Hence the vector bundle $\bigcup_{\tau \in \mathcal{M}}(I - R)N_{\tau} = \bigcup_{\tau \in \mathcal{M}}RD_{\tau}F(\tau,0,0)$ is also orientable, because $D_{\tau}F(\tau,0,0) : N_{\tau} \to RD_{\tau}F(\tau,0,0)$ is invertible. Since $S_{\tau} : \text{Fix}(R) \to RD_{\tau}F(\tau,0,0)$ is the orthogonal projection and the vector bundle $\bigcup_{\tau \in \mathcal{M}}\text{Fix}(R) = \mathcal{M} \times \text{Fix}(R)$ is orientable, we get the orientability of the vector bundle $\bigcup_{\tau \in \mathcal{M}}R(I - S_{\tau})\text{Fix}(R)$. Since $B(\tau)$ is a section of this vector bundle, the following assumption makes sense [7].

(H3) $\deg(B(\tau),\Omega,0) \neq 0$.

According to (H2), there is an open connected bounded neighbourhood $U_1 \subset \mathcal{M}$ of $\Omega$ such that $B(\lambda) \neq 0$, $\forall \tau \in U_1 \setminus \Omega$. Now we take an open subset $V_\varepsilon = \{(\tau,v) \in V \mid \tau \in U_1$ and $|v| < |\varepsilon| r_1\}$ for a positive constant $r_1$ and $0 < |\varepsilon| < \Delta/r_1$. Since $S_{\tau}(\lambda F(\tau,v,\varepsilon) + (1 - \lambda)D_{\tau}F(\tau,0,0)v) = S_{\tau}D_{\tau}F(\tau,0,0)v + o(|v|) + O(\varepsilon)$ and $S_{\tau}D_{\tau}F(\tau,0,0) : N_{\tau} \to RD_{\tau}F(\tau,0,0)$ is invertible, we get that $S_{\tau}(\lambda F(\tau,v,\varepsilon) + (1 - \lambda)D_{\tau}F(\tau,0,0)v) \neq 0$, $\forall (\tau,v) \in V_\varepsilon$, $|v| = r_1|\varepsilon|$ for $r_1$ sufficiently large and fixed. Furthermore, we have $(I - S_{\tau})(\lambda F(\tau,v,\varepsilon) + (1 - \lambda)\varepsilon D_{\tau}F(\tau,0,0)) = \varepsilon(I - S_{\tau})D_{\tau}F(\tau,0,0) + o(|v|) + o(\varepsilon)$ since $(I - S_{\tau})D_{\tau}F(\tau,0,0) = 0$. Hence $(I - S_{\tau})(\lambda F(\tau,v,\varepsilon) + (1 - \lambda)\varepsilon D_{\tau}F(\tau,0,0)) \neq 0$ for $(\tau,v) \in V_\varepsilon$, $|v| < r_1|\varepsilon|$ and $\tau \in U_1 \setminus \Omega$.

Summarizing, we see that $H(\tau,v,\varepsilon,\lambda) \neq 0$ for any $(\tau,v) \in \partial V_\varepsilon$, $\lambda \in [0,1]$ and $\varepsilon \neq 0$ sufficiently small. Consequently [6],

$$\deg(F,V_\varepsilon,0) = \deg(H(\cdot,\varepsilon,0),V_\varepsilon,0),$$

where $H(\cdot,\varepsilon,0) = S_{\tau}D_{\tau}F(\tau,v(\tau,\varepsilon),\varepsilon)v + \varepsilon(I - S_{\tau})D_{\tau}F(\tau,0,0)$. Since the linear map $S_{\tau}D_{\tau}F(\tau,v(\tau,\varepsilon),\varepsilon) : N_{\tau} \to RD_{\tau}F(\tau,0,0)$ is invertible and $U_1$ is connected, we get

$$\deg(H(\cdot,\varepsilon,0),V_\varepsilon,0) = \pm\deg(B(\tau),\Omega,0) \neq 0.$$ 

This implies $\#(\overline{W_p^s} \cap V_\varepsilon,\text{Fix } R \cap V_\varepsilon) \neq 0$. Summarizing we get the following result.

**Theorem 4.1.** Assume (H1), (H2) and (H3). Then there exists $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| \leq \varepsilon_0$, it is nonzero the local intersection number of the plain $R$ and the stable manifold of the hyperbolic fixed point of the map $x_{n+1} = f(x_n,\varepsilon)$ which is located near the fixed point $p$ of the map $x_{n+1} = f(x_n)$.
Hence Theorem 1.1 and the assumptions of Theorem 4.1 imply an infinite number of $R$-symmetric periodic orbits of $f(x, \varepsilon)$ accumulating on $R$-symmetric homoclinic orbits of $f(x, \varepsilon)$ for any $0 < \varepsilon \leq \varepsilon_0$.

By taking $\Omega = \mathcal{M}$, we see [2] that assumption (H3) holds if the Euler characteristic $\chi\left(\bigcup_{\tau \in \mathcal{M}} \mathcal{R}(I - S_{\tau})\text{Fix}(-R)\right)$ is nonzero. Then Theorem 4.1 holds under assumption (H1) for any $R$-reversible $C^2$-smooth perturbation $f(x, \varepsilon)$.

If $f(x, \varepsilon)$ is $C^3$-smooth, then $F$ is $C^2$-smooth. To solve $F(\tau, v, \varepsilon) = 0$, we follow the standard way [1] by splitting it as $F(\tau, v, \varepsilon) = S_{\tau}F(\tau, v, \varepsilon) + (I - S_{\tau})F(\tau, v, \varepsilon)$. By using the implicit function theorem, we can solve the equation $S_{\tau}F(\tau, v, \varepsilon) = 0$ in $v$ for $\varepsilon$ small and $\tau \in \mathcal{M}$ to get the $C^2$-smooth solution $v = v(\tau, \varepsilon) = O(\varepsilon)$. Then we consider the bifurcation equation $C(\tau, \varepsilon) = (I - S_{\tau})F(\tau, v(\tau, \varepsilon), \varepsilon) = 0$. Clearly $C(\tau, \varepsilon)/\varepsilon \to B(\tau)$ in the $C^1$-topology on $\mathcal{M}$ as $\varepsilon \to 0$. Consequently, a simple zero $\tau_0$ of $B(\tau)$, i.e. $B(\tau_0) = 0$ and $DB(\tau_0)$ is nonsingular, implies the solvability of $C(\tau, \varepsilon) = 0$ in $\tau$ near $\tau_0$ for $\varepsilon \neq 0$ small. Summarizing we see that $B$ is the Melnikov function for this problem, since its simple zero $\tau_0$ ensures the bifurcation of an $R$-symmetric homoclinic orbit of $f(x, \varepsilon)$ to $p_\varepsilon$ for $\varepsilon \neq 0$ small bifurcating from the $R$-symmetric homoclinic orbit of $f(x, 0) = f(x)$ which starts from $\tau_0 \in \mathcal{M}$.

5. AN EXAMPLE

In this section, we present an example, but first we simplify the formula of $B(\tau)$. If $a \in \mathcal{R}(I - S_{\tau})$, then $a \in \text{Fix}(-R)$ and $a \perp (I - R)T_\tau\tilde{W}_p^s$. Hence for any $w \in T_\tau\tilde{W}_p^s$ we have

$$0 = \langle a, (I - R)w \rangle = \langle a, w \rangle - \langle a, Rw \rangle = \langle a - Ra, w \rangle = 2\langle a, w \rangle.$$ 

Furthermore, since $RT_\tau\tilde{W}_p^s = T_\tau\tilde{W}_p^s + T_\tau\tilde{W}_p^u$ and for any $w \in T_\tau\tilde{W}_p^s$, we have

$$\langle a, Rw \rangle = -\langle Ra, w \rangle = -\langle a, w \rangle = 0,$$

we see that $a \in \mathcal{R}(I - S_{\tau})$ if and only if $a \perp (T_\tau\tilde{W}_p^s + T_\tau\tilde{W}_p^u + \text{Fix} R)$. We note that $	ext{Fix}(-R) = (\text{Fix} R)^{-1}$, and $\frac{1}{2}(I - R) : \mathbb{R}^{2N} \to \text{Fix}(-R)$ and $\frac{1}{2}(I + R) : \mathbb{R}^{2N} \to \text{Fix} R$ are the orthogonal projections. Consequently, if $a_i(\tau), i = 1, 2, \cdots, \dim \mathcal{M}$, is a continuous orthonormal vector field such that $a_i(\tau) \perp (T_\tau\tilde{W}_p^s + T_\tau\tilde{W}_p^u + \text{Fix} R)$, then the components of $B(\tau)$ are given by

$$B_i(\tau) = \langle a_i(\tau), (I - R)D_x x_0^+((\infty, \tau, 0)) \rangle = 2\langle a_i(\tau), D_x x_0^+((\infty, \tau, 0)) \rangle.$$ 

Now consider a perturbed second order differential equation

$$\ddot{z} = g(z) + \varepsilon h(z), \quad z \in \mathbb{R}^N,$$

where $g, h \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, $g(0) = h(0) = 0$. [2] has the form

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = g(z_1) + \varepsilon h(z_1).$$

Let $\phi(t, z_1, z_2, \varepsilon)$ be the flow of [2], then $f(x, \varepsilon) = \phi(T, x, \varepsilon)$, $x = (z_1, z_2)$ for a $T > 0$. Here $R(z_1, z_2) = (z_1, -z_2)$ and $\text{Fix} R = \{(z_1, 0) \mid z_1 \in \mathbb{R}^N\}$, $\text{Fix}(-R) = \{(0, z_2) \mid z_2 \in \mathbb{R}^N\}$. The inner product $\langle \cdot, \cdot \rangle$ is given by $\langle (z_1^1, z_2^1), (z_1^2, z_2^2) \rangle = \langle z_1^1, z_2^1 \rangle + \langle z_1^2, z_2^2 \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^N$. We assume that $p_\varepsilon = (0, 0)$ is a hyperbolic equilibrium of [2]. Let $\tau \in \text{Fix} R \cap \tilde{W}_p^s$. Then $\phi(t, \tau, 0) = (z_1^\tau(t), z_2^\tau(t))$ with $z_1^\tau(t)$ even and $z_2^\tau(t)$ odd. $\phi(t, \tau, 0)$ is a homoclinic solution of [2] with $\varepsilon = 0$. The linearization of [2] for $\varepsilon = 0$ along $\phi(t, \tau, 0)$ has the form

$$\dot{v} = w, \quad \dot{w} = Dg(z_1^\tau(t))v.$$
By (12), the function \(a\) is difficult to observe that now assumption (H1) holds and even bounded solution on \(\mathbb{R}\). To be more concrete, we consider the system (12) and the corresponding component of \(B(\tau)\) to \(a\) derived above is given by
\[
\dot{v}_\tau = w_\tau, \quad \dot{w}_\tau = Dg(z_\tau^1(t))v_\tau + h(z_\tau^1(t)).
\]
Consequently, the corresponding component of \(B(\tau)\) to \(a\) is given by
\[
2\langle a, D_\tau x_0^+(\infty, \tau, 0) \rangle = 2((0, a_2), (v_\tau(0), w_\tau(0))) = 2(a_2, w_\tau(0)) = 2(w_1(0), w_\tau(0)).
\]
On the other hand, since (11) holds along with \(lim_{t \to +\infty} w_1(t) = 0\), we have
\[
\int_0^\infty (h(z_\tau^1(t), w_1(t)) dt = (w_1(0), w_\tau(0)).
\]
To be more concrete, we consider the system (13)
\[
\ddot{x} = x - 2x(x^2 + y^2), \quad \ddot{y} = y - 2y(x^2 + y^2) + \varepsilon x^4, \quad x, y \in \mathbb{R}.
\]
has for \(\varepsilon = 0\) a homoclinic manifold \(x_\tau(t) = \sin \tau \tau(t), \ y_\tau(t) = \cos \tau \tau(t), \ r(t) = \text{sech} t\), which intersects Fix \(R\) in the circle \(M = (\sin \tau, \cos \tau, 0, 0)\). It is not difficult to observe that now assumption (H1) holds and \(w_1(t) = (y_\tau(t), -x_\tau(t))\). By (12), the function \(B(\tau)\) now has the form
\[
B(\tau) = -2\int_0^\infty \sin^5 \tau r^5(t) dt = -\frac{3}{8} \pi \sin^5 \tau.
\]
We note that \(x_0(t), y_0(t)\) are the even solutions of (13). The bifurcation equation \(C(\tau, \varepsilon) = 0\) from Section 4 is now analytical. Hence \(\tau = 0\) is its isolated solution for \(\varepsilon \neq 0\) small and fixed. The Brouwer degree of \(B(\tau)\) at \(\tau = 0\) is \(-1\), so Theorem 4.1 implies the following result.

**Theorem 5.1.** The point \((0, 1, 0, 0)\) is an isolated topologically transversal intersection of \(W_p^s\) and Fix \(R\) for (13) with \(\varepsilon \neq 0\) small. But this point is not a \(C^1\)-transversal intersection.

**Proof.** To prove the non-\(C^1\)-transversal intersection, we consider a \(C^3\)-perturbation of (13) given by
\[
\ddot{x} = x - 2x(x^2 + y^2), \quad \ddot{y} = y - 2y(x^2 + y^2) + \varepsilon \phi_\delta(x), \quad x, y \in \mathbb{R},
\]
where \(\delta > 0\) and \(\phi_\delta(x) = 0\) if \(|x| \leq \delta\), \(\phi_\delta(x) = (x - \delta \text{sgn} x)^4\) if \(|x| \geq \delta\). We see that (11) has even homoclinics \(x_\tau(t), y_\tau(t)\) for \(|\sin \tau| < \delta\) and any \(\varepsilon\). Hence \((0, 1, 0, 0)\) is not an isolated reversible homoclinic point for the \(C^3\)-perturbation (13) with \(\varepsilon \neq 0\), \(\delta > 0\) small of the system (13). The proof is finished.

**References**


REVERSIBLE HOMOCLINIC SETS


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