PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 11, Pages 3433–3437
S 0002-9939(02)06709-6
Article electronically published on May 29, 2002

AN INFINITE FAMILY OF SUMMATION IDENTITIES

S. F. KEATING

(Communicated by Dennis A. Hejhal)

Abstract. Theta functions have historically played a prominent role in number theory. One such role is the construction of modular forms. In this work, a generalized theta function is used to construct an infinite family of summation identities. Our results grew out of some observations noted during a presentation given by the author at the 1992 AMS-MAA-SIAM Joint Meetings in Baltimore.

1. Introduction

Theta functions have historically played a prominent role in number theory. One such role is the use of theta functions to construct modular forms. In this context, we now take the classical theta function [8]

$$\vartheta(z) = \sum_{m=-\infty}^{\infty} e^{i\pi m^2 z}, \qquad \mathrm{Im}(z) > 0, \tag{1}$$

which is absolutely convergent for $\mathrm{Im}(z) > 0$ and satisfies

$$\vartheta(z+2) = \vartheta(z) \tag{2}$$

and

$$\vartheta\left(\frac{-1}{z}\right) = \left(\sqrt{-iz}\right)\vartheta(z). \tag{3}$$

The square root is defined to be positive on the positive imaginary axis. Equations (2) and (3) establish that $\vartheta(z)$ is a modular form of weight 1/2 for the group G(2) generated by $z \to z+2$ and $z \to \frac{-1}{z}$ [9].

Take $z = it$ in (1) to obtain the Jacobi theta function

$$\theta(t) = \sum_{m\in\mathbf{Z}} e^{-\pi m^2 t} \quad \text{for } t > 0. \tag{4}$$

$\theta(t)$ satisfies the Jacobi inversion formula

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right), \qquad t > 0,\ \sqrt{t} > 0. \tag{5}$$

Received by the editors May 29, 2001.
2000 *Mathematics Subject Classification.* Primary 11-XX.

Equation (5) yields the summation identity

$$\frac{1}{s} + 2\sum_{m=1}^{\infty} \frac{1}{s+\pi m^2} = \frac{\sqrt{\pi}}{\sqrt{s}} + 2\frac{\sqrt{\pi}}{\sqrt{s}} \sum_{m=1}^{\infty} e^{-2m\sqrt{\pi s}} \quad \text{for } \operatorname{Re}(s) > 0. \tag{6}$$

We note that (6) is equivalent to the summation identity presented by Bellman in [1] and can be viewed as a partial fraction expansion of the hyperbolic cotangent function, $\coth(\sqrt{\pi s})$, for $\operatorname{Re}(s) > 0$.

In [5] the Jacobi theta function was generalized by

$$\theta_{n,l}(t) = \sum_{m\in\mathbf{Z}} e^{-\pi m^2 t} m^l H_n\left(\sqrt{2\pi t} m\right) \quad \text{if } l \neq 0, \tag{7}$$

$$\theta_{n,0}(t) = \sum_{m\in\mathbf{Z}} e^{-\pi m^2 t} H_n\left(\sqrt{2\pi t} m\right). \tag{8}$$

Equations (7) and (8) hold for $n = 0, 1, 2, \ldots$ and $t > 0$. $H_n(x)$ is the nth Hermite polynomial. We remark that theta functions *akin* to $\theta_{n,0}(t)$ occur in a variety of different contexts; cf., e.g., [7, pp. 428–429, 447], [10, pp. 1014, 1045], and [11].

By using these generalized theta functions, and expressing $H_n(x)$ as a series [6], one obtains a generalized Jacobi inversion formula [5]:

$$\theta_{n,l}(t) = \frac{(-1)^{\frac{(n-l)}{2}} n!l!}{t^{\frac{(2l+1)}{2}}} \sum_{k=0}^{\min(n,l)} \sum_{j=0}^{[\frac{(l-k)}{2}]} \left(\frac{(-1)^{n-k-j} (2)^{\frac{(k-4j)}{2}} \pi^{\frac{(-k-2j)}{2}}}{j!k!(l-k-2j)!(n-k)!} \right) \times \frac{1}{t^{\frac{(-k-2j)}{2}}} \theta_{n-k,l-k-2j}\left(\frac{1}{t}\right). \tag{9}$$

In (9) the symbol [ ] denotes the greatest integer function and the inversion formula holds for $t > 0$, $n, l = 0, 1, 2, \ldots,$ $n + l$ is an even, nonnegative integer.

Our primary aim in this note is to announce some new summation identities which are constructed via (9). Although these summation identities can be developed quite independently of modular forms, it is useful to recall the actual connection between (1) as a theta function and as a modular form and (4).

2. An infinite family of summation identities

We first define a constant c_q, for $q = 0, 1, 2, \ldots$, as

$$c_q = \frac{(-1)^q (2q)!}{q!}. \tag{10}$$

If we consider the generalized Jacobi inversion formula for the case where $n = 2q$, $q = 0, 1, 2, \ldots$, and $l = 0$, we obtain

$$\theta_{2q,0}(t) = \frac{(-1)^q}{t^{\frac{1}{2}}} \theta_{2q,0}\left(\frac{1}{t}\right). \tag{11}$$

We note that for $q = 0$ (11) is equal to (5).

We express (11) as a series and then apply the Laplace transform, interchanging summation and integral. The formulas found in [2], [3] and [4] are utilized and some algebraic simplification is implemented. We obtain the following infinite family of summation identities:

$$\begin{aligned}
&\frac{1}{s}c_q + 2c_q \sum_{m=1}^{\infty} \frac{1}{s+\pi m^2} \\
&\qquad + \sum_{m=1}^{\infty}\sum_{k=0}^{q-1} \frac{(-1)^k (2q)! 2^{(3q-3k+1)} \pi^{(q-k)} m^{(2q-2k)}}{k!(2q-2k)!} \frac{(q-k)!}{(s+\pi m^2)^{(q-k+1)}} \\
&\qquad = (-1)^q c_q \frac{\sqrt{\pi}}{\sqrt{s}} + 2(-1)^q c_q \frac{\sqrt{\pi}}{\sqrt{s}} \sum_{m=1}^{\infty} e^{-2m\sqrt{\pi s}} \\
&\qquad + \sum_{m=1}^{\infty}\sum_{k=0}^{q-1} \frac{2^{(3q-3k+1)}(2q)! \pi^{(\frac{q-k+1}{2})} m^{(q-k)} e^{-2m\sqrt{\pi s}}}{k!(2q-2k)!(-1)^{(-q-k)} s^{\frac{-(q-k-1)}{2}}} \\
&\qquad \times \sum_{r=0}^{q-k-1} \frac{(q-k-1+r)!}{r!(q-k-1-r)! 2^{(2r)} m^r \pi^{\frac{r}{2}} s^{\frac{r}{2}}},
\end{aligned} \tag{12}$$

for $q = 0, 1, 2, \ldots,\ \mathrm{Re}(s) > 0$.

If we consider the case where $q = 2\mu$ is an even, nonnegative integer in (12), expanding (11) and implementing algebraic simplification, we obtain

$$\begin{aligned}
&\frac{(4\mu)!}{(2\mu)!}\frac{1}{s} + 2\frac{(4\mu)!}{(2\mu)!} \sum_{m=1}^{\infty} \frac{1}{s+\pi m^2} \\
&\qquad + \sum_{m=1}^{\infty}\sum_{k=0}^{2\mu-1} \frac{(-1)^k (4\mu)! 2^{(6\mu-3k+1)} \pi^{(2\mu-k)} m^{(4\mu-2k)} (2\mu-k)!}{k!(4\mu-2k)!(s+\pi m^2)^{(2\mu-k+1)}} \\
&\qquad = \frac{(4\mu)!}{(2\mu)!}\frac{\sqrt{\pi}}{\sqrt{s}} + 2\frac{(4\mu)!}{(2\mu)!}\frac{\sqrt{\pi}}{\sqrt{s}} \sum_{m=1}^{\infty} e^{-2m\sqrt{\pi s}} \\
&\qquad + \sum_{m=1}^{\infty}\sum_{k=0}^{2\mu-1} \frac{(-1)^k 2^{(6\mu-3k+1)} (4\mu)! \pi^{(\frac{2\mu-k+1}{2})} m^{(2\mu-k)} e^{-2m\sqrt{\pi s}}}{k!(4\mu-2k)! s^{\frac{-(2\mu-k-1)}{2}}} \\
&\qquad \times \sum_{r=0}^{2\mu-k-1} \frac{(2\mu-k-1+r)!}{r!(2\mu-k-1-r)! 2^{(2r)} m^r \pi^{\frac{r}{2}} s^{\frac{r}{2}}},
\end{aligned} \tag{13}$$

for $\mu = 0, 1, 2, \ldots,\ \mathrm{Re}(s) > 0$.

As one might expect, when producing identities in the manner described, consideration of l equal to a positive, even integer in (9) is similar but more cumbersome.

3. Some examples

The value $\mu = 0$ applied to (13) yields the identity noted in (6). The value $\mu = 1$ applied to (13) yields the following identity:

$$\begin{aligned}\frac{12}{s} + 24\sum_{m=1}^{\infty}\frac{1}{s+\pi m^2} + \sum_{m=1}^{\infty}\frac{256\pi^2 m^4}{(s+\pi m^2)^3} - \sum_{m=1}^{\infty}\frac{192\pi m^2}{(s+\pi m^2)^2} \\ = \frac{12\sqrt{\pi}}{\sqrt{s}} + \frac{24\sqrt{\pi}}{\sqrt{s}}\sum_{m=1}^{\infty} e^{-2m\sqrt{\pi s}} + \sum_{m=1}^{\infty} 128\pi^{\frac{3}{2}} m^2 s^{\frac{1}{2}} e^{-2m\sqrt{\pi s}} \\ - \sum_{m=1}^{\infty} 64\pi m e^{-2m\sqrt{\pi s}}.\end{aligned} \tag{14}$$

Subtracting identity (6) from identity (14) and canceling common factors results in yet another identity:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{256\pi^2 m^4}{(s+\pi m^2)^3} - \sum_{m=1}^{\infty}\frac{192\pi m^2}{(s+\pi m^2)^2} + \sum_{m=1}^{\infty}\frac{22}{s+\pi m^2} + \frac{11}{s} \\ = \sum_{m=1}^{\infty} 128\pi^{\frac{3}{2}} m^2 s^{\frac{1}{2}} e^{-2m\sqrt{\pi s}} - \sum_{m=1}^{\infty} 64\pi m e^{-2m\sqrt{\pi s}} \\ + 22\frac{\sqrt{\pi}}{\sqrt{s}}\sum_{m=1}^{\infty} e^{-2m\sqrt{\pi s}} + 11\frac{\sqrt{\pi}}{\sqrt{s}}.\end{aligned} \tag{15}$$

Acknowledgment

We wish to thank Floyd L. Williams, University of Massachusetts at Amherst, and the editor for their kindness in providing suggestions which proved to be most helpful.

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Department of Mathematics and Computer Science, Eastern Connecticut State University, Willimantic, Connecticut 06226

E-mail address: keating@easternct.edu