PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 11, Pages 3125–3131
S 0002-9939(02)06648-0
Article electronically published on May 22, 2002

SELF-DUAL CODES OVER $\mathbb{Z}_4$ AND HALF-INTEGRAL WEIGHT MODULAR FORMS

YOUNGJU CHOIE AND PATRICK SOLÉ

(Communicated by Dennis A. Hejhal)

Abstract. In this paper, we find a connection between the weight enumerator of self-dual $\mathbb{Z}_4$ codes and half-integral weight modular forms. We generalize in that way results of Broué-Enguehard, Hirzebruch, Ozeki, Rains-Sloane, Runge.

1. Introduction

Since the times of Felix Klein and his work on the icosahedron, [11], there has existed a kind of dictionary between invariant polynomials of finite groups and the modular forms of number theory. More recently, since the seminal talk of Gleason at the ICM 1970 in Nice [10], invariant theory has been found to have important applications in combinatorics in connection with the study of weight enumerators of self-dual codes over finite fields [12].

In 1972, these two trends fused when Broué and Enguehard [2] established a ring isomorphism between, on the one hand, the space of invariant polynomials for the finite group that leaves weight enumerators of Type II binary codes invariant and, on the other, the ring of modular forms of weight a multiple of four.

In the 90's, this topic met with renewed interest for two main reasons:

- new alphabets (finite rings) appeared;
- new weight enumerators were discovered.

As an example of the latter, we can cite polynomial analogues for modular forms of a number of different types: Jacobi [1] (a generalized split weight enumerator); Hilbert [8] (a Lee weight enumerator); Siegel [7, 15] (a higher joint weight enumerator); and Jacobi forms of higher genus [4] (a higher complete and symmetrized weight enumerator). On the "flip" side (i.e., ring-theoretic front), one can mention the fact that the complete weight enumerator of Type II codes over the ring $\mathbb{Z}_{2m}$ determines a Jacobi form for the full Jacobi group [5].

The Broué-Enguehard map itself was extended by Ozeki [1, 13] and independently by Runge [15] to the ring of modular forms of even weight. A similar connection between the ring of modular forms of integral weight and self-dual binary codes was established by Rains and Sloane [14, p.282].

In this note, we study a Broué and Enguehard type map from the algebra of invariants, where the symmetric weight enumerators of self-dual codes over $\mathbb{Z}_4$ live,

Received by the editors August 5, 2000.
1991 *Mathematics Subject Classification.* Primary 05E20, 11F11, 11F50, 94B99, 05E99.
This research was partially supported by COM2MAC.

to the ring of *half* integral weight modular forms on the congruence group $\Gamma_0(4)$ and the theta group Γ_θ. We thus generalize to a wider class of modular forms the last two mentioned results. Note that the quaternary alphabet used here is instrumental in what seems to be the simplest known construction of the Leech lattice [3].

2. The weight enumerator of a code over $\mathbb{Z}_4$

A linear code C of length n over $\mathbb{Z}_4$ is an additive subgroup of $\mathbb{Z}_4^n$ and an element of C is called a codeword. We denote by $|C|$ the number of codewords in C. The inner product of x and y in $\mathbb{Z}_4^n$ is given by

$$x \cdot y = x_1y_1 + \cdots + x_ny_n \pmod 4.$$

The dual code $C^\perp$ of C is defined as $C^\perp = \{y \in \mathbb{Z}_4^n \mid x \cdot y = 0 \text{ for all } x \in C\}$. If $C = C^\perp$, then C is called self-dual.

The complete weight enumerator cwe_c of C over $\mathbb{Z}_4^n$ is defined by

$$cwe_c(W,X,Y,Z) = \sum_{u\in C} W^{n_0(u)}X^{n_1(u)}Y^{n_2(u)}Z^{n_3(u)},$$

where $n_k(u)$ denotes the number of components of u which are equal to k modulo 4. The symmetrized weight enumerator swe_C of C over $\mathbb{Z}_4$ is given by

$$swe_C(W,X,Y) = cwe_C(W,X,Y,X).$$

The MacWilliams identity of cwe_C of the code C over $\mathbb{Z}_4$ is known as

Theorem 2.1. 1.

$$cwe_{C^\perp}(W,X,Y,Z) = \frac{1}{|C|}cwe_C(W+X+Y+Z, W+iX-Y-iZ, \\ W-X+Y-Z, W-iX-Y+iZ).$$

2. $swe_{C^\perp}(W,X,Y) = \frac{1}{|C|}swe_C(W+2X+Y, W-Y, W-2X+Y)$.

Proof. See [5] or [6], for instance.

3. Jacobi forms and theta-series

Let Γ be an arbitrary subgroup of finite index in $SL_2(\mathbb{Z})$ and let χ be a character of Γ. A function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is said to be a Jacobi form of a weight ω and an index m for Γ with respect to χ (see [9]) if

1. $\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (\frac{c\tau+d}{\chi(M)})^\omega e^{2\pi i m\frac{z^2}{c\tau+d}}\phi(\tau,z), \forall M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$;
2. $\phi(\tau, z+\lambda\tau+\mu) = e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau,z), \forall(\lambda,\mu) \in \mathbb{Z}^2$; and
3. $\phi(\tau,z)$ has a Fourier expansion of the following type:

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}, r^2\le 4mn} c(n,r)e^{2\pi i n\tau}e^{2\pi i r z}.$$

We also recall the following theta-series $\theta_{2,\mu}(\tau,z)$ which was introduced to find the correspondence between Jacobi forms and modular forms of half integral weight (see [9]):

$$\theta_{2,\mu}(\tau,z) := \sum_{r\in\mathbb{Z}, r\equiv\mu \pmod 4} e^{\frac{\pi i r^2}{4}\tau}\xi^r, \xi = e^{2\pi i z}.$$

Remark 3.1. The theta-series $\theta_{2,\mu}(\tau,z) = \sum_{r\in\mathbb{Z}, r\equiv\mu \pmod 4} e^{\frac{\pi i r^2}{4}\tau}\xi^r$ satisfies the following relations:

(i) $\theta_{2,\mu}(\tau,z)$ is well-defined and holomorphic on $\mathbb{H}\times\mathbb{C}$;

(ii) $\theta_{2,\mu}(\tau+1,z) = e^{\frac{\pi i}{4}\mu^2}\theta_{2,\mu}(\tau,z), \mu\in\{0,1,2,3\}$;

(iii) $\theta_{2,\mu}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \sqrt{\frac{\tau}{4i}}\ e^{4\pi i\frac{z^2}{\tau}} \sum_{\nu(mod 4)} e^{(-\frac{\pi i}{4}\mu\nu)}\theta_{2,\nu}(\tau,z)$;

(iv) $\theta_{2,\mu}(\tau,z+\lambda\tau+\nu) = e^{-4\pi i(\lambda^2\tau+2\lambda z)}\theta_{2,\mu}(\tau,z), \forall(\lambda,\nu)\in\mathbb{Z}^2$.

The more general type of theta-series $\theta_{m,\mu}(\tau,z)$ has been used to study the connection between Jacobi forms and codes over the ring $\mathbb{Z}_{2m}$(see [5]). In particular, when $m=2$, we get the following relations among the theta-series:

$$\theta_{2,0}(\tau,0) = \frac{\theta_3(\tau)+\theta_4(\tau)}{2}, \theta_{2,1}(\tau,0) = \theta_{2,3}(\tau,0) = \frac{\theta_2(\tau)}{2},$$
$$\theta_{2,2}(\tau,0) = \frac{\theta_3(\tau)-\theta_4(\tau)}{2},$$

where

$$\theta_2(\tau) = \sum_{\ell\in\mathbb{Z}} q^{(\ell+\frac{1}{2})^2}, \theta_3(\tau) = \sum_{\ell\in\mathbb{Z}} q^{\ell^2},$$
$$\theta_4(\tau) = \theta_3(\tau+1) = \sum_{\ell\in\mathbb{Z}}(-q)^{\ell^2}, \text{ with } q = e^{\pi i\tau}.$$

The following relations are well-known identities among theta series.

Proposition 3.2.

$$\theta_2(\tau+1) = \sqrt{i}\theta_2(\tau), \theta_3(\tau+1) = \theta_4(\tau), \theta_4(\tau+1) = \theta_3(\tau),$$
$$\theta_2(-\frac{1}{\tau}) = (\frac{\tau}{i})^{\frac{1}{2}}\theta_4(\tau), \theta_3(-\frac{1}{\tau}) = (\frac{\tau}{i})^{\frac{1}{2}}\theta_3(\tau), \theta_4(-\frac{1}{\tau}) = (\frac{\tau}{i})^{\frac{1}{2}}\theta_2(\tau)$$
$$\theta_3(\tau)\theta_4(\tau) = \theta_4(2\tau)^2, \theta_2(\tau)^4+\theta_4(\tau)^4 = \theta_3(\tau)^4, \theta_3(\tau)^2-\theta_4(\tau)^2 = 2\theta_2(2\tau)^2.$$

Proof. See, for instance, [6].

4. Jacobi forms and codes over $\mathbb{Z}_4$

We adapt two theorems derived in [5] in a different context to the class of self-dual $\mathbb{Z}_4$-codes.

Theorem 4.1. *Let $C\subset\mathbb{Z}_4^n$ be a linear code with complete weight enumerator $cwe_C(W,X,Y,Z)$. Let*

$$\theta_{\Lambda(C),2}(\tau,z) = \sum_{x\in\Lambda(C)} e^{\pi i(x\cdot x)\tau}e^{2\pi i(P\cdot x)z}, P = \frac{1}{2}(1,1,...,1)\in\Lambda(C),$$

where $\Lambda(C)$ is the lattice associated to the code C. Then

$$\theta_{\Lambda(C),2}(\tau,z) = cwe_C(\theta_{2,0}(\tau,z),\theta_{2,1}(\tau,z),\theta_{2,2}(\tau,z),\theta_{2,3}(\tau,z)).$$

Proof. This is an elaboration of Theorem 4 given in [5]. Let $u=(u_1,\cdots,u_n)$ be any given codeword of C and $n_k(u)=N_k$. For $\hat{u}=(\hat{u}_1,\cdots,\hat{u}_n)\in\mathbb{Z}^n$, if $\varphi(\hat{u})=u$ and $0\le\hat{u}_k<4$ for all k, then it is easy that the set $\varphi^{-1}(u)$ equals the set $\varphi^{-1}(0)+\hat{u}$,

and the number of k $(0 \le k < 4)$ of $\hat{u}_1, \cdots, \hat{u}_n$ is exactly N_k. Thus, for given $u \in C$,

$$\begin{aligned}
&\textstyle\sum_{Y\in\varphi^{-1}(u)} e^{2\pi i\left(\frac{1}{8}(Y\cdot Y)\tau+(Y\cdot 1)z\right)}\\
&= \textstyle\sum_{Y\in\varphi^{-1}(0)} e^{2\pi i\left(\frac{1}{8}((Y+\hat{u})\cdot(Y+\hat{u}))\tau+((Y+\hat{u})\cdot 1)z\right)}\\
&= \textstyle\sum_{Y_1\in 4Z+\hat{u}_1}\cdots\sum_{Y_n\in 4Z+\hat{u}_n}\\
&\quad\cdot e^{2\pi i(\frac{1}{8}(Y_1^2+\cdots+Y_n^2)\tau+(Y_1+\cdots+Y_n)z)}\\
&= (\textstyle\sum_{Y_1\in 4\mathbb{Z}+\hat{u}_1} e^{2\pi i(\frac{Y_1^2}{8}\tau+Y_1 z)})\\
&\quad\cdots(\textstyle\sum_{Y_n\in 4\mathbb{Z}+\hat{u}_n} e^{2\pi i(\frac{Y_n^2}{8}\tau+Y_n z)})\\
&= (\textstyle\sum_{Y_1\in 4\mathbb{Z}+\hat{u}_1} q^{\frac{Y_1^2}{8}}\xi^{Y_1})\cdots(\sum_{Y_n\in 4\mathbb{Z}+\hat{u}_n} q^{\frac{Y_n^2}{8}}\xi^{Y_n})\\
&= \theta_{2,0}(\tau,z)^{N_0}\theta_{2,1}(\tau,z)^{N_1}\theta_{2,2}(\tau,z)^{N_2}\theta_{2,3}(\tau,z)^{N_3}
\end{aligned}$$

implies that

$$\theta_{\Lambda(C),2}(\tau,z) = cwe_C(\theta_{2,0}(\tau,z), \theta_{2,1}(\tau,z)\theta_{2,2}(\tau,z), \theta_{2,3}(\tau,z)).$$

Theorem 4.2. *If $C \subset \mathbb{Z}_4^n$ is a self-dual linear code, then the theta-series $\theta_{\Lambda(C),2}(\tau,z)$ satisfies the following functional equations:*

(1) $\theta_{\Lambda(C),2}(\tau+2,z) = \theta_{\Lambda(C),2}(\tau,z)$;
(2) $\theta_{\Lambda(C),2}(-\frac{1}{\tau},\frac{z}{\tau}) = e^{\frac{2\pi i n}{8}\frac{z^2}{\tau}}(\frac{\tau}{i})^{\frac{n}{2}}\theta_{\Lambda(C),2}(\tau,z)$;
(3) $\theta_{\Lambda(C),2}(\tau, z+\lambda\tau+\nu) = e^{\frac{-2\pi i n}{8}(\lambda^2\tau+2\lambda z)}\theta_{\Lambda(C),2}(\tau,z)$.

So, $\theta_{\lambda(C),2}(\tau,z)$ is a Jacobi form of weight $\frac{n}{2}$ and index $\frac{n}{8}$ over Γ_θ with respect to χ_0. Here, χ_0 is the character on Γ_θ defined as $\chi_0(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)) = 1, \chi_0(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)) = i$.

Proof. We adapt Theorem 3.2 and Proposition 4.2 in [5] (Proposition 4.2 in [5] is valid only for Type II code C over $\mathbb{Z}_{2m}$) to the situation at hand. The first functional equation (1) is immediate from the definition of the theta-series $\theta_{\Lambda(C),2}(\tau,z)$. The second functional equation follows from Poisson-summation formula,

$$\theta_{\Lambda(C),2}(-\frac{1}{\tau},\frac{z}{\tau}) = \frac{1}{vol(\mathbb{R}^n/\Lambda(C))}(\frac{\tau}{i})^{\frac{n}{2}}e^{2\pi i\frac{nz}{8}}\theta_{\Lambda(C)^*,2}(\tau,z),$$

and self-duality of $\Lambda(C)$. The last equation (3) follows from

$$\begin{aligned}
e^{2\pi i\frac{n}{8}}\theta_{\Lambda(C),2}(\tau, z+\lambda\tau+\mu) &= \sum_{x\in\Lambda(C)} e^{((x+\lambda P)\cdot(x+\lambda P))\tau}e^{2\pi i(P\cdot(x+\lambda P))z}\\
&= \theta_{\Lambda(C),2}(\tau,z),
\end{aligned}$$

since λP is again in $\Lambda(C)$. Now, to claim that $\theta_{\Lambda(C),2}(\tau,z)$ is a Jacobi form, one needs to check only the condition at infinity. The condition at infinity can be checked from the fact that, for each $x = (x_1, ..., x_n) \in \Lambda(C)$, $\frac{(x_1+x_2+...+x_n)^2}{n} \le x_1^2 + x_2^2 + ... + x_n^2$. The result follows.

Regarding invariants of finite groups, we use the following piece of notation. If G is a matrix group acting on a polynomial ring R, we denote by R^G (resp. R_n^G) the ring of invariants; that is,

$$R^G = \{f \in R \,\forall g \in G \mid g.f = f\}$$

(resp. the set of elements of R^G of degree n).

Corollary 4.3. *Let C be a self-dual code over $\mathbb{Z}_4^n$.*

1. *Let G_4 be the group generated by*

$$M_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

   Then $cwe_C(W,X,Y,Z) \in \mathbb{C}[W,X,Y,Z]^{G_4}$.

2. *Let G_3 be the matrix group generated by the matrices*

$$N_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \frac{1}{2}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

   Then $swe_C(W,X,Y)$ is in $\mathbb{C}[W,X,Y]^{G_3}$.

Proof. The results follow directly from the functional equation of $\theta_{2,\mu}(\tau,z)$ in Remark 3.1 and the functional equation $\theta_{\Lambda(C),2}(\tau,z)$ with Theorem 2.1. So, we omit the detailed proof.

Theorem 4.4. *If $h \in \mathbb{C}[W,X,Y,Z]_n^{G_4}$ is a degree n homogeneous polynomial invariant under G_4, then $h(\theta_{2,0}(\tau,z),\theta_{2,1}(\tau,z),\theta_{2,2}(\tau,z),\theta_{2,3}(\tau,z))$ is a Jacobi form of weight $\frac{n}{2}$ and index $\frac{n}{8}$ on Γ_θ with respect to χ_0. Here, χ_0 is the character defined in the statement of Theorem 4.2.*

Proof. Let $H(\tau,z) = h(\theta_{2,0}(\tau,z),\theta_{2,1}(\tau,z),\theta_{2,2}(\tau,z),\theta_{2,3}(\tau,z))$. Then one can check the two functional equations of $H(\tau,z)$ by using the relations among $\theta_{2,\mu}(\tau,z)$ given in Remark 3.1. The condition of the Fourier expansion is valid since h is a polynomial in theta-series.

Theorem 4.5. *Let $g \in \mathbb{C}[W,X,Y,Z]^{G_4}$ be a degree n homogeneous polynomial. Then $g(\theta_{2,0}(2\tau,z),\theta_{2,1}(2\tau,z),\theta_{2,2}(2\tau,z),\theta_{2,3}(2\tau,z))$ is a Jacobi form of weight $\frac{n}{2}$ and index $\frac{n}{8}$ on $\Gamma_0(4)$ with respect to the trivial character.*

Proof. Let $G(\tau,z) = g(\theta_{2,0}(2\tau,z),\theta_{2,1}(2\tau,z),\theta_{2,2}(2\tau,z),\theta_{2,3}(2\tau,z))$. Since $\Gamma_0(4)$ is generated by two elements $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, one only needs to check two functional equations of $G(\tau,z)$ under the action of $\Gamma_0(4)$. This can be done using the relations among $\theta_{2,\mu}(2\tau,z)$ given in Remark 3.1. The condition on the Fourier expansion holds since g is a polynomial in theta-series.

5. Half integral weight modular forms and codes over $\mathbb{Z}_4$

Let Γ be an arbitrary subgroup of finite index in $SL_2(\mathbb{Z})$ and let χ be a character of Γ. A complex valued function $f : \mathcal{H} \to \mathbb{C}$ is called a modular form of weight ω for Γ with respect to χ if

1. $f(\tau)$ is holomorphic for $Im(\tau) > 0$;
2. $f(\frac{a\tau+b}{c\tau+d}) = (\frac{c\tau+d}{\chi(M\tau)})^\omega f(\tau), \forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$;
   and
3. $f(\tau)$ is "holomorphic" at every cusp of Γ.

First, let us recall the following classical result studied by Hecke (see [6], p. 187).

Theorem 5.1. 1. *Let Γ_θ be the group generated by $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and the character χ_0 be given by $\chi_0(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)) = 1, \chi_0(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)) = i$. Let $\mathcal{M}_{\frac{n}{2}}(\Gamma_\theta, \chi_0)$ be the space of modular forms of weight $\frac{n}{2}$ on Γ_θ with respect to χ_0. Then*

$$\mathcal{M}(\Gamma_\theta, \chi_0) = \bigoplus_{n=0}^{\infty} \mathcal{M}_{\frac{n}{2}}(\Gamma_\theta, \chi_0) = \mathbb{C}[\theta_3(\tau), \Delta_8(\tau)].$$

Here,

$$\theta_3(\tau) = \sum_{\ell \in \mathbb{Z}} q^{\ell^2},$$

$$\Delta_8(\tau) = \frac{1}{16}\theta_2(\tau)^4\theta_4(\tau)^4 = q\prod_{m=1}^{\infty}\{(1-q^{2m-1})(1-q^{4m})\}^8, \quad \text{with } q = e^{\pi i \tau}.$$

2. *Let $\mathcal{M}_{\frac{n}{2}}(\Gamma_0(4), id)$ be the space of modular forms of weight $\frac{n}{2}$ on $\Gamma_0(4)$ with trivial character $\chi = id$. Then*

$$\mathcal{M}(\Gamma_0(4), id) = \bigoplus_{n=0}^{\infty} \mathcal{M}_{\frac{n}{2}}(\Gamma_0(4), id) = \mathbb{C}[\theta_3(2\tau), g(\tau)].$$

Here, $g(\tau)$, defined as

$$g(\tau) := \theta_2^4(2\tau) - 2\theta_2^2(2\tau)\theta_4^2(2\tau), \tag{5.1}$$

is a modular form of weight 2 in $\Gamma_0(4)$.

Remark 5.2. 1. In general, it is known that the ring $\mathcal{M}_{\frac{n}{2}}(\Gamma_0(4), id)$ is isomorphic to $\mathbb{C}[\theta_3(\tau), F_2(\tau)]$, where F_2 is any modular form of weight 2, which is linearly independent from $\theta_3(\tau)^4$, on $\Gamma_0(4)$.

2. One can check, from Proposition 3.2, that $g(\tau) := \theta_2^4(\tau) - 2\theta_2^2(2\tau)\theta_4^2(2\tau)$ is a modular form of weight 2 on $\Gamma_0(4)$.

Remark 5.3. If $\phi(\tau, z)$ is a Jacobi form of weight ω and index m on Γ with respect to χ, then $\phi(\tau, 0)$ is a modular form of weight ω on Γ with respect to χ.

Theorem 5.4. *The map $\phi : \mathbb{C}[W, X, Y]^{G_3} \to \mathbb{C}[\theta_3, \Delta_8]$ defined by*

$$\phi(f(W, X, Y)) = f(\frac{\theta_3(\tau) + \theta_4(\tau)}{2}, \frac{\theta_2(\tau)}{2}, \frac{\theta_3(\tau) - \theta_4(\tau)}{2})$$

is an algebra epimorphism.

Proof. From Theorem 4.4 one notes that for any homogeneous polynomial f of degree n in $\mathbb{C}[W, X, Y]^{G_3}$, $f(\theta_{2,0}(\tau, z), \theta_{2,1}(\tau, z), \theta_{2,2}(\tau, z))$ is a Jacobi form of weight $\frac{n}{2}$ and index $\frac{n}{8}$. This means that, by specializing to $z = 0$, the quantity $f(\theta_{2,0}(\tau, 0), \theta_{2,1}(\tau, 0), \theta_{2,2}(\tau, 0))$ is a modular form of weight $\frac{n}{2}$ on Γ_θ with respect to χ_0. The structure Theorem 5.1 with Proposition 3.2 implies that $\phi(f(W, X, Y)) \in \mathbb{C}[\theta_3, \Delta_8]$. Also, ϕ is onto because

$$\phi(W + Y) = \theta_3(\tau), \qquad \phi(X^4(W - Y)^4) = \Delta_8(\tau).$$

Theorem 5.5. *The map*

$$\tilde{\phi} : \mathbb{C}[W, X, Y]^{G_3} \to \mathbb{C}[\theta_3(2\tau), g(\tau)],$$

given by $\tilde{\phi}(f(W, X, Y)) = f(\frac{\theta_3(2\tau)+\theta_4(2\tau)}{2}, \frac{\theta_2(2\tau)}{2}, \frac{\theta_3(2\tau)-\theta_4(2\tau)}{2})$, is an algebra homomorphism. Here $g(\tau)$ is a modular form of weight 2 defined by (5.1).

Proof. Theorem 4.5 implies that $\tilde{\phi}$ is well defined.

References

[1] E. Bannai, M. Ozeki, Construction of Jacobi forms from certain combinatorial polynomials, Proceedings of the Japan Ac., Vol. 72, Ser. A, No 1 (1996) 12-15. MR **98a:**11060

[2] M. Broué, M. Enguehard, Polynômes des poids de certains codes et fonctions theta de certains réseaux, Ann. Sc. ENS 5 (1972) 157-181. MR **48:**3596

[3] A. Bonnecaze, P. Solé, A.R.C. Calderbank, Quaternary Constructions of Unimodular Lattices, IEEE Trans. on Information Theory, IT-41 (1995) 366-377. MR **96b:**94027

[4] Y. Choie and H. Kim, Codes over $\mathbb{Z}_{2m}$ and Jacobi forms of genus n, Journal of Combinatorial Theory-Series A 95 (2001) 335-348.

[5] Y. Choie and N. Kim, The complete weight enumerator of Type II codes over $\mathbb{Z}_{2m}$ and Jacobi Forms, IEEE Transactions on Information Theory, Vol. IT-47, pp.396-399 (2001). MR **2001m:**11072

[6] J.H. Conway, N.J.A. Sloane, *Sphere Packings, Lattices and Groups* Springer (1998) 3rd edition. MR **2000b:**11077

[7] W. Duke, Codes and Siegel Modular Forms, International Research Notices in Math., 3, p.125 in Duke Math. J. 70 (1993). MR **94d:**11029

[8] W. Ebeling, *Lattices and Codes*, Vieweg (1994). MR **95c:**11084

[9] M. Eichler, D. Zagier, *The theory of Jacobi forms*, Birkhäuser (1985). MR **86j:**11043

[10] A. M. Gleason, Weight enumerators of self-dual codes and MacWilliams identities, Actes du Congrés International de mathématiques, vol. 3, Gauthier-Villars (1970) 211-215. MR **54:**12354

[11] F. Klein, Vorlesungen uber das Ikosaeder und die Auflosung der Gleichungen vom funften Grade. (German) [Lectures on the icosahedron and the solution of equations of fifth degree] Reprint of the 1884 original. Edited, with an introduction and commentary by Peter Slodowy. Birkhauser Verlag, Basel; B. G. Teubner, Stuttgart, (1993). MR **96g:**01046

[12] F.J. MacWilliams, N.J.A. Sloane, *The theory of Error Correcting Codes*, North-Holland second edition (1981). MR **57:**5408a; MR **57:**5408b (review of original volumes)

[13] M. Ozeki, On the notion of Jacobi polynomials for codes, Math. Proc. Cambridge Phil. Soc. (1997). MR **98b:**11126

[14] E.M. Rains, N.J.A. Sloane, Self-dual codes, in *The Handbook of Coding Theory I*, V.S. Pless and W.C. Huffman, eds., North-Holland (1998). MR **2000h:**94001

[15] B. Runge, On Siegel Modular forms I, Crelle 436 (1993) 57-85. MR **94c:**11041

Department of Mathematics, Pohang University of Science and Technology, Pohang, 790–784, Korea

E-mail address: `yjc@postech.ac.kr`

CNRS-I3S, ESSI, Route des Colles, 06 903 Sophia Antipolis, France

E-mail address: `ps@essi.fr`