NON-VANISHING OF SYMMETRIC SQUARE $L$-FUNCTIONS

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Abstract. Given a complex number $s$ with $0 < \Re s < 1$, we study the existence of a cusp form of large even weight for the full modular group such that its associated symmetric square $L$-function $L(\text{sym}^2 f, s)$ does not vanish. This problem is also considered in other articles.

1. Introduction

Let $k$ be an even positive integer and $f$ a holomorphic cusp form of weight $k$ with respect to the full modular group. We represent the Fourier expansion of $f$ (at the cusp $\infty$) by

$$f(z) = \sum_{n=1}^{\infty} \psi_f(n) n^{(k-1)/2} e(nz)$$

where $e(\alpha) = e^{2\pi i \alpha}$. Assume that $f(z)$ is an eigenfunction for all Hecke operators $T_n$, with $T_n f = \lambda_f(n) n^{(k-1)/2} f$. Note that $\lambda_f(n)$ is real and has the Deligne’s bound

$$|\lambda_f(n)| \leq \tau(n) \quad \text{for } n
divides \tau(n) = \sum_{d|n} 1$$

is the divisor function. We normalize $f$ so that $\psi_f(1) = 1$; then we have $\psi_f(n) = \lambda_f(n)$. Such an $f$ is called a primitive form. Associated to each primitive $f$, the Rankin-Selberg convolution $L$-function $L(f \otimes f, s)$ and the symmetric square $L$-function $L(\text{sym}^2 f, s)$ are respectively defined as, for $\Re s > 1$,

$$L(f \otimes f, s) = \sum_{n=1}^{\infty} \lambda_f(n)^2 n^{-s}$$

and

$$L(\text{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s}$$

where $\zeta(s)$ is the Riemann zeta-function. These two $L$-functions are closely linked by the relation (see [5] (0.2) and (0.4))

$$\zeta(s) L(\text{sym}^2 f, s) = \zeta(2s) L(f \otimes f, s).$$

In this paper, we are concerned with the non-vanishing results of $L(\text{sym}^2 f, s)$ in the critical strip. Li [4] showed that for a given complex number $\rho \neq 1/2$
satisfying $0 < \Re \rho < 1$ and $\zeta(\rho) \neq 0$, there are infinitely many primitive forms $f$ of different weight such that $\zeta(2s)L(f \otimes f, s)$ do not vanish at $s = \rho$, or equivalently, $L(\text{sym}^2 f, \rho) \neq 0$. In addition, Kohnen and Sengupta [3] have recently showed that for any fixed $s = \sigma + it$ with $0 < \sigma < 1$ and $\sigma \neq 1/2$, and for all sufficiently large $k$, there exists a primitive form $f$ of weight $k$ such that $L(\text{sym}^2 f, s) \neq 0$. The approaches used in [4] and [3] are different: the former utilizes an approximate formula of Zagier. Here, we shall use another method to prove the theorem below, which includes the results in [3] and [4].

**Theorem.** For any fixed $s \in \mathbb{C}$ with $0 < \Re s < 1$, there exist infinitely many even $k$ such that $L(\text{sym}^2 f, s) \neq 0$ for some primitive form $f$ of weight $k$. Furthermore, when $\Re s \neq 1/2$ or $s = 1/2$, there exists a constant $k_0(s)$ depending on $s$ such that for all even $k \geq k_0(s)$, $L(\text{sym}^2 f, s)$ does not vanish for some primitive form $f$ of weight $k$.

**Remark.** The case $s = 1/2$ is not treated in either [3] or [4]. Moreover, our alternative proof is somewhat simpler than [4], and seems more 'elementary' than [3] (without using Zagier’s formula).

## 2. Preliminaries

Let $S_k(1)$ be the linear space of cusp forms of weight $k$ for the full modular group $\Gamma = SL_2(\mathbb{Z})$. Then $S_k(1)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$(f, g) = \int_{\Gamma \backslash \mathbb{H}} y^k f(z) \overline{g(z)} \frac{dx dy}{y^2}$$

and the set of all primitive forms $\mathcal{B}_k$ forms an orthogonal basis for $S_k(1)$. Moreover, we have the Petersson trace formula: define $w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} (f, f)$ and $S(m, n, c) = \sum_{ad \equiv 1 (c)} e((am + dn)/c) / \Gamma(2)$ (the classical Kloosterman sum); then

$$(2.1) \sum_{f \in \mathcal{B}_k} w_f \lambda_f(m) \lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c \geq 1} c^{-1} S(m, n, c) J_{k-1}(\frac{4\pi \sqrt{mnt}}{c})$$

where $\delta_{m,n} = 1$ or 0 according to whether $m = n$ or not, and $J_{k-1}(x)$ is the Bessel function. From [6] (5) in Section 2.13], we have the integral representation

$$(2.2) J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-1)\theta + ix \sin \theta} d\theta.$$

Bounding trivially, using integration by parts or the Poisson integral representation $J_{k-1}(x) = (\sqrt{\pi}(k-1/2))^{-1}(x/2)^{k-1} \int_{-1}^{1} (1-t^2)^{k-3/2} e^{i\pi t} dt$ (6 (3) in 2.3], we have the following estimates: for $x \geq 0$,

$$(2.3) \begin{align*}
(i) & \quad J_{k-1}(x) \ll 1, \\
(ii) & \quad J_{k-1}(x) \ll \frac{x}{k}, \\
(iii) & \quad J_{k-1}(x) \ll \frac{1}{\Gamma(k-1/2)} (\frac{x}{2})^{k-1}.
\end{align*}$$

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Using the Weil bound

\[(2.4)\quad |S(m, n, c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c),\]

and \(\lambda_f(1) = 1\), we have with (2.3)(ii)

\[(2.5)\quad \sum_{f \in B_k} w_f \ll 1 + k^{-1} \sum_{c \geq 1} c^{-3/2} \tau(c) \ll 1.\]

Define

\[(2.6)\quad \Delta(s) = \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right) = \pi^{-1-3s} 2^{1-s-k} \Gamma(s+k-1) \Gamma\left(\frac{s+1}{2}\right)\]

(as \(\Gamma(s) \Gamma(s+1/2) = \sqrt{\pi} 2^{1-2s} \Gamma(2s))\) and \(\Lambda(\text{sym}^2 f, s) = \Delta(s) L(\text{sym}^2 f, s)\). Then \(\Lambda(\text{sym}^2 f, s)\) is entire and satisfies the functional equation (shown by Shimura [5])

\[(2.7)\quad \Lambda(\text{sym}^2 f, s) = \Lambda(\text{sym}^2 f, 1-s).\]

Moreover one can show that \(\Lambda(\text{sym}^2 f, s) \to 0\) as \(|\text{Im } s| \to \infty\) in any vertical strip \(|\text{Re } s| \ll 1\).

Finally, let us explain the approach here (which is quite widely used in non-vanishing problems). Using residue theorem and the functional equation of \(L(\text{sym}^2 f, \cdot)\), we can express \(L(\text{sym}^2 f, s)\) as a convergent series. The averaging process (over all primitive forms) with Petersson trace formula yields that the (averaged) sum consists of two parts: the diagonal terms (contributed by \(\delta_{m,n}\) in (2.1)) and the off-diagonal terms. (See (3.6) below.) We then obtain the asymptotic formula (3.13) after giving an estimation to the off-diagonal terms. Our result is deduced from this formula.

### 3. Proof of the Theorem

Assume \(0 < \text{Re } s \leq 1/2\). Consider the integral \((2\pi i)^{-1} \int_{\mathcal{R}} \Lambda(\text{sym}^2 f, s+w) \frac{dw}{w}\) where \(\mathcal{R}\) is the positively oriented rectangular contour with vertices at \(\pm 2 \pm iT\), we have, by residue theorem and taking \(T \to \infty\), that

\[\Lambda(\text{sym}^2 f, s) = \frac{1}{2\pi i} \left( \int_{(2)} - \int_{(-2)} \right) \Delta(\text{sym}^2 f, s+w) \frac{dw}{w}\]

\[= \frac{1}{2\pi i} \int_{(2)} \Lambda(\text{sym}^2 f, s+w) \frac{dw}{w} + \frac{1}{2\pi i} \int_{(2)} \Lambda(\text{sym}^2 f, 1-s+w) \frac{dw}{w}\]

after using the functional equation (2.7) and changing \(w\) to \(-w\). Hence, if we write

\[(3.1)\quad V_z(y) = \frac{1}{2\pi i} \int_{(2)} \zeta(\text{sym}^2 f, z+w) \Delta(z+w) y^{-w} \frac{dw}{w},\]

we get from (1.2) that

\[(3.2)\quad L(\text{sym}^2 f, s) \Delta(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^{1-s}} V_{1-s}(n) + \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} V_s(n).\]

Let \(z = 1-s\) or \(s\). From (2.6),

\[(3.3)\quad \Delta(z+w) \ll |\text{s}| \ 2^{-k} \Gamma(\text{Re } (z+w) + k - 1) |\Gamma\left(\frac{z+w}{2} + 1\right)|\]
for $\Re (z + w) \geq -3/4$. Moving the line of integration to $\Re (z + w) = A$, we have for $A > \max(\Re z, 1/2)$,

$$V_2(y) \ll y^{\Re z - A} 2^{-k} \Gamma(k + A - 1).$$

(3.4)

Shifting to $\Re (z + w) = -1/2$ (across the poles at $w = 0, 1/2 - z$), we obtain with (3.3)

$$V_2 = \left\{ \begin{array}{l}
\zeta(2s) \Delta(2s) + \Delta(1/2)(1/2 - z)^{-1} \\
\gamma \Delta(1/2) + 2^{-1} \Delta'(1/2)
\end{array} \right. + O(2^{-k} \Gamma(k - 3/2))$$

(3.5)

where $\gamma$ is the Euler constant. The second case corresponds to $z = 1/2$. As will be seen, the main term is given by

$$V_{1-s}(1) + V_s(1) = \left\{ \begin{array}{l}
\zeta(2s) \Delta(1-s) + \zeta(2s) \Delta(s) \\
2\gamma \Delta(1/2) + \Delta'(1/2)
\end{array} \right. + \ldots$$

according to $s \neq 1/2$ or $s = 1/2$. Its order of magnitude is about $2^{-k} \Gamma(k - \Re s)$. Let $0 < \nu \leq 10^{-3}$ be a fixed number. Both sums in (3.4) over $n > k^{1+5 \nu}$ can be evaluated as follows: choosing $A = 1 + \nu^{-1}$ in [3.2], we have ($z = 1 - s$ or $s$)

$$\sum_{n > k^{1+5 \nu}} \frac{\lambda_f(n^2)}{n^z} V_2(n) \ll 2^{-k} \Gamma(k + \nu^{-1}) \sum_{n > k^{1+5 \nu}} \frac{\tau(n^2)}{n^{1+1/\nu}}$$

$$\ll 2^{-k} k^{-4-1/\nu} \Gamma(k + \nu^{-1}) \ll 2^{-k} k^{-4/4} \Gamma(k - 1/2)$$

by (1.1) and Stirling’s formula ([1, Chapter 10]). Summing over all primitive forms and using (2.1), with $\lambda_f(1) = 1$,

$$\Delta(s) \sum_{f \in B_k} w_f L(\text{sym}^2 f, s)$$

$$= V_{1-s}(1) + V_s(1) + \sum_{z = 1-s, s} 2\pi i^{-k} \sum_{n \leq k^{1+5 \nu}} n^{-\bar{z}} V_2(n) J(n)$$

$$+ O(\sum_{f} w_f) 2^{-k} k^{-4} \Gamma(k - 1/2))$$

(3.6)

where $J(n) = \sum_{c > 1} c^{-1} S(1, n^2, c) J_{k-1}(4\pi n/c)$. We give an estimate for $J(n)$. From (2.3) (ii) and (2.4),

$$\sum_{c > k^{1+20 \nu}} c^{-1} S(1, n^2, c) J_{k-1}(4\pi n/c) \ll nk^{-1} \sum_{c > k^{1+20 \nu}} c^{-3/2} \tau(c) \ll n/k^{3/2+9 \nu}.$$

By (2.3) (iii), when $n \leq k^{1-\nu}$,

$$\sum_{c \leq k^{1+20 \nu}} c^{-1} S(1, n^2, c) J_{k-1}(4\pi n/c) \ll \Gamma(k-1/2)^{-1} \sum_{c \leq k^{1+20 \nu}} c^{-1/2} \tau(c)(2\pi k^{1-\nu})^{k-1}$$

$$\ll k^{1-\nu} \Gamma(k-1/2)^{-1} \ll n/k^{3/2+9 \nu},$$

by Stirling’s formula. Similarly, for $k^{1-\nu} < n \leq k^{1+5 \nu}$ we have

$$\sum_{k^{6 \nu} \leq c \leq k^{1+20 \nu}} c^{-1} S(1, n^2, c) J_{k-1}(4\pi n/c) \ll n/k^{3/2+9 \nu}.$$
Hence,

\[ J(n) = \delta(n, k) \sum_{c \leq k^{6\nu}} e^{-\frac{3}{2}S(1, n^2, c)}J_{k-1}(\frac{4\pi n}{c}) + O(nk^{-3/2-9\nu}) \]

where \(\delta(n, k) = 0\) if \(n \leq k^{1-\nu}\), and \(1\) if \(k^{1-\nu} < n \leq k^{1+5\nu}\). Inserting (3.7) into (3.6), together with (2.5) and the estimate

\[ k^{-3/2-9\nu} \sum_{n \leq k^{1+5\nu}} |n^{-\frac{1}{2}}V_z(n)| \ll 2^{-k}k^{-3/2-9\nu}(\log k)\Gamma(k+1) \ll 2^{-k}k^{-2\nu}\Gamma(k-1/2) \]

(following from (3.4) with \(A = 2\)), we see that (3.6) becomes

\[ \Delta(s) \sum_f w_f L(\text{sym}^2 f, s) \]

(3.8) = \( V_1-s(1) + V_s(1) + i^{-k}(\mathcal{E}_k(1-s) + \mathcal{E}_k(s)) + O(2^{-k}k^{-8\nu}\Gamma(k-1/2)) \)

where \(\mathcal{E}_k(z) = 2\pi \sum_{k^{1-\nu} < n \leq k^{1+5\nu}} n^{-z}V_z(n) \sum_{c \leq k^{6\nu}} e^{-\frac{3}{2}S(1, n^2, c)}J_{k-1}(\frac{4\pi n}{c}) \).

From (2.2), we have

(3.9) \[ \mathcal{E}_k(z) = \sum_{k^{1-\nu} < n \leq k^{1+5\nu}} n^{-\frac{1}{2}}V_z(n) \sum_{c \leq k^{6\nu}} e^{-\frac{3}{2}S(1, n^2, c)} \int_{0}^{\pi/2} 2\Re f_k(\theta, \frac{4\pi n}{c}) d\theta \]

where \(f_k(\theta, x) = e^{ix \sin \theta} (e^{-i(k-1)\theta} - e^{i(k-1)\theta})\). When \(|x| \leq k^{6/5}\), we have

\[ \left| \frac{d}{d\theta} (x \sin \theta \pm (k-1)\theta) \right| \geq k \quad \text{for} \pi/2 - k^{-1/4} \leq \theta \leq \pi/2, \]

whence \( \int_{\pi/2-k^{-1/4}}^{\pi/2} f_k(\theta, \frac{4\pi n}{c}) d\theta \ll k^{-1} \) for \(4\pi n/c \leq k^{6/5}\) by integration by parts. From (3.4) with \(A = 1\) and (2.4),

\[ \ll 2^{-k}k^{-1}\Gamma(k) \sum_{k^{1-\nu} < n \leq k^{1+5\nu}} n^{-1} \sum_{c \leq k^{6\nu}} c^{-1/2} \tau(c) \ll 2^{-k}k^{-8\nu}\Gamma(k-1/2). \]

We put this estimate into (3.9). Then we interchange the sums in the remaining part and use the periodicity of \(S(1, \cdot, c)\) to give

(3.10) \[ \mathcal{E}_k(z) = \sum_{c \leq k^{6\nu}} c^{-\frac{1}{2}} \sum_{k^{1-\nu} < n \leq k^{1+5\nu}} S(1, n^2, c)n^{-\frac{1}{2}}V_z(n) \int_{0}^{\pi/2-k^{-1/4}} 2\Re f_k(\theta, \frac{4\pi n}{c}) d\theta \]

\[ + O(2^{-k}k^{-8\nu}\Gamma(k-1/2)) \]
\[ = \sum_{c \leq k^{6\nu}} \sum_{0 \leq r < c} c^{-\frac{1}{2}}S(1, n^2, c)T_z(r, c) + O(2^{-k}k^{-8\nu}\Gamma(k-1/2)) \]

where \(\mathcal{J}(n) = \delta(n, k) \sum_{c \leq k^{6\nu}} e^{-\frac{3}{2}S(1, n^2, c)}J_{k-1}(\frac{4\pi n}{c}) + O(nk^{-3/2-9\nu}) \).
with
\[ T_z(r, c) = 2 \sum_{n \in \mathbb{N}} n^{-z} V_z(n) \int_0^{\pi/2-k^{-1/4}} \Re f_k(\theta, \frac{4\pi n}{c}) d\theta. \]

From the definition of \( f_k(\cdot, \cdot) \) (the line below (3.9)), we see that
\[ T_z(r, c) \ll \int_0^{\pi/2-k^{-1/4}} \left| \int_{n \in \mathbb{N}} n^{-z} V_z(n) e(\frac{2n}{c} \sin \theta) \right| d\theta \]

by (3.11) with the path moved from \( \Re w = 2 \) to \( \kappa = 2 - \Re z \). By (3.3), \( \Delta(z + w) \ll 2^{-k} \Gamma(k + 1)(|w| + 1)^{-2} \) for \( \Re w = \kappa \). Hence,
\[ T_z(r, c) \ll 2^{-k} \Gamma(k + 1) \]

(3.12)
where \( K_1 = (k^{1-\nu} - r)/c \) and \( K_2 = (k^{1+5\nu} - r)/c \). Using \( \sum_{m \leq M} e(2m\alpha) \ll |\sin(2\pi\alpha)|^{-1} \) with partial summation, or bounding trivially, the sum in (3.11) is

(3.13)
which is absorbed by the \( O \)-term in (3.8). Therefore, (3.8) and (3.5) yield
\[ \Delta(s) \sum_{f \in \mathcal{B}_s} w_f L(\text{sym}^2 f, s) \]

(3.13)
From Stirling’s formula, we have \(\Gamma(k + z - 1) = \Gamma(k + a - 1)e^{ib\log k + O(1/k)}\) for \(z = a + ib\) for \(|z| \leq k^{1/3}\) and \(\Gamma'(k - 1/2)/\Gamma(k - 1/2) = \log k + O(1)\). Hence for the case \(s = 1/2\), the dominating term in (3.14) is \(\Delta'(1/2)\), of order \(2^{-k}(\log k)\Gamma(k-1/2)\), for all large \(k\), and we can thus conclude \(\sum_{f \in B_k} w_f L(\text{sym}^2 f, 1/2) \neq 0\). For the case \(\Re s < 1/2\), the term \(\zeta(2 - 2s)\Delta(1 - s) \approx 2^{-k}\Gamma(k - \Re s)\) dominates others for all large \(k\). (Note that \(\zeta(2 - 2s)\) is non-zero.) When \(s = 1/2 + it\) and \(t \neq 0\), denoting \(a(t) = 2^{1/2 - it}\pi^{-1/4 - 3it/2}\zeta(1 + 2it)\Gamma(3/4 + it/2)\), the main term in (3.13) is

\[
\zeta(1 + 2it)\Delta(1/2 + it) + \zeta(1 - 2it)\Delta(1/2 - it)
\]

\[
= 2^{-k}\left( a(t)\Gamma(k - 1/2 + it) + a(-t)\Gamma(k - 1/2 - it) \right)
\]

\[
= 2^{-k}\Gamma(k - 1/2)\left( 2|a(t)|\cos(t \log k + \vartheta(t)) + O(k^{-1}) \right)
\]

where \(\vartheta(t)\) is the argument of \(a(t)\). Suppose \((2\pi)^{-1}t \log 2\) is irrational. Then by Kronecker’s theorem ([2 Theorem 438]), there exist infinitely many \(r_i\) (depending on \(t\)) satisfying \(|r_i t \log 2 + \vartheta(t) - 2\pi m_i| \leq \pi/4\) for some integer \(m_i\). Thus, we take \(k = 2^r\) for those sufficiently large \(r_i\) so that the right side of (3.13) is \(\gg 2^{-k}|a(t)|\Gamma(k - 1/2) > 0\). If \((2\pi)^{-1}t \log 2\) is rational, we consider instead \((2\pi)^{-1}t \log 3\) which must then be irrational. Our result follows with the previous argument. The case \(1/2 < \Re s < 1\) is done because of the functional equation (2.7).

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